Stability Analysis of Fuzzy Cellular Neural Networks with Time-Varying Delays and Impulses*

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Summary
In this paper, a generalized model of fuzzy cellular neural networks (FCNN) with time-varying delays and impulses is investigated. By employing the delay differential inequality with impulses initial conditions and using the properties of M-cone and eigenspace of the spectral radius of nonnegative matrices, some sufficient conditions for global exponential stability of FCNN with time-varying delays and impulses are obtained. An example is given to show the effectiveness of the obtained results.
Keywords:
Fuzzy cellular neural networks; delays; impulses; global exponential stability.

1. Introduction

Since cellular neural networks (CNN) was introduced by Chua and Yang in [1, 2], many researchers have done extensive works on this subject due to their comprehensive applications in classification of patterns, associative memories, image processing, quadratic optimization, and other areas, e.g.,Refs.[3-10]. However, in mathematical modeling of real world problems, we encounter inconveniences, namely, the complexity and the uncertainty or vagueness. In order to take vagueness into consideration, fuzzy theory is considered as a suitable setting. Based on traditional CNN, Yang and Yang proposed the fuzzy cellular neural networks (FCNN)[11, 12], which integrates fuzzy logic into the structure of the traditional CNN and maintains local connectedness among cells. Unlike previous CNN structures, FCNN has fuzzy logic between its template input and/or output besides the sum of product operation. FCNN is very useful paradigm for image processing problems, which is a cornerstone in image processing and pattern recognition. In such applications, it is of prime importance to ensure that the designed FCNN be stable. In [11, 12], the authors have obtained some conditions for the existence and the global stability of the equilibrium point of FCNN without delays. In [13], Liu and Tang have considered FCNN with either constant delays or time-varying delays, several sufficient conditions have been obtained to ensure the existence and uniqueness of the equilibrium point and its global exponential stability. Yuan, Cao and Deng have given several novel criteria of exponential stability and periodic solutions for FCNN with time-varying delays [14]. Recently, Huang has considered the stability of FCNN with diffusion terms and time-varying delay [16], at the same time, Huang has investigated the exponential stability of FCNN with distributed delay [15].

However, besides delay effect, impulsive effect likewise exists in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time, involving such fields as medicine and biology, economics, mechanics, electronics and telecommunications, etc. Many interesting results on impulsive effect have been gained, e.g., Refs. [17-23]. As artificial electronic systems, neural networks such as CNN, bidirectional neural networks and recurrent neural networks often are subject to impulsive perturbations which can affect dynamical behaviors of the systems just as time delays. Therefore, it is necessary to consider both impulsive effect and delay effect on the stability of neural networks. Xu and Yang in [24] have established a new delay differential inequality with impulsive initial conditions, and using the properties of M-cone and eigenspace of the spectral radius of nonnegative matrices, some new sufficient conditions for global exponential stability of impulsive delay neural networks are obtained. To the best of our knowledge, few authors have considered FCNN with time-varying delays and impulses.

Motivated by the above discussions, in this paper, we intend to study the global exponential stability of FCNN with time-varying delays and impulses, by applying the impulsive delay differential inequality and using the properties of M-cone and eigenspace of the spectral radius of nonnegative matrices, we shall give some sufficient conditions for the globally exponential stability of the equilibrium point of impulsive delay FCNN.

The remainder part of this paper is organized as follows. Model description and preliminaries are given in

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section 2. In section 3, main results and their proofs are presented. An example is given to illustrate our theory in section 4. Finally, in section 5， we give the conclusion.

2. Model description and preliminaries

In this section, we will consider the model of fuzzy cellular neural networks involving time-varying delays and impulses, it is described by the following impulsive system:

$$\begin{cases} y_i'(t) = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} F_j(y_j(t)) + \sum_{j=1}^{n} \tilde{a}_{ij} u_j \\ \quad + \bigwedge_{j=1}^{n} b_{ij} F_j(y_j(t-\tau_{ij}(t))) \\ \quad + \bigvee_{j=1}^{n} \tilde{b}_{ij} F_j(y_j(t-\tau_{ij}(t))) \\ \quad + \bigwedge_{j=1}^{n} T_{ij} u_j + \bigvee_{j=1}^{n} H_{ij} u_j + I_i, \quad t \neq t_k, \\ y_i(t) = V_{ik}(y_1(t^-), \cdots, y_n(t^-)) + J_{ik} \\ \quad + W_{ik}(y_1(t-\tau_{i1}(t))^-, \cdots, y_n(t-\tau_{in}(t))^-), \\ \qquad t = t_k, \end{cases} \tag{1}$$

