Exponential Stability of Generalized Cohen-Grossberg Neural Networks with Time-Varying Delays and Reaction-Diffusion term

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Summary
In this paper, a generalized model of Cohen-Grossberg neural networks (CGNNs) with time-varying delays and reaction-diffusion term is investigated. By constructing suitable Lyapunov functional, inequality technique and M -matrix theory, some sufficient conditions for global exponential stability of generalized CGNNs with time-varying delays and reaction-diffusion term are obtained. An examples are given to show the effectiveness of the obtained results.

Key words:
Cohen-Grossberg neural networks; time-varying delays; reaction-diffusion; global exponential stability.

1. Introduction

In 1983, Cohen and Grossberg studied the following neural networks (see [1]), which is described by the ordinary differential equations:

$$\frac{dx_i}{dt} = -a_i(x_i)\left[b_i(x_i) - \sum_{j=1}^{m} t_{ij} S_j(x_j) + J_i\right], \qquad (1)$$

for $i = 1,2,\cdots,m$, where x_i denotes the state variable associated with the ith neuron, a_i represents an amplification function, b_i is an appropriately behaved function, and $m \geq 2$ is the number of neurons in the network. $T = (t_{ij})_{m\times m}$ denotes the $m \times m$ connection matrix, which tells how the neurons are connected in the network, and the activation functions $S_j(x_j)$ show how neurons respond to each other, and J_i denotes the ith component of an external input source introduced from outside the network to the cell *i*. This type of network has been widely studied in recent years and has been found applications in many areas.

The systems (1) are usually called as Cohen-Grossberg neural networks (CGNNs), clearly, the CGNNs include the well-known Hopfield neural networks (see [2,3,5]):

$$\frac{dx_i}{dt} = d_i x_i - \sum_{j=1}^{m} t_{ij} S_j(x_j) + I_i,\ i = 1,\cdots,m, \qquad (2)$$

Due to their promising potential applications in areas such as pattern recognition and optimization. The network (1) have attracted increasing interest in scientific community, see, for example, [6,10-12,15-20,23,25,26] and references cited therein.

In reality, time delays inevitably exist in biological and artificial neural networks due to the finite switching speed of neurons and amplifiers. It is also important to incorporate time delay in various neural networks. In recent years, there exist some results on global asymptotical stability, global exponential stability and periodic solutions for the neural networks with constant delays or time-varying delays (see [4,6-26]). However, the diffusion phenomena could not be ignored in neural networks and electric circuits once electrons transport in a nonuniform electromagnetic field. Hence, it is essential to consider the state variables are varying with the time and space variables. The neural networks with diffusion terms can commonly be expressed by partial differential equations. The study on the stability of reaction–diffusion neural networks, for instance, see [21-24], and references therein.

In this paper, we intend to study the global exponential stability of generalized CGNNs with reaction-diffusion terms and time-varying delays in a very general setting. Under quite general conditions, we apply the idea of vector function, M-matrix theory and inequality technique, by constructing suitable Lyapunov functional, several new sufficient conditions are obtained to ensure the existence, uniqueness, and global exponential stability of equilibrium point for the generalized CGNNs neural networks with reaction-diffusion terms and time-varying delays. These results generalize and improve the earlier publications.

The paper is organized as follows. Model description and preliminaries are given in section 2. In section 3, main results and their proofs are presented. Some remarks are given to compare with the earlier references, and an example are given to illustrate our theory in section 4. Finally, in section 5 we give the conclusion.

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2. Model description and preliminaries

Consider the generalized Cohen-Grossberg neural networks with time-varying delays and reaction-diffusion term:

$$\begin{cases} \dfrac{\partial u_i(t,x)}{\partial t} = \sum_{k=1}^{m} \dfrac{\partial}{\partial x_k}\left(D_{ik} \dfrac{\partial u_i(t,x)}{\partial x_k}\right) \\ \qquad - a_i(u_i(t,x))[b_i(u_i(t,x)) \\ \qquad - \sum_{j=1}^{n} c_{ij} f_j(u_j(t,x)) \\ \qquad - \sum_{j=1}^{n} d_{ij} g_j(u_j(t-\tau_{ij}(t),x) + J_i],\ x \in \Omega \\ \dfrac{\partial u_i}{\partial \tilde{n}} = \left(\dfrac{\partial u_i}{\partial x_1}, \cdots, \dfrac{\partial u_i}{\partial x_m}\right)^T = 0,\ \ t \ge 0,\ x \in \partial\Omega, \\ u_i(s,x) = \phi_i(s,x),\ \ -\tau \le s \le 0, \end{cases} \tag{3}$$

