Ultraproduct of first-order lattice-valued logic LF(X) based on finite lattice implication algebra

Wang Xuefang$^1$ and Liu Peishun$^2$

1. Department of Mathematics, Ocean University of China, 266071, China
2. Department of Computer Science, Ocean University of China, 266071, China

Summary
In recent years, model theory has had remarkable success in solving important problems. Its importance lies in the observation that mathematical objects can be cast as models for a language. Ultraproduct is a method of constructing a new model from a family of models. In this paper, we deal with a new form of ultraproduct model for first-order lattice-valued logic LF(X) whose truth-value field is a finite lattice implication algebra. At the same time, Expansion theorem, two forms of fundamental theorem of ultraproducts and consistent theorem are obtained. Finally, another application of ultraproduct to algebra is discussed.

Key words:
Lattice-valued logic; lattice implication algebra; ultrafilter; ultraproduct; consistent theorem

1. Introduction
Lattice-valued logic is an important form of many-valued logic which extends the field of truth-values to lattices. More important, lattice-valued logic can represent the uncertainty, specially the incomparable property of people's thinking, judging and decision. Therefore, lattice-valued logic is studied by many scholars [2,3,4]. However, their work are limited to the interval [0,1] or the finite chain of truth values. In order to establish a logic system with truth-values in a relatively general lattice, lattice implication algebras were defined by Xu Yang in [7] and its many properties were discussed. Based on conclusions on lattice implication algebra, Xu Yang et al established several lattice-valued logic systems [8-12], where they established a first-order lattice-valued logic system LF(X) based on lattice implication algebra. On the basis of a set of grouping sentences or a theory, Wang Shiqiang et al in [2] proved the fundamental theorem of ultraproducts in lattice-valued model whenever the set of truth-values L is finite. Ying Mingsheng gave another form of the fundamental theorem of ultraproducts [13]. We have discussed one form of ultraproduct model for LF(X) [15], which is based on an ultrafilter on L', where L is any lattice implication algebra, I is an index set. Moreover, the main difference between the ultraproduct models for classical logic and for LF(X) lies in the ultrafilter, the former is a generalization of the latter. In this paper, by virtue of the idea of [6, 13], we construct another form of ultraproduct model for LF(X) based on finite lattice implication algebra by a classical ultrafilter on the index set I.

2. Preliminaries
Definition 1[7] Let ($L, \lor, \land, \neg$) be a complemented lattice with universal bounds 0, 1. If the mapping $\rightarrow : L \times L \rightarrow L$ satisfies the following conditions: for all $x, y, z \in L$,

(1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z),$  
(2) $x \rightarrow x = 1,$  
(3) $x \rightarrow y = y' \rightarrow x',$  
(4) If $x = y = x = 1$, then $x = y,$  
(5) $x \rightarrow y = (y \rightarrow x) \rightarrow x,$  
(6) $(x \lor y) \rightarrow z = (x \rightarrow z) \land (y \rightarrow z),$  
(7) $(x \land y) \rightarrow z = (x \rightarrow z) \lor (y \rightarrow z),$  

Then ($L, \lor, \land, \neg$) is said to be a lattice implication algebra (shortly as L).

Definition 2 [1] Assume I is a nonempty set and $S(I) = \{X: X \subseteq I\}$. A filter D over I is defined to be a set $D \subseteq S(I)$ such that:

(1) $I \in D$;  
(2) if $X, Y \in D$, then $X \cap Y \in D$;  
(3) if $X \in D$ and $X \subseteq Y \subseteq I$, then $Y \in D$.  

D is called an ultrafilter over I, if D is a filter over I and satisfies the following conditions:

(4) For any $X \in S(I), X \in D$ if and only if $I - X \notin D$.

