Weak LI-ideal in Lattice Implication Algebra

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Summary
In this paper, the notions of weak LI-ideals (briefly, WLI-ideals) and maximal weak LI-ideals of lattice implication algebra are introduced, respectively. The properties of weak LI-ideals are investigated. Several characterizations of weak LI-ideals are given. Therefore, this article aims at discussing new development in LI-ideals and properties.
Keywords:
Lattice implication algebra; LI-ideal; WLI-ideals.

1. Introduction

Non-classical logic has become a considerable formal tool for artificial intelligence to deal with uncertainty information and automated reasoning. Many-valued logic a great extension and development of classical logic (see e.g. [2]), it provide an interesting alternative to the classical logic for modeling and reasoning about systems. In the field of many-valued logic, lattice valued plays an important role (see e.g. [5, 7]). Hence Goguen [1], Pavelka [3], and Novak [4] researched on this lattice-valued logic formal systems. Moreover, in order to research the many-valued logical systems whose propositional value is given in a lattice, in 1990, Xu [6, 17] proposed the notion of lattice implication algebras and investigated many useful properties. Since then this logical algebra has been extensively investigated by several researchers (see e.g. [8, 10, 15, 16]). In [9], Jun et al. defined the concept of LI-ideals in lattice implication algebras and discussed its some properties. For the general development of lattice implication algebras, the ideal theory plays an important role (see e.g. [11, 12, 13, 14]). In this article, as an extension of above-mention work we propose the concept of WLI-ideals in lattice implication algebras, and investigate the properties of WLI-ideals.

2. Preliminaries

Definition 2.1. [6] A bounded lattice (L, $\vee$, $\wedge$, $'$, O, I) with ordered-reversing involution $'$ and a binary operation $\rightarrow$ is called a lattice implication algebra if it satisfies the following axioms:

(L_1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(L_2) $x \rightarrow x = I$,

(L_3) $x \rightarrow y = y' \rightarrow x'$,

(L_4) $x \rightarrow y = y \rightarrow x = I$ imply $x = y$,

(L_5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,

(L_6) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,

(L_7) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$,

for all $x, y, z \in L$.

A lattice implication algebra L is called lattice H implication algebra if it satisfies:

$x \vee y \vee ((x \wedge y) \rightarrow z) = 1$ for all $x, y, z \in L$.

Theorem 2.1. [17] $(L, \vee, \wedge, ', \rightarrow)$ is a lattice H implication algebra if and only if $(L, \vee, \wedge, ')$ is a Boolean lattice, x' is the complement of x and $x \rightarrow y = x' \vee y$ for any $x, y \in L$.

Definition 2.2. [9] Let A be a lattice implication algebra. An LI-ideal A is non-empty subset of L such that for any $x, y \in L$,

(1). $O \in A$;

(2). $(x \rightarrow y)' \in A$ and $y \in A$ imply $x \in A$.

Theorem 2.2. [9] Let A be an LI-ideal of a lattice implication algebra L. if $x \le y$ and $y \in A$ then $x \in A$.

Theorem 2.3. Let A be a non-empty subset of a lattice implication algebra L. Then A is an ILI-ideal of L if and

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only if it satisfies for all $z \in A$ and $y, (x \to y)' \in L$, $((x \to y)' \to y)' \le z$ imply $(x \to y)' \in A$.

Proof. Now suppose first that A is an ILI-ideal of L. Let $z \in A$ and $(x \to y)' \in L$. Then

$$((x \to y)' \to y)' \le z$$

Implies

$$(((x \to y)' \to y)' \to z)' = O \in A.$$

Using definition we obtain $(x \to y)' \in A$.

Conversely, suppose that $((x \to y)' \to y)' \le z$ implies $(x \to y)' \in A$ for all $z \in A$, $y \in L$, and $(x \to y)' \in L$. Since A is a non-empty subset. Hence $((o \to o)' \to z)' \le z$ implies $o \in A$ holds. On the other hand,

$$((x \to y)' \to y)' \le z$$
$$\Leftrightarrow ((x \to y)' \to y)' \to z = I$$
$$\Leftrightarrow (((x \to y)' \to y)' \to z)' = O \in A.$$

Then we have

$$(((x \to y)' \to y)' \to z)' \in A$$

implies $(x \to y)' \in A$ holds. Moreover A is an ILI-ideal of L. this completes the proof.

3. WLI-ideals of lattice implication algebras

Definition 3.1. Let L be a lattice implication algebra, a subset A of L is called a weak LI-ideals (briefly, WLI-ideal) of L if it satisfies the following condition

$$(x \to y)' \in A$$

implies

$$((x \to y)' \to y)' \in A \quad \text{holds for all} \quad x, y \in L.$$

The following example shows that there exists the WLI-ideal in lattice implication algebra.