for $i = 1,2,\cdots,n$. Where $n \ge 2$ is the number of neurons in the network, x_i are space variables, $u_i(t,x)$ is the state variable of the *i*th neuron at time t and in space x, $f_j(u_j(t,x))$ and $g_j(u_j(t,x))$ denote the output of the *j*th unit at time t and in space x, smooth function $D_{ik} = D_{ik}(t,u,x) \ge 0$ is diffusion operator, $\tau_{ij}(t)$ is the transmission delay along the axon of jth unit from the *i*th unit at time t, and $0 \le \tau_{ij}(t) \le \tau_{ij} \le \tau = \max_{1\le i,j\le n}\{\tau_{ij}\}$, $\tau'_{ij}(t) \le 0$. Ω is a compact set with smooth boundary $\partial\Omega$ and measure $\text{mes}\Omega > 0$ in R^m. $\phi_i(s,x)$ is the initial boundary value. a_i presents an amplification function, b_i can include a constant term indicating a fixed input to the network, J_i denotes the external bias on the *i*th unit, constants c_{ij} and d_{ij} weight the strength of the *j*th unit on the *i*th unit at time t and $t - \tau_{ij}(t)$, respectively.

To obtain our results, we give the following assumptions:

(**A1**) Each function $a_i(u)$ is bounded, positive and continuous, furthermore $0 < \underline{a}_i \le a_i(u) \le \overline{a}_i$ for all $u \in R,\ i = 1,2,\cdots,n$.

(**A2**) For each function $b_i(u) \in C^1(R,R)$, there exists a positive constant $b_i > 0$ such that $b_i'(u) \ge b_i > 0$, for $i = 1,2,\cdots,n$.

(**A3**) For functions f_i and g_i, there exist two positive diagonal matrices $F = \text{diag}(F_1, F_2, \cdots, F_n)$ and $G = \text{diag}(G_1, G_2, \cdots, G_n)$ such that

$$F_i = \sup_{u_1 \ne u_2} \left| \frac{f_i(u_1) - f_i(u_2)}{u_1 - u_2} \right|,$$

$$G_i = \sup_{u_1 \ne u_2} \left| \frac{g_i(u_1) - g_i(u_2)}{u_1 - u_2} \right|$$

for all $u_1 \ne u_2$, $i = 1,2,\cdots,n$.

For convenience, we introduce several notations. For a $n \times n$ matrix A, $|A|$ denotes the absolute value matrix given by $|A| = (|a_{ij}|)_{n\times n}$, For

$$u(t,x) = (u_1(t,x), u_2(t,x), \cdots, u_n(t,x))^T \in R^n,$$

denotes

$$\|u_i(t,x)\|_2 = \left(\int_\Omega |u_i(t,x)|^2\, dx\right)^{\frac{1}{2}},\ i = 1,2,\cdots,n.$$

For $\phi(s,x) = (\phi_1(t,x), \phi_2(s,x), \cdots, \phi_n(s,x))^T \in C[(-\tau,0] \times R^m, R^n]$, the norm is defined by

$$\|\phi\| = \sum_{i=1}^{n} \sup_{-\tau\le s\le 0} \|\phi_i(s,x)\|_2,\ r \ge 1,$$

It can be easily proved that the $C[(-\tau,0] \times R^m, R^n]$ is a Banach space.

Definition 1. The equilibrium point $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^T \in R^n$ of model (3) is said to be globally exponentially stable if there exist two positive constants M and λ such that

$$\sum_{i-1}^{n} \left\|u_i(t,x) - u_i^*\right\|_2 \le Me^{-\lambda t} \left\|\phi - u^*\right\|,\ t \ge 0.$$

Definition 2. ([27]) A real matrix $A = (a_{ij})_{n\times n}$ is said to be a nonsingular M-matrix if $a_{ij} \le 0, i, j = 1,\cdots,n,$ $i \ne j$, and all successive principal minors of A are positive.

Lemma 1. ([27]) D is a nonsingular M-matrix if and only if the diagonal elements of D are all positive and there exists a positive vector l such that $Dl > 0$ or $D^T l > 0$.

3. Main results

Theorem 1. Under assumptions (**A1**), (**A2**) and (**A3**), model (3) has a unique equilibrium point, which is globally exponentially stable if

$$\underline{A}B-\overline{A}(|C|F+|D|G)$$

is a nonsingular M -matrix, where

$\underline{A}=\mathrm{diag}(\underline{a}_1,\underline{a}_2,\cdots,\underline{a}_n)$, $\overline{A}=\mathrm{diag}(\overline{a}_1,\overline{a}_2,\cdots,\overline{a}_n)$,

$B=\mathrm{diag}(b_1,b_2,\cdots,b_n)$,

$|C|=(|c_{ij}|)_{n\times n}$, $F=\mathrm{diag}(F_1,F_2,\cdots,F_n)$,

$|D|=(|d_{ij}|)_{n\times n}$, $G=\mathrm{diag}(G_1,G_2,\cdots,G_n)$.