In the language of first-order lattice-valued logic LF(X), we will use the following symbols:

(1) An infinite collection $V$ of variables $x_i, i \in N$;  
(2) Constant symbols: $c_k, k \in K$, where K is an index set. The set of constant symbols is written as C;  
(3) Relation symbols: $P_{m_r}, r \in R$, where R is an index set, and $P_{m_r}$ is $m_r$-ary, $m_r \in N,$;
(4) Function symbols: \( F_{m_j}, \ j \in J \), where \( J \) is an index set and \( F_{m_j} \) is \( m_j \)-ary, \( m_j \in \mathbb{N}_+ \);

(5) Logical connectives: \( \lor, \land, \rightarrow \); \( \neg \);

(6) Quantifiers: \( \forall, \exists \);

(7) Technical symbols: \( (\cdot) \).

We now define terms and formulas.

A term is defined as follows:

1. Each element of \( V \cup C \) is a term, which is said to be a non-superscript term;
2. For any \( j \in J \), if \( t_1, \ldots, t_{m_j} \) are non-superscript terms or terms, then \( F_{m_j}(t_1, \ldots, t_{m_j}) \) is a term;
3. Any term can be obtained from (1), (2).

A well-formed formula (shortly as formula) is defined as follows:

1. For any \( \alpha \in L, \alpha \in F' \);
2. For any \( r \in R \), if \( t_1, \ldots, t_{m_r} \) are terms, then \( P_r^{(r)}(t_1, \ldots, t_{m_r}) \in F' \);
3. If \( p, q \in F' \), then \( p \land q, p \lor q, p \rightarrow q, p' \in F' \);
4. If \( p \in F' \), \( x \in V \), then \( \forall x p, \exists x p \in F' \).

Definition 3 [11] A model for LF(X) is defined as follows:

\[ \mu = \{ A, \{ P_r; r \in R \}, \{ F_j; j \in J \}, \{ c_k; k \in K \} \} \]

where:

1. \( A \) is a nonempty set which is said to be the universe of the model;
2. For any \( r \in R \), \( P_r : A^{m_r} \rightarrow L \) is an \( m_r \)-ary relation assigned to each relation symbol \( P_r^{(r)} \);
3. For any \( j \in J \), \( F_j : A^{m_j} \rightarrow A \) is a \( m_j \)-ary function assigned to each function symbol \( F_{m_j} \);
4. For any \( k \in K, c_k \) is a constant assigned to each constant symbol \( c_k \).

Definition 4 [11] Let

\[ \mu = \{ A, \{ P_r; r \in R \}, \{ F_j; j \in J \}, \{ c_k; k \in K \} \} \]

and

\[ \lambda = \{ B, \{ P_r; r \in R \}, \{ F_j; j \in J \}, \{ c_k; k \in K \} \} \]

be two models for LF(X).

1. If \( A \subseteq B \) and for any \( j \in J \), \( F_{m_j} \) is an \( m_j \)-ary function symbol, \( F_{j,k} = F_{j,B} | A^{m_j} \), for any \( k \in K \), \( c_{k,k} = c_{k,B} \) and for any \( r \in R \), \( P_{m_j}^{(r)} \) is an \( m_j \)-ary relation symbol, \( P_{j,k} = P_{j,B} | A^{m_j} \), then \( \mu \) is called a submodel of \( \lambda \) or \( \lambda \) is called an extension of \( \mu \), denoted as \( \mu \subseteq \lambda \).

2. \( \mu \) and \( \lambda \) are said to be isomorphic, whenever there is a bijection \( g : A \rightarrow B \) such that:

For each \( m_j \)-ary function symbol \( F_{m_j} \) and any sequence of elements \( a_1, a_2, \ldots, a_{m_j} \in A \), we have

\[ g(F_{m_j}(a_1, a_2, \ldots, a_{m_j})) = F_{m_j}(g(a_1), g(a_2), \ldots, g(a_{m_j})) \]

For each \( m_j \)-ary relation symbol \( P_{m_j}^{(r)} \) and any sequence of elements \( a_1, a_2, \ldots, a_{m_j} \in A \), we have

\[ P_{m_j}^{(r)}(a_1, a_2, \ldots, a_{m_j}) = P_{m_j}^{(r)}(g(a_1), g(a_2), \ldots, g(a_{m_j})) \]

For each constant symbol \( c_k \), we have \( g(c_{k,B}) = c_{k,B} \).