for $i = 1,2,\cdots,n$. Where $c_i > 0,\ 0 \le \tau_{ij} \le \tau$, the fixed moments of time t_k satisfy $t_1 < t_2 < \cdots$, $\lim_{k\to\infty} t_k = \infty$, $i, j = 1,2,\cdots,n$ and $k = 1,2,\cdots$. The first part (called continuous part) of model (1) describes the continuous processes of FCNN. n corresponds to the number of units in the neural network; y_i corresponds to the state variable; $F_j(y_j(t))$ denotes the activation function of the jth neurons; u_i and I_i denote input and bias of the ith neuron, respectively. c_i represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the networks and external inputs; a_{ij} and $\tilde{a}_{ij}$ are elements of feedback template and feedforward template, respectively; b_{ij}, $\tilde{b}_{ij}$ are elements of the delay fuzzy feedback MIN template, the delay fuzzy feedback MAX template, respectively; T_{ij} and H_{ij} are elements of fuzzy feedforward MIN template and fuzzy feedforward MAX template, respectively; $\tau_{ij}(t)$ corresponds to the transmission delay. $\wedge$ and $\vee$ denote the fuzzy AND and fuzzy OR operation, respectively. The second part (called discrete part) of model (1) describes that the evolution processes experience abrupt change of states at the moments of time t_k (called impulsive moments). $V_{ik}(y_1(t^-), \cdots, y_n(t^-))$ represents impulsive perturbations of the ith unit at time t_k and $y_j(t^-)$ denotes the left limit of $y_j(t)$ for $j = 1,2,\cdots n$, $W_{ik}(y_{i1}(t-\tau_{i1}(t))^-, \cdots, y_{in}(t-\tau_{in}(t))^-)$ denotes impulsive perturbations of the ith unit at time t_k which caused by the transmission delays, J_{ik} represents external impulsive input at time t_k.

To begin with, we introduce some notation and recall some basic definitions. Let R^n be the space of n-dimensional real column vectors and $R^{m\times n}$ denote the set of $m\times n$ real matrices. Usually E denotes an $n\times n$ unit matrix and $e_n = (1,\cdots,1)^T \in R^n$. For $A, B \in R^{m\times n}$ or $A, B \in R^n$, $A \ge B$ ($A \le B, A > B, A < B$) means that each pair of corresponding elements of A and B satisfies the inequality "$\ge$ ($\le, >, <$)". Especially, A is called a nonnegative matrix if $A \ge 0$, and z is called a positive vector if $z > 0$.

For $x = (x_1, \cdots, x_n)^T \in R^n$, we denote

$$\mathrm{sgn}(x) = diag(\mathrm{sgn}(x_1), \cdots, \mathrm{sgn}(x_n)),$$

$$\mathrm{sgn}(x_i) = \begin{cases} -1, & if\ x_i < 0, \\ 0, & if\ x_i = 0, \\ +1, & if\ x_i > 0. \end{cases}$$

$C[X,Y]$ denotes the space of continuous mappings from the topological space X to the topological space Y. Especially, let $C_\tau = C[[-\tau,0], R^n]$, where $\tau > 0$.

$C[I, R^n] = \{\psi : I \to R^n \mid \psi(t^+) = \psi(t)$ for $t \in I$, $\psi(t^-)$ exists for $t \in (t_0, +\infty)$, $\psi(t^-) = \psi(t)$ for all but points $t_k \in (t_0, +\infty)$ }, where $I \subset R$ is an interval, $\psi(t^+)$ and $\psi(t^-)$ denote the left-hand and right-hand limits of scalar function $\psi(t)$, respectively.

For $x \in R^n, A \in R^{n\times n}$ or $\varphi \in C_\tau$, we define

$$|x| = (|x_1|, \cdots, |x_n|)^T, \qquad |A| = (|a_{ij}|)_{n\times n},$$

$$[\varphi(t)]_\tau = ([\varphi_1(t)]_\tau, \cdots, [\varphi_n(t)]_\tau)^T,$$

$$|\varphi(t)|_\tau = [|\varphi(t)|]_\tau,$$

where $[\varphi_i(t)]_\tau = \sup_{-\tau\le s\le 0}\{\varphi_i(t+s)\}$. And we introduce the corresponding norm for them as follows:

$$\|x\| = \max_{1\le i\le n}\{|x_i|\},$$

$$\|A\| = \max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|,$$

$$\|\varphi\| = \max_{1\le i\le n}\{[|\varphi_i(t)|]_\tau\}.$$

Definition 1 For any given $t_0 \in R, \varphi \in C_\tau$, a function $y(t) \in C[[t_0-\tau,+\infty), R^n]$ is called a solution of model (1) through (t_0,φ), if $y(t)$ satisfies the initial condition in form

$$y(t_0+s) = \varphi(s), \qquad s\in[-\tau,0], \tag{2}$$

and satisfies model (1) for $t > t_0$, denoted by $y(t,t_0,\varphi)$. Especially, a point y^* is called an equilibrium point of (1), if $y(t) = y^*$ is a solution of (1).