Proof. Let $u^*=(u_1^*,u_2^*,\cdots,u_n^*)^T\in R^n$ be an equilibrium point of model (3), then we have

$$a_i(u_i^*)[b_i(u_i^*)-\sum_{j=1}^n c_{ij}f_j(u_j^*)-\sum_{j=1}^n d_{ij}g_j(u_j^*)-J_i]=0.$$

From (**A1**), it follows that

$$b_i(u_i^*)-\sum_{j=1}^n c_{ij}f_j(u_j^*)-\sum_{j=1}^n d_{ij}g_j(u_j^*)-J_i=0.$$

Since $\underline{A}B-\overline{A}(|C|F+|D|G)$ is a nonsingular M -matrix, and $\underline{A}\overline{A}^{-1}=diag(\underline{a}_1\overline{a}_1^{-1},\underline{a}_2\overline{a}_2^{-1},\cdots,\underline{a}_n a_n^{-1})\le E$, $B-(|C|F+|D|G)$ is a nonsingular M -matrix. From Theorem 1 in [25], we know that model (3) has a unique equilibrium point.

Set $u(t,x)=(u_1(t,x),u_2(t,x),\cdots,u_n(t,x))^T$ is any solution of model (3), then we have

$$\begin{aligned}\frac{\partial(u_i(t,x)-u_i^*)}{\partial t}&=\sum_{k=1}^m\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)-a_i(u_i(t,x))\{b_i(u_i(t,x))-b_i(u_i^*)\\&-\sum_{j=1}^n c_{ij}[f_j(u_j(t,x))-f_j(u_j^*)]-\sum_{j=1}^n d_{ij}[g_j(u_j(t-\tau_{ij}(t),x))-g_j(u_j^*)]\}\end{aligned}\tag{4}$$

Multiply both side of (4) by $u_i(t,x)-u_i^*$ and integrate it, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega(u_i(t,x)-u_i^*)^2dx&=\sum_{k=1}^m\int_\Omega(u_i(t,x)-u_i^*)\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)dx\\&-\int_\Omega a_i(u_i(t,x))(u_i(t,x)-u_i^*)\{b_i(u_i(t,x))-b_i(u_i^*)\\&-\sum_{j=1}^n c_{ij}[f_j(u_j(t,x))-f_j(u_j^*)]-\sum_{j=1}^n d_{ij}[g_j(u_j(t-\tau_{ij}(t),x))-g_j(u_j^*)]\}dx\\&=\sum_{k=1}^m\int_\Omega(u_i(t,x)-u_i^*)\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)dx\\&-\int_\Omega a_i(u_i(t,x))b_i'(\xi_i)(u_i(t,x)-u_i^*)^2dx\\&+\sum_{j=1}^n c_{ij}\int_\Omega a_i(u_i(t,x))(u_i(t,x)-u_i^*)[f_j(u_j(t,x))-f_j(u_j^*)]dx\\&+\sum_{j=1}^n d_{ij}\int_\Omega a_i(u_i(t,x))(u_i(t,x)-u_i^*)[g_j(u_j(t-\tau_{ij}(t),x))-g_j(u_j^*)]dx,\end{aligned}\tag{5}$$

where ξ_i locates between u_i^* and $u_i(t,x)$. From the boundary condition of (1), we get

$$\begin{aligned}
&\sum_{k=1}^{m}\int_{\Omega}(u_i(t,x)-u_i^*)\frac{\partial}{\partial x_k}\left(D_{ik}\,\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)dx=\int_{\Omega}(u_i(t,x)-u_i^*)\nabla\cdot\left(D_{ik}\,\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)_{k=1}^{m}dx\\
&=\int_{\Omega}\nabla\cdot\left((u_i(t,x)-u_i^*)D_{ik}\,\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)_{k=1}^{m}dx-\int_{\Omega}\left(D_{ik}\,\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)_{k=1}^{m}\cdot\nabla(u_i(t,x)-u_i^*)dx\\
&=\int_{\partial\Omega}\left((u_i(t,x)-u_i^*)D_{ik}\,\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)_{k=1}^{m}d\sigma-\sum_{k=1}^{m}\int_{\Omega}D_{ik}\left(\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)^2dx\\
&=-\sum_{k=1}^{m}\int_{\Omega}D_{ik}\left(\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)^2dx,
\end{aligned}\tag{6}$$

in which $\nabla=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\cdots\frac{\partial}{\partial x_m}\right)^T$ is the gradient operator, and

$$\left(D_{ik}\,\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\right)_{k=1}^{m}=\left(D_{i1}\,\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_1},\cdots,D_{im}\,\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_m}\right)^T.$$