We write as \( \mu \cong \lambda \) and call \( g \) an isomorphic mapping.

3. Expansion theorem and the fundamental theorem of ultraproducts

Suppose the set of truth-values is a finite lattice implication algebra \((L, \lor, \land, \rightarrow, 0, 1)\), and \( L = \{ \alpha_1, \ldots, \alpha_n \} \), i.e., \( L \) has \( n \) elements, \( n \in \mathbb{N} \) and \( 1 \) is any nonempty index set, \( D \) is any proper filter on \( D \) (in classical sense), \( \mu \) is an \( m \)-placed functional symbol in LF(X) and its corresponding interpretation in any model \( \mu \), \( \mu \in \mathbb{M}(i) \), \( i = 1, \ldots, n \), is a family of models for LF(X) whose universes are denoted \( A_i \), respectively. Now we give a definition of reduced product.

Definition 5 A reduced product \( \prod_{D} \mu_i \) of \( \mu_i (i = 1) \) is defined as follows:

1. Its universe is \( \prod_{D} A \);
2. Suppose \( \mu_i (r \in R, R \) is an index set) is an \( m_i \)-placed relation symbol in LF(X) and its corresponding interpretation in any model \( \mu_i \) is \( P_i^{(r)} \) \((i = 1)\), then its interpretation in \( \prod_{D} \mu_i \) is the following relation \( P' \):
   for any \( f_{D}^1, \ldots, f_{D}^{m_i} \in \prod_{D} A \), let
   \( P(f_{D}^1, \ldots, f_{D}^{m_i}) = \alpha_s \) if and only if
   \( i \in I; P_i^{(r)}(f_{D}^1(i), \ldots, f_{D}^{m_i}(i)) = \alpha_s \) \( i \in D \), \( 1 \leq s \leq n \).
3. Suppose \( F_{m_j} \) is an \( m_j \)-placed functional symbol in LF(X) and its corresponding interpretation in any
model $\mu_i$ is $F_i^j$ (i\(\in\)I), then its interpretation in $\prod D\mu_i$ is the following function $F$: for any $f^1_D, \ldots, f^m_D \in \prod D A_i$.

$$F^j (f^1_D, \ldots, f^m_D) = \leq F^j_i (f^1 (i), \ldots, f^m (i)); i \in I \not\supset D.$$  

(4) Suppose $c_k$ (k\(\in\)K, K is an index set ) is a constant symbol in LF(X), and its interpretation in any model $\mu_i$ is $c_k^i$, then its interpretation in $\prod D\mu_i$ is the constant $c = <c_k^i; i > D$.

In addition, if $D$ is an ultrafilter of $I$, then the reduced product $\prod D\mu_i$ of $\mu_i$ (i\(\in\)I) is called an ultraproduct of $\mu_i$ (i\(\in\)I).

**Lemma 1** Let $I$ be any nonempty index set, $D$ be a proper filter on $I$, $\mu_i$ (i\(\in\)I) be a family of models for LF(X).

(1) If $f^1 = \prod D g^1, \ldots, f^m = \prod D g^m$, then for any $s, l \leq n,$

$$\{i \in I; P^i (f^1 (i), \ldots, f^m (i)) = \alpha_i \} \in D$$

if and only

$$\{i \in I; P^i (g^1 (i), \ldots, g^m (i)) = \alpha_i \} \in D;$$

(2) If $f^1 = \prod D g^1, \ldots, f^m = \prod D g^m$, then

$$< F^1_i (f^1 (i), \ldots, f^m (i)); i \in I \supset D$$

$$< F^1_i (g^1 (i), \ldots, g^m (i)); i \in I \not\supset D.$$  

According to Lemma 1, the definition of reduced product $\prod D\mu_i$ is well defined, $P(f^1_D, \ldots, f^m_D)$ and $F^j_i (f^1_D, \ldots, f^m_D)$ depend only on the equivalence classes $f^1_D, \ldots, f^m_D$ and $f^1_D, \ldots, f^m_D$, but not on the representatives of these equivalence classes.