Example 3.1. Let A= {I, O} be a set. Now it takes $x = O, y = I$ then $(O \to I)' = O \in A$ implies $((O \to I)' \to I)' = O \in A$;

if $x = I, y = O$ then $(I \to O)' = I \in A$ implies Example 3.1 $((I \to O)' \to O)' = I \in A$. Hence A is a WLI-ideal.

Example 3.2. Let B={O} be a set. B={O} can be check similarly.

Theorem 3.1. Let L be a lattice implication algebra, $A \subseteq L$ is an LI-ideal of L. Then A is a WLI-ideal of L.

Proof. Suppose that A is an LI-ideal of L and $(x \to y)' \in A$ for all $x, y \in L$. Then

$$(((x \to y)' \to y)' \to (x \to y)')'$$
$$= ((x \to y) \to ((x \to y)' \to y)')'$$
$$= ((x \to y) \to (y' \to (x \to y)))'$$
$$= (y' \to ((x \to y) \to (x \to y)))'$$
$$= O \in A,$$

i.e., $(((x \to y)' \to y)' \to (x \to y)')' \in A$. Thus $((x \to y)' \to y)' \in A$ as A is an LI-ideal and $(x \to y)' \in A$. Therefore, A is a WLI-ideal of L. This completes the proof.

Theorem 3.2. Let L be a lattice implication algebra. Every ILI-ideal of is a WLI-ideal.

Proof. Suppose that A is an ILI-ideal of lattice implication algebra L, and $(x \to y)' \in A$ for al**l** $x, y \in L$.

Since

$$(((((x \to y)' \to y)' \to o)' \to o)' \to (x \to y)')'$$
$$= ((((x \to y)' \to y)' \to o)' \to (x \to y)')'$$
$$= (((x \to y)' \to y)' \to (x \to y)')'$$
$$= ((x \to y) \to ((x \to y)' \to y))'$$
$$= (y \to I)' = o \in A.$$

Hence we obtain $((x \to y)' \to y)' \in A$ by A is an ILI-ideal of L. Therefore A is a WLI-ideal of L.

Theorem 3.3. A is a non-empty subset of lattice implication algebra L and $A' = \{x' : x \in A\}$, then A' is a WLI-ideal of L if and only if A is a weak filter of L.

Proof. Assume that for any $x, y \in L$, A is a weak filter of L and $x \to y \in A$

implies $x \to (x \to y) \in A$ holds.

Then $(x \to y)' \in A'$ implies $(x \to (x \to y))' \in A'$,

i.e., if $(y' \to x')' \in A'$

then $((y' \to x')' \to x')' \in A'$.

Thus A' is a WLI-ideal of L.

Conversely, let A' is a WLI-ideal of L and $(x \to y)' \in A'$ implies $((x \to y)' \to y)' \in A'$ for all $x, y \in L$. Since $x \to y = y' \to x' \in A$;

$$(((x \to y)' \to y)')'$$
$$= (x \to y)' \to y$$
$$= (y' \to (y' \to x')) \in A.$$

Moreover, we get
$y' \to x' \in A$ implies $(y' \to (y' \to x')) \in A$ holds. Hence A is a weak filter of L. Ending the proof.

Theorem 3.4. Every lattice ideal in lattice H implication algebra L is a WLI-ideal of L.

Proof. Let L be a lattice H implication algebra, A is a lattice ideal and $(x \to y)' \in A, y \in A$ for all $x, y \in L$. For

$$y \vee (x \to y)' = y \vee (x' \vee y)' = x \vee y.$$

Hence $x \vee y \in A$. It follows that

$$\begin{aligned} & y \vee ((x \to y)' \to y)' \\ &= y \vee (((x' \vee y)')' \vee y)' \\ &= y \vee ((x' \vee y) \vee y)' \\ &= y \vee (x \wedge y') = x \vee y. \end{aligned}$$

So that $y \vee ((x \to y)' \to y)' \in A$. Since

$$((x \to y)' \to y)' \leq y \vee ((x \to y)' \to y)'.$$

Therefore $((x \to y)' \to y)' \in A$ by A is lattice ideal of L. This completes the proof.

Corollary 3.5. Let L be a lattice H implication algebra, then LI-ideal {o} of L is WLI-ideal.

Theorem 3.6. Let L be a lattice H implication algebra, if $A(t) = \{x \in L : (x \to t)' = o\}$ for all the element t of L. Then $A(t)$ is a WLI-ideal of L.

Proof. Suppose that $(x \to y)' \in A(t)$ for all $x, y \in L$, then

$$((x \to y)' \to t)' = o$$
$$\Leftrightarrow ((x \to y) \vee t)' = o,$$

i.e., $(x \to y)' \wedge t' = o$.