Throughout this paper, we assume that for any $\varphi\in C_\tau$, there exists at least one solution with the initial values φ of model (1). Let y^* be an equilibrium point of model (1) and $y(t)$ be any solution of (1). Let $x(t) = y(t) - y^*$, substituting them into model (1), we can get

$$\begin{cases} x_i'(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) \\ \qquad + \bigwedge_{j=1}^{n} b_{ij}F_j(y_j(t-\tau_{ij}(t))) - \bigwedge_{j=1}^{n} b_{ij}F_j(y_j^*) \\ \qquad + \bigvee_{j=1}^{n} \tilde{b}_{ij}F_j(y_j(t-\tau_{ij}(t))) - \bigvee_{j=1}^{n} \tilde{b}_{ij}F_j(y_j^*), \\ \qquad\qquad t\ne t_k, \\ x_i(t) = v_{ik}(x_1(t^-),\cdots,x_n(t^-)) \\ \qquad + w_{ik}(x_1(t-\tau_{i1}(t))^-,\cdots,x_n(t-\tau_{in}(t))^-), \\ \qquad\qquad t = t_k, \end{cases} \tag{3}$$

where

$$f_j(x_j(t)) = F_j(x_j(t)+y_j^*) - F_j(y_j^*),$$

$$v_{ik}(x_i(t)) = V_{ik}(x_i(t)+y_i^*) - V_{ik}(y_i^*),$$

$$w_{ik}(x_i(t-\tau_{ij}(t))) = W_{ik}(x_i(t-\tau_{ij}(t)) + y_i^*) - W_{ik}(y_i^*).$$

It is clear that the stability of the zero solution of model (3) is equivalent to the stability of the equilibrium point y^* of model (1). Therefore, we may mainly discuss the stability of the zero solution of system (3).

Definition 2 The zero solution of system (3) is said to be globally exponentially stable if for any solution $x(t,t_0,\varphi)$ with the initial condition $\varphi\in C_\tau$, there exist constants $\alpha>0$ and $\kappa>1$ such that

$$\|x(t,t_0,\varphi)\| \le \kappa\|\varphi\| e^{-\alpha(t-t_0)}, \qquad t\ge t_0. \tag{4}$$

Definition 3 [25] A real matrix $D=(d_{ij})_{n\times n}$ is said to be a non-singular M-matrix if $d_{ij}\le 0, i,j=1,2,\cdots,n,$ $i\ne j$, and all successive principal minors of D are positive.

To the non-singular M-matrix, we have

Lemma 1 [25] Each of the following conditions is equivalent:

(i) D is a nonsingular M-matrix.

(ii) $D = C - M$ and $\rho(C^{-1}M) < 1$, where $M\ge 0$, $C = diag(c_1,\cdots,c_n)$ and $\rho(\cdot)$ is the spectral radius of the matrix $(\cdot)$.

(iii) The diagonal elements of D are all positive and there exists a positive vector d such that $Dd>0$ or $D^T d > 0$.

Especially, the matrix D is a nonsingular M-matrix if it is row or column strictly dominant diagonal, that is, $De_n > 0$ or $D^T e_n > 0$. For a nonsingular M-matrix D, we denote

$$\Omega_M(D) = \{z\in R^n \mid Dz>0, z>0\},$$

which is a nonempty set by (iii) of Lemma 1, and satisfying that $k_1z_1 + k_2z_2 \in \Omega_M(D)$ for any scalars $k_1,k_2>0$ and vectors $z_1,z_2\in\Omega_M(D)$. So $\Omega_M(D)$ is a cone without vertex in R^n. We call it an "M-cone".

For a nonnegative matrix $A\in R^{n\times n}$, let $\rho(A)$ be the spectral radius of A. Then $\rho(A)$ is an eigenvalue of A and its eigenspace is denoted by

$$\Omega_\rho(A) = \{z\in R^n \mid Az = \rho(A)z\},$$

which includes all positive eigenvectors of A provided that the nonnegative matrix A has at least one positive eigenvector (see Refs. [25, 26]).

Lemma 2 [11] Suppose y and $\bar{y}$ are two state of model (1), then we have

$$\left|\bigwedge_{j=1}^{n} \alpha_{ij} F_j(y_j) - \bigwedge_{j=1}^{n} \alpha_{ij} F_j(\bar{y}_j)\right|$$

$$\leq \sum |\alpha_{ij}| |F_j(y_j) - F_j(\bar{y}_j)|,$$

$$\left|\bigvee_{j=1}^{n} \alpha_{ij} F_j(y_j) - \bigvee_{j=1}^{n} \alpha_{ij} F_j(\bar{y}_j)\right|$$

$$\leq \sum |\alpha_{ij}| |F_j(y_j) - F_j(\bar{y}_j)|.$$

Lemma 3 [24] Let $P = (p_{ij})_{n\times n}$ and $p_{ij} \geq 0$ for $i \neq j$, $Q = (q_{ij})_{n\times n} \geq 0$, and let $D = -(P+Q)$ be a nonsingular M-matrix. For $b \in (t_0, +\infty)$, let $u(t) = (u_1(t), \cdots, u_n(t))^T \in C[[t_0, b), R^n]$ be a solution of the following delay differential inequality with the initial condition $u(s) \in C_\tau, t_0 - \tau \leq s \leq t_0$:

$$D^+ u(t) \leq Pu(t) + Q[u(t)]_\tau, \quad t \geq t_0. \tag{5}$$

Then

$$u(t) \leq z\, e^{-\lambda(t-t_0)}, \quad t \geq t_0, \tag{6}$$

provided that the initial conditions satisfies

$$u(s) \leq z\, e^{-\lambda(s-t_0)}, \quad t_0 - \tau \leq s \leq t_0, \tag{7}$$

where $z = (z_1, \cdots, z_n)^T \in \Omega_M(D)$ and the positive number λ is determined by the following inequality:

$$[\lambda E + P + Qe^{\lambda\tau}]z < 0. \tag{8}$$

To obtain our results, we give the following assumptions.