Now we give the first conclusion, namely, expansion theorem.

**Theorem 1** (Expansion Theorem) Let LF(X)' be an expansion of the language of LF(X), $\lambda_i$ be a nonempty index set, D a proper filter on $I$. If for any i\(\in\)I, $\mu_i$ is a model for LF(X), $\lambda_i$ is an expansion of the model $\mu_i$ on LF(X)', then the reduced product $\prod D\lambda_i$ is an expansion of the model $\prod D\mu_i$ on LF(X)'.

Proof. It can be proved similarly to that of classical logic.

The first form of the fundamental theorem of ultraproducts is given as follows:

**Theorem 2** (the fundamental theorem of ultraproducts I) Let L be a finite lattice implication algebra, $L= [\alpha_1, \ldots, \alpha_n]$ (n is a finite integer), I a nonempty index set, D an ultrafilter on I, $\mu_i$ (i\(\in\)I) a family of models for LF(X), and $\pi = \prod D\mu_i$ an ultraproduct model for LF(X) whose universe is B, then

(1) For any term $(x_1, \ldots, x_m)$ and any $f^1_D, \ldots, f^m_D \in B,$

$$\pi (f^1, \ldots, f^m) = \mu_i (f^1 (i), \ldots, f^m (i)); i \in I \not\supset D;$$

(2) For any formula $p(x_1, \ldots, x_m)$ and any $f^1_D, \ldots, f^m_D \in B,$

$$\pi (p[f^1_D, \ldots, f^m_D]) = \alpha_s$$ if and only if

$$\{i \in I; \mu_i (p[f^1 (i), \ldots, f^m (i)]) = \alpha_s \} \in D (1 \leq s \leq n);$$

(3) For any sentence $p,$

$$\pi (p) = \alpha_s$$ if and only if

$$\{i \in I; \mu_i (p) = \alpha_s \} \in D (1 \leq s \leq n).$$

Proof. It can be proved by induction over the construction of terms, and formulas in similar way to the proof of Theorem 1 [6].

**Theorem 3** (the fundamental theorem of ultraproducts II) Let L be a finite lattice implication algebra, $L= [\alpha_1, \ldots, \alpha_n]$ (n is a finite integer), I a nonempty index set, D an ultrafilter on I, $\mu_i$ (i\(\in\)I) a family of models for LF(X), and $\pi = \prod D\mu_i$ an ultraproduct model for LF(X) whose universe is B, then

(1) For any formula $p(x_1, \ldots, x_m)$, any $f^1_D, \ldots, f^m_D \in B$ and any $\alpha_s \in L,$

$$\pi (p[f^1_D, \ldots, f^m_D]) \supset \alpha_s$$ if and only if

$$\{i \in I; \mu_i (p[f^1 (i), \ldots, f^m (i)]) \supset \alpha_s \} \in D (1 \leq s \leq n);$$

(2) For any sentence $p$ and any $\alpha_s \in L,$

$$\pi (p) \supset \alpha_s$$ if and only if

$$\{i \in I; \mu_i (p) \supset \alpha_s \} \in D (1 \leq s \leq n).$$

Proof. (1) Suppose there are t elements greater than $\alpha_s$ in $L$, denoted as $\alpha_{s_1}, \ldots, \alpha_{s_t}$. If $\pi (p[f^1_D, \ldots, f^m_D]) \supset \alpha_s$, then there exists $\alpha_{s_0}$, such that $\pi (p[f^1_D, \ldots, f^m_D]) = \alpha_{s_0}$. By theorem 2, $X_{\alpha_s} = \{i \in I; \mu_i (p[f^1 (i), \ldots, f^m (i)]) \supset \alpha_{s_0} \} \in D.$

Since $X_{\alpha_s} \subseteq \{i \in I; \mu_i (p[f^1 (i), \ldots, f^m (i)]) \supset \alpha_s \}$ and D is a filter, we have
\{i \in I; \mu_i(p[f^1(i), \ldots, f^m(i)]) \geq \alpha_s \} \in D.