Since

$$\begin{aligned} & (((x \to y)' \to y)' \to t)' \\ &= (((x \to y) \vee y)' \to t)' \\ &= \ (((x \to y) \vee y) \vee t)' \\ &= \ ((x \to y)' \wedge y') \wedge t' \\ &= \ ((x \to y)' \wedge t') \wedge y' \\ &= \ o \wedge y' = o, \end{aligned}$$

we have $((x \to y)' \to y)' \in A(t)$. $A(t)$ is a WLI-ideal of L by Definition 3.1.

Theorem 3.7. Let L be a lattice H implication algebra, if A is an LI-ideal of L then $A_t = \{x \in L : (x \to t)' \in A\}$ is a WLI-ideal for any $t \in L$.

Proof. Assume that $(x \to y)' \in A_t$ for all $x, y \in L$, then $((x \to y)' \to t)' \in A$. Since

$$\begin{aligned} & ((((x \to y)' \to y)' \to t)' \to ((x \to y)' \to t)')' \\ &= (((x \to y)' \to t) \to (((x \to y)' \to y)' \to t))' \\ &= (((x \to y)' \to y)' \to (((x \to y)' \to t) \to t))' \\ &= (((x \to y)' \to y)' \to ((x \to y)' \vee t))' \\ &= ((((x \to y)' \to y)' \to (x \to y)') \\ & \vee (((x \to y)' \to y)' \to t))' \\ &= (((x \to y) \to ((x \to y)' \to y)) \\ & \vee (t' \to ((x \to y)' \to y))' \\ &= (((x \to y) \to ((x \to y)' \to y)) \\ & \vee (t' \to ((x \to y)' \to y))' \\ &= ((y' \to I) \vee (y' \to (t' \to (x \to y))))' \\ &= (I \vee (y' \to (t' \to (x \to y))))' \\ &= o \wedge (((x \to y)' \to t)' \to y)' \\ &= o \wedge ((x \to y) \vee t)' \wedge y' = o \in A. \end{aligned}$$

Note that if A is an LI-ideal of L, then

$$(((x \to y)' \to y)' \to t)' \in A.$$

Hence $((x \to y)' \to y)' \in A_t$. Consequently, the result is valid.

Theorem 3.8. Let L be a lattice implication algebra, $\{A_i : i \in I\}$ is the set of WLI-ideal of L for I is a index set, then $\bigcup_{i \in I} A_i$ and $\bigcap_{i \cup I} A_i$ are WLI-ideals.

Proof. Let $(x \to y)' \in \bigcup_{i \in I} A_i$ for all $x, y \in L$, then there exist $i \in I$ such that $(x \to y)' \in A_i$. Since A_i is WLI-ideal, which imply that $((x \to y)' \to y)' \in A_i$ for some $i \in I$. Hence we get $((x \to y)' \to y)' \in \bigcup_{i \in I} A_i$. By Definition 3.1, $\bigcup_{i \in I} A_i$ is a WLI-ideal of L.

Suppose that $(x \to y)' \in \bigcap_{i \cup I} A_i$ for any $x, y \in L$, then $(x \to y)' \in A_i$ for any $i \in I$. Since A_i is a WLI-ideal of L, we have $((x \to y)' \to y)' \in A_i$ for any $i \in I$. Thus $((x \to y)' \to y)' \in \bigcap_{i \cup I} A_i$. Therefore $\bigcap_{i \cup I} A_i$ is a WLI-ideal.

Remark: Let L be a lattice implication algebra, the intersection of WLI-ideals of L is also a WLI-ideal by Theorem 3.8. Suppose $A \subseteq L$, the maximal WLI-ideal containing A is called the WLI-ideal generated by A and denoted by L < A >.

Definition 3.2 Let L be a lattice implication algebra, a WLI-ideal is called a maximal WLI-ideal if it is not L, and it is a maximal element of the set of all WLI-ideals with respect to set inclusion.

In what follows, for any $a \in L$,

$L_a^1 = \{((x \to y)' \to y)' : x, y \in L, (x \to y)' = a\}$;

$L_a^2 = \{((x \to y)' \to y)' : x, y \in L, (x \to y)' = L_a^1\}$;

$L_a^3 = \{((x \to y)' \to y)' : x, y \in L, (x \to y)' = L_a^2\}$;

$L_a^4 = \{((x \to y)' \to y)' : x, y \in L, (x \to y)' = L_a^3\}$;

$\vdots \qquad \vdots$

$L_a^n = \{((x \to y)' \to y)' : x, y \in L, (x \to y)' = L_a^{n-1}\}$.