(**H1**) For any $x, \bar{x} \in R^n$ there exists a nonnegative diagonal matrix $\bar{F}$ such that

$$|F(x) - F(\bar{x})| \leq \bar{F}\, |x - \bar{x}|, \tag{9}$$

where $F(x) = (F_1(x_1), \cdots, F(x_n))^T$, $x = (x_1, \cdots, x_n)^T$, $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)^T$.

Note. (**H1**) is a vector form of the globally Lipschitz conditions used by many researchers. Since $f_j(x_j(t)) = F_j(x_j(t) + y_j^*) - F_j(y_j^*)$, $j = 1, 2, \cdots, n$. i.e, $f(x(t)) = F(x(t) + y^*) - F(y^*)$, and $f(0) = 0$, it is obvious that $|f(x)| \leq \bar{F}\, |x|$.

(**H2**) For any $x \in R^n$ there exist nonnegative matrices R_k, S_k such that

$$|v_k(x)| \leq R_k\, |x|, \; |w_k(x)| \leq S_k\, |x|, \tag{10}$$

for $k = 1, 2, \cdots$. Where $v_k(x) = (v_{k1}(x_1), \cdots, v_{kn}(x_n))^T$, $w_k(x) = (w_{k1}(x_1), \cdots, w_{kn}(x_n))^T$.

3. Main results

In this section, we will give several sufficient conditions on the global exponential stability of equilibrium point for the fuzzy cellular neural network (1).

Theorem 1 Under assumptions (**H1**) and (**H2**), if

(i) $D = C - (|A| + |B| + |\tilde{B}|)\bar{F}$ is a non-singular M-matrix, where $C = diag(c_1, \cdots, c_n)$, $|A| = (|a_{ij}|)_{n\times n}$, $|B| = (|b_{ij}|)_{n\times n}$, $|\tilde{B}| = (|\tilde{b}_{ij}|)_{n\times n}$.

(ii) $\Omega = \bigcap_{k=1}^{\infty} [\Omega_\rho(R_k) \cap \Omega_\rho(S_k)] \cap \Omega_M(D)$ is nonempty.

(iii) There exists a constant μ such that

$$\frac{\ln \mu_k}{t_k - t_{k-1}} \leq \mu \leq \lambda, \quad k = 1, 2, \cdots, \tag{11}$$

where the scalar λ is determined by the following inequality

$$\left[\lambda E - C - [|A| + (|B| + |\tilde{B}|)\, e^{\lambda\tau}]\bar{F}\right] z < 0, \tag{12}$$

for a given $z \in \Omega$, and

$$\mu_k = \max\{1, \rho(R_k) + \rho(S_k)\, e^{\lambda\tau}\}. \tag{13}$$

Then the zero solution of system (3) is globally exponentially stable and the exponential converging index is $\lambda - \mu$.

Proof. Calculating the upper right derivative $D^+ |x(t)|$ along the solutions of system (3), from Lemma 2, we have

$$
\begin{aligned}
& D^{+} | x(t) | \\
& \leq \operatorname{sgn}(x_i'(t)) x_i'(t) \\
& \leq -c_i | x_i(t) | + \sum_{i=1}^{n} | a_{ij} \| f_j(x_j(t) | \\
& \quad + | \mathop{\wedge}_{j=1}^{n} b_{ij} F_j(x_j(t-\tau_{ij}(t)) + y_j^*) - \mathop{\wedge}_{j=1}^{n} b_{ij} F_j(y_j^*) | \\
& \quad + | \mathop{\vee}_{j=1}^{n} \tilde{b}_{ij} F_j(x_j(t-\tau_{ij}(t)) + y_j^*) - \mathop{\vee}_{j=1}^{n} \tilde{b}_{ij} F_j(y_j^*) | \\
& \leq -c_i | x_i(t) | + \sum_{i=1}^{n} | a_{ij} \| f_j(x_j(t) | \\
& \quad + \sum_{j=1}^{n} | b_{ij} \| f_j(x_j(t-\tau_{ij}(t))) | \\
& \quad + \sum_{j=1}^{n} | \tilde{b}_{ij} \| f_j(x_j(t-\tau_{ij}(t))) | \\
& = -c_i | x_i(t) | + \sum_{i=1}^{n} | a_{ij} \| f_j(x_j(t) | \\
& \quad + \sum_{j=1}^{n} (| b_{ij} | + | \tilde{b}_{ij} |) | f_j(x_j(t-\tau_{ij}(t))) |
\end{aligned} \tag{14}
$$

for $t_{k-1} \leq t < t_k$, $k = 1,2,\cdots$. Rewriting (14) as following matrix form, we get

$$
\begin{aligned}
D^{+} | x(t) | &\leq -C | x(t) | + | A \| f(x(t)) | \\
&\quad + (| B | + | \tilde{B} |) | f(x(t)) |_{\tau}
\end{aligned} \tag{15}
$$

for $t_{k-1} \leq t < t_k$, $k = 1,2,\cdots$.