Conversely, suppose 
\{i \in I; \mu_i(p[f^1(i), \ldots, f^m(i)]) \geq \alpha_s \} \in D, then for any \(i \in \mathcal{X}, \mu_i(p[f^1(i), \ldots, f^m(i)]) \geq \alpha_s \).

Let 
\(X_\theta = \{i \in I; \mu_i(p[f^1(i), \ldots, f^m(i)]) = \alpha_{s\theta}, 1 \leq \theta \leq t, \text{ then } X = X_1 \cup \cdots \cup X_t. \) Since D is an ultrafilter, then there exists exactly one among \(X_1, \ldots, X_t\) belonging to D, we can suppose \(X_\theta \in D\), i.e.,
\[X_\theta = \{i \in I; \mu_i(p[f^1(i), \ldots, f^m(i)]) = \alpha_{s\theta} \} \in D.\]

By Theorem 2,
\[\pi(p[f^1, \ldots, f^m]) = \alpha_{s\theta} \geq \alpha_s.\]

(2) It is inferred from (1) directly.

**Definition 6** Let I be an any nonempty index set, D an ultrafilter on I, \(\mu_i, (i \in I)\) a family of models for LF(X), and \(\pi = \prod_{\mu_i,\mu} \) an ultraproduct model for LF(X). If \(\mu = \mu \) holds for any \(i \in I\), then denote \(\pi\) as \(\prod_{D,\mu}\) and call it as an ultrapower of \(\mu\).

**Corollary 1** Let L be a finite lattice implication algebra and \(\prod_{D,\mu}\) an ultrapower of \(\mu\). Then

1. \(\prod_{D,\mu} = \mu;\)
2. The natural embedding of \(\mu\) into the ultrapower \(\prod_{D,\mu}\) is an elementary embedding, i.e.,
\[\mu \subset \prod_{D,\mu}.\]

Proof. (1) It is inferred from Theorem 2 (3) directly.

(2) For any \(a \in A\), let \(\hat{a} \in \prod_{A,\mu}\) and \(\hat{a} = \{a; i \in I > \}\), i.e., for any \(i \in I\), \(\hat{a}(i) = a\). Let \(d: A \to \prod_{D,\mu}\) be a mapping satisfying: for any \(a \in A\), \(d(a) = \hat{a}_D\), where \(\hat{a}_D = \{a; i \in I > D\}\). Obviously, \(d\) is a one-to-one mapping into \(\prod_{D,\mu}\). For any formula \(p(x_1, \ldots, x_m)\) and \(a_1, \ldots, a_m \in A\), by Theorem 2, we have
\[\prod_{D,\mu} p(d[a_1], \ldots, d[a_m]) = \alpha_s\] if and only if
\[\{i \in I; \mu_i(p[\hat{a}_i(i), \ldots, \hat{a}_m(i)]) = \alpha_s \} \in D\] if and only if
\[\mu_i(p[a_1, \ldots, a_m]) = \alpha_s.\] Hence, \(\mu \subset \prod_{D,\mu}.\)

### 4. Applications of the fundamental theorem of ultraproducts

**Lemma 2** [1] Let I be a index set, \(S(I)\) be its power set. Then D is an ultrafilter over I if and only if there exists \(i \in I\) such that \(D = \{X \in S(I) | i \in X\}\).

**Theorem 3** Let L be a finite lattice implication algebra and I a nonempty index set, D an ultrafilter generated by \(i_0 \in I\), and \(\{\mu_i; i \in I\}\) be any family of models for LF(X). If \(\pi = \prod_{i \in I} \mu_i\) is an ultraproduct of \(\{\mu_i; i \in I\}\), then \(\pi \equiv \mu_{i_0}\).

Proof. Suppose the universe of \(\pi\) is B. Define a mapping as follows:
\[\gamma: B \to A_{i_0}\]
\[f_D \mapsto f(i_{i_0}).\]

(1) First we prove the mapping \(\gamma\) is a bijection. Let \(f_D, g_D \in B\) and \(f_D \neq g_D\), then \(f = D g\) does not hold, \(i_0 \notin \{i \in I | f(i) = g(i)\}\), i.e.,
\[f(i_{i_0}) \neq g(i_{i_0}),\] hence \(\gamma\) is an injection.