It is easy to check

$$((x \to y)' \to y)' = (((x \to y)' \to y)' \to o)';$$

$$(((x \to y)' \to y)' \to y)' \leq ((x \to y)' \to y)'.$$

Hence $L_a^n \subseteq L_a^{n-1} \cdots \subseteq L_a^4 \subseteq L_a^3 \subseteq L_a^2 \subseteq L_a^1$ and denoted by $T_a = \bigcap_{i=1}^{\infty} L_a^i$.

Theorem 3.9. Let L be a lattice implication algebra, then T_a is a WLI-ideal for any $a \in L$.

Proof. Suppose that $(x \to y)' \in T_a$ for any $x, y \in L$, then there exists $i \geq 1$ and it is the element of the set of {0,1,2,3…,…} such that $(x \to y)' \in L_a^i$. Hence $((x \to y)' \to y)' \in L_a^{i+1}$, i.e., $((x \to y)' \to y)' \in T_a$. Therefore, T_a is a WLI-ideal of L by Definition 3.1.

Theorem 3.10. Let L be a lattice implication algebra, $x, a \in L$, then $x \in T_a$ if and only if there exist $k \in N^+$, x_k, x_{k-1}, $x_2, x_1 \in L$, and y_k, y_{k-1}, $y_2, y_1 \in L$, if it satisfies the follows conditions:

(1). $(x \to y)' = a$;

(2). $(x_i \to y_i)' \in L_a^{i-1}$

and $(x_i \to y_i)' = ((x_{i-1} \to y_{i-1})' \to y_{i-1})'$;

(3). $((x_k \to y_k)' \to y_k)' = x$.

Proof. Assume that conditions hold. Obviously, $x \in T_a$.

Let $x \in T_a$, then there exist $k \in N^+$ such that $x \in L_a^k$ by $T_a = \bigcap_{i=1}^{\infty} L_a^i$, i.e., $\exists x_k, y_k \in L$ such that $x = ((x_k \to y_k)' \to y_k)'$. Thus we have $(x \to y)' \in L_a^{k-1}$. Since there exist $x_{k-1}, y_{k-1} \in L$ such that $(x_k \to y_k)' = ((x_{k-1} \to y_{k-1})' \to y_{k-1})'$ for $x_k \to y_k \in L_a^{k-1}$, and so we get $x_{k-1} \to y_{k-1} \in L_a^{k-2}$. It follows that we can be obtain sequences x_k, x_{k-1}, $x_2, x_1 \in L$, and y_k, y_{k-1}, $y_2, y_1 \in L$ such that three conditions hold. Ending the proof.

Theorem 3.11. Let L be a lattice implication algebra, then $T_a = < a >$ for any $a \in L$.

Proof. Suppose that $a \in T_a$ then $< a > \subseteq T_a$ by Theorem 3.9. On the other hand, let $a \in T_a$ then there exist $k \in N^+$ such that x_k, x_{k-1}, $x_2, x_1 \in L$, and y_k, y_{k-1}, y_2, $y_1 \in L$ satisfy the following conditions:

(1). $(x \to y)' = a$;

(2). $(x_i \to y_i)' \in L_a^{i-1}$

and $(x_i \to y_i)' = ((x_{i-1} \to y_{i-1})' \to y_{i-1})'$ (i= 2,3…);

(3). $((x_k \to y_k)' \to y_k)' = x$.

Moreover, we have $(x_i \to y_i)' \in < a >$ (i= 1, 2, 3,..., k), i.e., $T_a \subseteq < a >$. Consequently, the result is valid.

Theorem 3.12. Let L be a lattice implication algebra, $A \subseteq L$. Then $< A > = \bigcap_{a \in A} < a >$.

Proof. Since $a \in < a >$ for all $a \in A$, we have $A \subseteq \bigcap_{a \in A} < a >$. Thus $< A > \subseteq \bigcap_{a \in A} < a >$. On the other hand, if $\forall a \in A$ then $< a > \subseteq < A >$. Hence we obtain $\bigcap_{a \in A} < a > \subseteq < A >$. Thus we have $< A > = \bigcap_{a \in A} < a >$

Corollary 3.13. Let L be a lattice implication algebra, $A \subseteq L$, $B \subseteq L$ and $A \subseteq B$. then $< A > \supseteq < B >$.

4. Conclusion

In order to provide a logical foundation for uncertain information processing theory, especially for the fuzziness, the incomparability in uncertain information in the reasoning, Xu initiated the notion of lattice implication algebras. Hence for development of non-classical logical system, it is needed to make clear the structure of lattice implication algebra. It well known that to investigate the structure of a logical system, the ideals with special properties play an important role. The aim of this article is to introduce the concept of WLI-ideal in lattice implication algebra and investigate related properties. Hence the research for the properties of ideals of lattice implication algebra will advance the research of logical system with proporsitional value.

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