We obtain from assumption (**H1**) that

$$
\begin{aligned}
D^{+} | x(t) | &\leq -C | x(t) | + | A | \overline{F} | x(t) | \\
&\quad + (| B | + | \tilde{B} |) \overline{F} | x(t) |_{\tau} \\
&= P | x(t) | + Q | x(t) |_{\tau}
\end{aligned} \tag{16}
$$

for $t_{k-1} \leq t < t_k, k = 1,2,\cdots$, where $P = -C + | A | \overline{F}$, $Q = (| B | + | \tilde{B} |)\overline{F}$. Since $D = -(P+Q)$ is a nonsingular M-matrix and the set Ω is nonempty, there exists at least one vector $z \in \Omega \subset \Omega_M(D)$ such that

$$Dz > 0, \quad \text{or} \quad (P+Q)z < 0.$$

By using of continuity, we know that (12) has at least one positive solution λ.

For the initial conditions: $x(t_0 + s) = \varphi(s)$, $s \in [-\tau, 0]$, where $\varphi \in C_{\tau}$ and $t_0 \in R$ (no loss of generality, we assume $t_0 \leq t_1$), we can get

$$| x(t) | \leq d \| \varphi \| e^{-\lambda(t-t_0)}, \quad t_0 - \tau \leq t \leq t_0, \tag{17}$$

where

$$d = \frac{z}{\min\limits_{1 \leq i \leq n} z_i} \geq e_n.$$

From the property of M-cone and $z \in \Omega \subset \Omega_M(D)$, we have $d\|\varphi\| \in \Omega_M(D)$. Then all conditions of Lemma 3 are satisfied by (16), (17) and (i) in Theorem 1, it follows that

$$| x(t) | \leq d \| \varphi \| e^{-\lambda(t-t_0)}, \quad t_0 \leq t \leq t_1. \tag{18}$$

Suppose that for all $m (1 \leq m \leq k)$ the inequalities

$$| x(t) | \leq \mu_0 \mu_1 \cdots \mu_{m-1} d \| \varphi \| e^{-\lambda(t-t_0)}, \quad t_{m-1} \leq t \leq t_m \tag{19}$$

hold, where $\mu_0 = 1$. Then, from (**H2**) and (19), the discrete part of system (3) satisfies that

$$
\begin{aligned}
| x(t_k) | &\leq | v_k(x(t_k^-)) | + | w_k(x(t_k - \tau(t_k))^-) | \\
&\leq R_k | x(t_k^-) | + S_k | x(t_k - \tau(t_k))^- | \\
&\leq R_k \mu_0 \mu_1 \cdots \mu_{k-1} d \| \varphi \| e^{-\lambda(t_k - t_0)} \\
&\quad + S_k \mu_0 \mu_1 \cdots \mu_{k-1} d \| \varphi \| e^{-\lambda(t_k - \tau - t_0)} \\
&\leq [R_k + S_k e^{\lambda \tau}] d \mu_0 \mu_1 \cdots \mu_{k-1} \| \varphi \| e^{-\lambda(t_k - t_0)}.
\end{aligned} \tag{20}
$$

Since $d \in \Omega \subset \Omega_{\rho}(R_k) \cap \Omega_{\rho}(S_k)$, we have $R_k d = \rho(R_k) d$ and. $S_k d = \rho(S_k) d$ Hence we obtain from (13) and (20) that

$$
\begin{aligned}
& | x(t_k) | \\
& \leq [\rho(R_k) + \rho(S_k) e^{\lambda \tau}] d \mu_0 \mu_1 \cdots \mu_{k-1} \| \varphi \| e^{-\lambda(t_k - t_0)} \\
& \leq \mu_0 \mu_1 \cdots \mu_{k-1} d \| \varphi \| e^{-\lambda(t_k - t_0)}.
\end{aligned} \tag{21}
$$

From (21) and (19), we derive

$$| x(t) | \leq \mu_0 \cdots \mu_{k-1} \mu_k d \| \varphi \| e^{-\lambda(t-t_0)} \tag{22}$$

for all $t \in [t_k - \tau, t_k]$. By the property of M-cone again, the vector $\mu_0 \cdots \mu_{k-1} \mu_k d \| \varphi \| \in \Omega_M(D)$. It follows from (22) and Lemma 3 that

$$| x(t) | \leq \mu_0 \cdots \mu_{k-1} \mu_k d \| \varphi \| e^{-\lambda(t-t_0)}, t_k \leq t < t_{k+1}. \tag{23}$$

By the mathematical induction, we can conclude that

$$| x(t) | \leq \mu_0 \cdots \mu_{k-1} \mu_k d \| \varphi \| e^{-\lambda(t-t_0)}, t_{k-1} \leq t < t_k. \tag{24}$$

for $k = 1,2,\cdots$.