For any \(a_{i_0} \in A_{i_0}\), define a mapping \(f\) as follows: for any \(i \in I\),
\[f(i) = \begin{cases} a_{i_0}, & i = i_0, \\ a_i, & i \neq i_0, \end{cases}\]
randomly in \(A_s, i \in I \setminus \{i_0\}\). So \(\gamma(f_D) = f(i_{i_0}) = a_{i_0}\), then \(\gamma\) is a surjection.

(2) Let \(F_{m_j}\) be an \(m_j\) -placed functional symbol, \(j \in J\), for any \(f_D^1, f_D^2, \ldots, f_D^{m_j} \in B\), suppose
\[f = \langle F_{m_j}^1(f_D^1(i), \ldots, f_D^{m_j}(i)) | i \in I > \rangle.\]

By Definition 5, we have
\[\gamma(F_{m_j}^1(f_D^1(i), \ldots, f_D^{m_j}(i))) = \gamma(f_D) = f(i_{i_0}) = F_{m_j}^1(f_D^1(i_{i_0}), \ldots, f_D^{m_j}(i_{i_0})) = F_{m_j}^1(\gamma(f_D^1), \gamma(f_D^2), \ldots, \gamma(f_D^{m_j})).\]

(3) Let \(P_{r_{m_j}}\) be an \(m_j\) -placed relation symbol, \(r \in R\), for any \(f_D^1, f_D^2, \ldots, f_D^{m_j} \in B\), suppose
\[P(f_D^1, \ldots, f_D^{m_j}) = \alpha_s,\]
by definition 5 we have
\[\{i \in I; P_{r_{m_j}}(f_D^1(i), \ldots, f_D^{m_j}(i)) = \alpha_s \} \in D, \text{ so } P_{r_{m_j}}^D(f_D^1(i_{i_0}), f_D^2(i_{i_0}), \ldots, f_D^{m_j}(i_{i_0})) = \alpha_s.\]

(4) Let \(c_\xi, k \in K\) be a constant symbol,\(\gamma(c_{\xi_k}) = c_{\xi_{i_0}}\) holds obviously.
To sum up, $\gamma$ is an isomorphism between the models $\pi$ and $\mu_\alpha$, so $\pi \cong \mu_\alpha$.

**Definition 7**[14] A mapping from the set of all the sentences in $\text{LF}(X)$ to lattice implication algebra $L$ is called a (L-valued ) theory in $\text{LF}(X)$. If $T$ is a theory in $\text{LF}(X)$, $\mu$ is an (L-valued ) model for $\text{LF}(X)$, then for every sentence $p$, $\mu(p) \geq T(p)$, then it is called that $\mu$ satisfies $T$ or $\mu$ is a model of $T$.

**Definition 8**[14] Let $A$, $B$ be $L$-fuzzy subsets of the set of formulas in $\text{LF}(X)$. $B$ is called a finite limit of $A$, if there exists a set of finite formulas $\{p_1, \ldots, p_m\} \subseteq \text{Supp}A$ (the support of $A$), $m \in N$ is finite, such that for any formula $p$,

$$B(p) = \begin{cases} A(p_i), & \text{if } p = p_i, 1 \leq i \leq m \\ O, & \text{otherwise.} \end{cases}$$

Now we prove the following consistent theorem mathematically by the fundamental theorem of ultraproducts, which is very important application.

**Theorem 4** (Consistent Theorem) Let $L$ be a finite lattice implication algebra, and $T$ a theory in $\text{LF}(X)$, and $I = S_\alpha(\text{Supp}T)$ a set consisting of all finite subsets of $\text{Supp}T$. If for any $i \in I$, $\mu_i$ is a model of $T_i$, then there exists an ultrafilter $D$ on $I$, such that for any formula $p$,

$$\mu(p) = \begin{cases} T_i(p), & \text{if } p \in i, \\ O, & \text{otherwise,} \end{cases}$$

then there exists an ultrafilter on $I$, such that the ultraproduct $\prod_D \mu_i$ is a model of $T$.