Also, from (11), we have

$$\mu_k \leq e^{\mu(t_k - t_{k-1})}, k = 1,2,\cdots,$$

Applying these inequalities to (24), we can conclude that

$$|x(t)| \le e^{\mu(t_1-t_0)} \cdots e^{\mu(t_{k-1}-t_{k-2})} d\|\varphi\| e^{-\lambda(t-t_0)}$$
$$\le d\|\varphi\| e^{\mu(t-t_0)} e^{-\lambda(t-t_0)}$$
$$= d\|\varphi\| e^{-(\lambda-\mu)(t-t_0)}, \ \forall t \in [t_0, t_{k+1}), k = 1,2,\cdots.$$

This implies that the zero solution of model (3) is globally exponentially stable, and the exponential converging index is $\lambda - \mu$.

Theorem 2 Under assumptions (**H1**) and (**H2**), if $D = C - (|A| + |B| + |\tilde{B}|)\bar{F}$ is a row strictly dominant diagonal matrix, and the following inequalities

$$\mu_k \ge \max\left\{1, \left\|R_k + S_k e^{\lambda\tau}\right\|\right\}$$

and

$$(\lambda E + P + Qe^{\lambda\tau})e_n < 0 \tag{25}$$

hold, where $P = C - |A|\bar{F}$, $Q = (|B| + |\tilde{B}|)\tilde{F}$. Then the zero solution of system (3) is globally exponentially stable and the exponential converging index is $\lambda - \mu$.

Proof. Since D is row strictly dominant diagonal, we have

$$(P+Q)e_n < 0.$$

By using continuity, we obtain that the strictly inequality in (25) has at least one positive solution λ. The remainder of the proof of Theorem 2 is essentially the same as the proof of Theorem 1 except that one choose $z = e_n$ and notices that the inequality

$$|x(t)| \le \mu_0\mu_1 \cdots \mu_{m-1} e_n \|\varphi\| e^{-\lambda(t-t_0)}, \tag{26}$$

holds for $t_{m-1} \le t < t_m$, $m = 1,2,\cdots,k$, it can derive from (**H2**) and (25) that

$$|x(t_k)| \le |v_k(x(t_k^-))| + |w_k(x(t_k - \tau(t_k))^-)|$$
$$\le R_k |x(t_k^-)| + S_k |x(t_k - \tau(t_k))^-|$$
$$\le R_k \mu_0\mu_1 \cdots \mu_{k-1} e_n \|\varphi\| e^{-\lambda(t_k-t_0)}$$
$$+ S_k \mu_0\mu_1 \cdots \mu_{k-1} e_n \|\varphi\| e^{-\lambda(t_k-\tau-t_0)}$$
$$\le [R_k + S_k e^{\lambda\tau}] e_n \mu_0\mu_1 \cdots \mu_{k-1} \|\varphi\| e^{-\lambda(t_k-t_0)}$$
$$\le \left\|R_k + S_k e^{\lambda\tau}\right\| e_n \mu_0\mu_1 \cdots \mu_{k-1} \|\varphi\| e^{-\lambda(t_k-t_0)}.$$
$$\le \mu_0 \cdots \mu_{k-1}\mu_k e_n \|\varphi\| e^{-\lambda(t_k-t_0)}. \tag{27}$$

In Theorem 1, we may properly choose the matrices R_k and S_k in (**H2**) to guarantee $\Omega \ne \varnothing$. In particular, when $R_k = \alpha_k E$ and $S_k = \beta_k E$ (α_k, β_k are nonnegative constants), Ω is nonempty. It follows from Theorem 1 that we have

Corollary 1 Under assumption (**H1**), if

(i) $D = C - (|A| + |B| + |\tilde{B}|)\bar{F}$ is a non-singular M-matrix, where $C = diag(c_1, \cdots, c_n)$, $|A| = (|a_{ij}|)_{n\times n}$, $|B| = (|b_{ij}|)_{n\times n}$, $|\tilde{B}| = (|\tilde{b}_{ij}|)_{n\times n}$.

(ii) For any $x \in R^n$ there exist nonnegative constants α_k, β_k such that

$$|v_k(x)| \le \alpha_k |x|, \ |w_k(x)| \le \beta_k |x|, k = 1,2,\cdots. \tag{28}$$

(iii) There exists a constant μ such that

$$\frac{\ln \mu_k}{t_k - t_{k-1}} \le \mu \le \lambda, \qquad k = 1,2,\cdots, \tag{29}$$

where the scalar λ is determined by the following inequality

$$\left[\lambda E - C - [|A| + (|B| + |\tilde{B}|)e^{\lambda\tau}]\bar{F}\right]z < 0, \tag{30}$$

for a given $z \in \Omega_M(D)$, and

$$\mu_k \ge \max\{1, \alpha_k + \beta_k e^{\lambda\tau}\}. \tag{31}$$

Then the zero solution of system (3) is globally exponentially stable and the exponential converging index is $\lambda - \mu$.

Proof. From (ii), we have $R_k = \alpha_k E$ and $S_k = \beta_k E$, it follows that

$$\rho(R_k) = \alpha_k, \qquad \rho(S_k) = \beta_k,$$
$$\Omega_\rho(R_k) = \Omega_\rho(S_k) = R^n,$$
$$\Omega = \bigcap_{k=1}^{\infty}[\Omega_\rho(R_k) \cap \Omega_\rho(S_k)] \cap \Omega_M(D)$$
$$= \Omega_M(D).$$

Since The M-cone $\Omega_M(D) \ne \varnothing$, the condition (ii) in Theorem 1 holds. By using Theorem 1, we can deduce the conclusion.

Remark 1. If $V_{ik}(y_1(t^-), \cdots y_n(t^-)) = (y_1, \cdots, y_n)^T$, $W_{ik}(y_1((t-\tau_{i1}(t))^-), \cdots y_n((t-\tau_{in}(t))^-) = 0, J_k = 0,$ then model (1) becomes delay FCNN without impulses. Here, we have

Corollary 2 (Theroem 1 in [13]) Under assumption (**H1**), if $D = C - (|A| + |B| + |\tilde{B}|)\overline{F}$ is a non-singular M-matrix, then the fuzzy system (1) has exactly one equilibrium point.

Thus it can be seen that Theorem 1 ensures that the fuzzy impulsive system (1) has a unique equilibrium point, which is globally exponentially stable.

Remark 2. Let us consider as following system

$$\begin{cases} y_i'(t) = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} F_j(y_j(t)) + \sum_{j=1}^{n} \tilde{a}_{ij} u_j \\ \quad + \bigwedge_{j=1}^{n} b_{ij} G_j(y_j(t - \tau_{ij}(t))) \\ \quad + \bigvee_{j=1}^{n} \tilde{b}_{ij} G_j(y_j(t - \tau_{ij}(t))) \\ \quad + \bigwedge_{j=1}^{n} T_{ij} u_j + \bigvee_{j=1}^{n} H_{ij} u_j + I_i, \quad t \neq t_k, \\ y_i(t) = V_{ik}(y_1(t^-), \cdots, y_n(t^-)) + J_{ik} \\ \quad + W_{ik}(y_1(t - \tau_{i1}(t))^-, \cdots, y_n(t - \tau_{in}(t))^-), \\ \qquad t = t_k, \end{cases} \tag{32}$$

for i = 1, 2, · · · , n. where $G_j(y_j(t))$, the same as $F_j(y_j(t))$, denotes the activation function of the jth neuron, other symbols posses the same meaning as that of (1). We assume that

(**H** $1'$) For any $x, \overline{x} \in R^n$, there exists nonnegative diagonal matrices $\overline{F}$ and $\overline{G}$ such that

$$\begin{aligned} |F(x) - F(\overline{x})| &\le \overline{F}\,|x - \overline{x}|, \\ |G(x) - G(\overline{x})| &\le \overline{G}\,|x - \overline{x}|. \end{aligned} \tag{33}$$

Let $f(x(t)) = F(x(t) + y^*) - F(y^*)$, $g(x(t)) = G(x(t) + y^*) - G(y^*)$, then we have $|f(x)| \le \overline{F}\,|x|$, $|g(x)| \le \overline{G}\,|x|$.

By a minor modification of the proof of Theorem 1, we can easily derive

Theorem 3 Under assumptions (**H**$1'$) and (**H2**), if

(i) $D = C - (|A| + |B| + |\tilde{B}|)\overline{F}$ is a non-singular M-matrix, where $C = diag(c_1, \cdots, c_n)$, $|A| = (|a_{ij}|)_{n\times n}$, $|B| = (|b_{ij}|)_{n\times n}$, $|\tilde{B}| = (|\tilde{b}_{ij}|)_{n\times n}$.

(ii) $\Omega = \bigcap_{k=1}^{\infty}[\Omega_\rho(R_k) \cap \Omega_\rho(S_k)] \cap \Omega_M(D)$ is nonempty.

(iii) There exists a constant μ such that

$$\frac{\ln \mu_k}{t_k - t_{k-1}} \le \mu \le \lambda, \qquad k = 1, 2, \cdots \tag{34}$$

where the scalar λ is determined by the following inequality

$$\left[\lambda E - C - [|A|\,\overline{F} + (|B| + |\tilde{B}|)\overline{G}\, e^{\lambda\tau}]\right] z < 0, \tag{35}$$

for a given $z \in \Omega$, and

$$\mu_k = \max\{1, \rho(R_k) + \rho(S_k)\, e^{\lambda\tau}\}. \tag{36}$$

Then the equilibrium point y^* of system (32) is globally exponentially stable and the exponential converging index is $\lambda - \mu$.

Remark 3. In general, ones always assume that there is an equilibrium point for the impulsive systems to study their stability. However, we show in Corollary 2 that there is a unique equilibrium point y^* of the continuous part of the system (1) under the conditions (**H1**) and (i) (in Theorem 1). In many cases, y^* may not be a solution of the discrete part of the system (1) without the external impulsive input. That is, the entire system (1) may have no equilibrium point. In order to guarantee that the entire system (1) has an equilibrium point, we introduce the external impulsive input J_k so that y^* is also an equilibrium point of the discrete part of the system (1).