Proof. For any sentence $p \in \text{Supp}T$, let $\hat{p} = \{i \in I; p \in i\}$, i.e., $\hat{p}$ is a set consisting of finite subsets of $\text{Supp}T$ containing $p$. Suppose $E = \{ \hat{p}; p \in \text{Supp}T\}$. Since for any $p_1, \ldots, p_n \in \text{Supp}T$, $\{p_1, \ldots, p_n\} \subseteq \hat{p_1} \cap \cdots \cap \hat{p_n}$, $E$ has the finite intersection property. By Ultrafilter Theorem ([20], Proposition 4.1.3), there exists an ultrafilter $D$ on $I$, such that $E \subseteq D$. Construct ultraproduct model $\prod_D \mu_i$ of a family of models $\mu_i$, $i \in I$, we need to prove $\prod_D \mu_i$ is a model of $T$. In fact, for any sentence $p$, let $T(p) = \alpha_s$ and $\|p\| = \{i \in I; \mu_i(p) \geq \alpha_s\}$. Since for any $i \in \hat{p}$, we have $p \in i$, then $\mu_i(p) \geq T_i(p) = T(p) = \alpha_s$, i.e., $i \in \|p\|$, hence $\hat{p} \subseteq \|p\|$. By $\hat{p} \in E$, $E \subseteq D$ and since $D$ is an ultrafilter, we have $\|p\| \in D$. It is inferred from Theorem 2 that $\prod_D \mu_i(p) \geq \alpha_s$, i.e., $\prod_D \mu_i(p) \geq T(p)$. Therefore, $\prod_D \mu_i$ is a model of $T$.

Obviously, Theorem 4 can be described as follows: Let $L$ be a finite lattice implication algebra, and $T$ a theory in $\text{LF}(X)$. If every finite limit of $T$ has a model, then $T$ has a model.

Now we give another application of the fundamental theorem of ultraproducts. First, we give a notion.

**Definition 9** Let $\mu$ be a model for $\text{LF}(X)$, its universe be $A$, $\{\mu_i; i \in I\}$ be a family of submodels of $\mu$ and their universe be $\Lambda_i$, respectively. If $A = \bigcup_{i \in I} A_i$ and for any $i, j \in I$, there exists $k \in I$, such that $A_i \cup A_j \subseteq A_k$, then $\{\mu_i; i \in I\}$ is said to be a local system of $\mu$.

For example, all finitely generated submodels of $\mu$ consist of a local system of $\mu$.

**Theorem 5** Let $\mu$ be a model for $\text{LF}(X)$, $\{\mu_i; i \in I\}$ a local system of $\mu$. If there exists a family of models $\{A_i; i \in I\}$, such that for any $i, j \in I$, $\mu_i$ is isomorphically embedded in $\Lambda_j$, i.e., $\mu_i \subseteq \Lambda_j$, then there exists an ultrafilter $D$ on $I$, such that $\mu$ is isomorphically embedded in $\prod_D \Lambda_i$, i.e., $\mu \subseteq \prod_D \Lambda_i$.

Proof. Similar to the proof of the corresponding theorem of classical logic and omit it.

5. Conclusions

In this paper, we developed model theory of lattice-valued first-order logic $\text{LF}(X)$ and gave an important conclusion, the fundamental theorem of ultraproducts. All of the above work will lay a solid foundation for further study on lattice-valued logic. More important, since a model in many-valued logic is namely a control process, so all the results on model theory that have been obtained here can help us to interpret the results of approximate reasoning on semantic level in some model. One of our future work is to further study model theory and construct approximate reasoning method by it.

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References


Wang Xuefang received the B.S. degree in Department of Mathematics and Computer Science from Qufu Normal University from 1994 and 1998. During 1998-2004, received the M.S. degree and Ph.D degree in Department of Applied Mathematics and College of Computer and Communication from Southwest Jiaotong University, respectively. Finally, work in Department of Mathematics, Ocean University of China.