4 An illustrative example

Example Consider the following model

$$\begin{cases} y_i'(t) = -c_i y_i(t) + \sum_{j=1}^{2} a_{ij} f_j(y_j(t)) + \sum_{j=1}^{2} \tilde{a}_{ij} u_j \\ \quad + \bigwedge_{j=1}^{2} b_{ij} f_j(y_j(t-\tau_{ij}(t))) \\ \quad + \bigvee_{j=1}^{2} \tilde{b}_{ij} f_j(y_j(t-\tau_{ij}(t))) \\ \quad + \bigwedge_{j=1}^{2} T_{ij} u_j + \bigvee_{j=1}^{2} H_{ij} u_j + I_i, \quad t \neq t_k, \\ y_i(t) = V_{ik}(y_1(t^-), y_2(t^-)) \\ \quad + W_{ik}(y_1(t-\tau_{i1}(t))^-, y_2(t-\tau_{i2}(t))^-), \\ \qquad t = t_k, \end{cases} \tag{37}$$

for $i = 1,2,\ \ t_1 = 0.3,\ \ t_k = t_{k-1} + 0.3k,\ k = 1,2,\cdots$, and where

$$C = \begin{pmatrix} 6 & \\ & 6 \end{pmatrix},\quad A = \begin{pmatrix} \frac{1}{4} & -1 \\ 3 & -1 \end{pmatrix},\quad \tilde{A} = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}\quad \tilde{B} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix},\quad I_1 = I_2 = 2,$$

$$u_1 = u_2 = 1,\quad T = (T_{ij}) = E,\quad H = (H_{ij}) = E,$$

$$f_1(y) = f_2(y) = \frac{|y+1| + |y-1|}{2},$$

$$\tau_{ij}(t) = |\sin(i+j)t| \le 1 = \tau \text{ for } i, j = 1,2.$$

(i) If $V_{ik}(y_1, y_2) = y_i$, $W_{ik}(y_1, y_2) = 0$, $J_k = 0$ for $i = 1,2$ and $k = 1,2,\cdots$, then the system (37) becomes delay fuzzy cellular neural networks without impulses. Here, model (37) satisfies all assumptions of Corollary 2 in this paper with $\overline{F} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$. It is easy computing that $D = C - (|A| + |B| + |\tilde{B}|)\overline{F} = \begin{pmatrix} 5 & -2 \\ -4 & 4 \end{pmatrix}$ is a nonsingular M-matrix. By Corollary 2, we know that the system (37) has exactly one globally exponentially stable equilibrium point, which is actually $\left(\frac{25}{24}, \frac{7}{6}\right)^T$.

(ii) If

$$V_{1k}(y_1, y_2) = 0.2e^{0.05k} y_1 - 0.1e^{0.05k} y_2,$$
$$V_{2k}(y_1, y_2) = -0.4e^{0.05k} y_1 + 0.2e^{0.05k} y_2,$$
$$W_{1k}(y_1, y_2) = 0.4e^{0.05k} y_1, \quad J_{1k} = 1 - 0.5e^{0.05k},$$
$$W_{2k}(y_1, y_2) = -0.4e^{0.05k} y_2, \quad J_{2k} = 1 + 0.6e^{0.05k},$$

we can verify that the point $\left(\frac{25}{24}, \frac{7}{6}\right)^T$ is also an equilibrium point of the impulsive system (37), and the parameters of (**H2**) and condition (ii) (in Theorem 1) are as follows:

$$R_k = 0.1e^{0.05k} \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix},\quad \rho(R_k) = 0.4e^{0.05k},$$

$$S_k = 0.2e^{0.05k} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},\quad \rho(S_k) = 0.4e^{0.05k},$$

$$\Omega_\rho(R_k) = \{(z_1, z_2)^T \mid z_2 = 2z_1\},$$
$$\Omega_\rho(S_k) = R^2,$$
$$\Omega_M(D) = \{(z_1, z_2)^T > 0 \mid z_1 < z_2 < \frac{5}{2} z_1\}.$$

So $\Omega = \{(z_1, z_2)^T \mid z_2 = 2z_1\}$ is nonempty. Let $d = (1,2)^T \in \Omega$ and $\lambda = 0.2427$ which satisfies the inequality $(\lambda E + P + Qe^{\lambda\tau})d < 0$ for $k = 1,2,\cdots$, we can get

$$\mu_k = e^{0.05k} \ge \max\{1,\ 0.4e^{0.05k} + 0.4e^{0.05k} e^{0.2427}\},$$

$$\frac{\ln \mu_k}{t_k - t_{k-1}} \le \frac{\ln e^{0.05k}}{0.3k} \le 0.1667 = \mu < \lambda.$$

Obviously, all conditions of Theorem 1 are satisfied, hence the equilibrium point is globally exponentially stable and the exponential converging index is approximately equal to 0.076.

5 Conclusions

Stability is important in the applications and theories of neural networks. By employing the delay differential inequality with impulses initial conditions and using the properties of M-cone and eigenspace of the spectral radius of nonnegative matrices, we have obtained some sufficient conditions of the global exponential stability of FCNN with delays and impulses. It is believed that these results are significant and useful for the design and applications of the fuzzy cellular neural networks.

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