From assumptions (A1) and (A2), we have

$$-\int_{\Omega}a_i(u_i(t,x))b_i'(\xi_i)(u_i(t,x)-u_i^*)^2dx\le-\int_{\Omega}\underline{a}_ib_i(u_i(t,x)-u_i^*)^2dx=-\underline{a}_ib_i\left\|u_i(t,x)-u_i^*\right\|_2^2.\tag{7}$$

By use of assumptions (A1) and (A3), we have

$$\begin{aligned}
&\sum_{j=1}^{n}c_{ij}\int_{\Omega}a_i(u_i(t,x))(u_i(t,x)-u_i^*)[f_j(u_j(t,x))-f_j(u_j^*)]dx\\
&\le\overline{a}_i\sum_{j=1}^{n}|c_{ij}|F_j\int_{\Omega}|u_i(t,x)-u_i^*|\cdot|u_j(t,x)-u_j^*|dx\\
&\le\overline{a}_i\sum_{j=1}^{n}|c_{ij}|F_j\left\|u_i(t,x)-u_i^*\right\|_2\left\|u_j(t,x)-u_j^*\right\|_2.
\end{aligned}\tag{8}$$

We get from the same reason

$$\begin{aligned}
&\sum_{j=1}^{n}d_{ij}\int_{\Omega}a_i(u_i(t,x))(u_i(t,x)-u_i^*)[g_j(u_j(t-\tau_{ij}(t),x))-g_j(u_j^*)]dx\\
&\le\overline{a}_i\sum_{j=1}^{n}|d_{ij}|G_j\left\|u_i(t,x)-u_i^*\right\|_2\left\|u_j(t-\tau_{ij}(t),x)-u_j^*\right\|_2.
\end{aligned}\tag{9}$$

By applying (6)-(9) to (5), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\|u_i(t,x)-u_i^*\right\|_2^2\le&-\underline{a}_ib_i\left\|u_i(t,x)-u_i^*\right\|_2^2+\overline{a}_i\left\|u_i(t,x)-u_i^*\right\|_2\sum_{j=1}^{n}|c_{ij}|F_j\left\|u_j(t,x)-u_j^*\right\|_2\\
&+\overline{a}_i\left\|u_i(t,x)-u_i^*\right\|_2\sum_{j=1}^{n}|d_{ij}|G_j\left\|u_j(t-\tau_{ij}(t),x)-u_j^*\right\|_2
\end{aligned}$$

i.e.

$$D^+\left\|u_i(t,x)-u_i^*\right\|_2 \le -\underline{a}_i b_i\left\|u_i(t,x)-u_i^*\right\|_2 + \overline{a}_i\sum_{j=1}^{n}\left[|c_{ij}|F_j\left\|u_j(t,x)-u_j^*\right\|_2 + |d_{ij}|G_j\left\|u_j(t-\tau_{ij}(t),x)-u_j^*\right\|_2\right]. \quad (10)$$

Since $\underline{A}B-\overline{A}(|C|F-|D|G)$ is a nonsingular M -matrix, there exists a positive vector $l=(l_1,l_2,\cdots,l_n)^T>0$ such that

$$\underline{a}_i b_i l_i - F_i\sum_{j=1}^{n} l_j\overline{a}_j|c_{ji}| - G_i\sum_{j=1}^{n} l_j\overline{a}_j|d_{ji}|>0,$$

it follows that

$$-\underline{a}_i b_i l_i + F_i\sum_{j=1}^{n} l_j\overline{a}_j|c_{ji}| + G_i\sum_{j=1}^{n} l_j\overline{a}_j|d_{ji}|<0. \quad (11)$$

From (11), we can choose a sufficient small $\varepsilon>0$ such that

$$-l_i(\underline{a}_i b_i-\varepsilon)+\left[F_i\sum_{j=1}^{n} l_j\overline{a}_j|c_{ji}| + G_i\sum_{j=1}^{n} l_j\overline{a}_j|d_{ji}|e^{\varepsilon\tau}\right]<0, \quad (12)$$

where $\tau=\max_{1\le i,j\le n}\{\tau_{ij}\}$.

Now we construct a Lyapunov functional:

$$V(t)=\sum_{i=1}^{n} l_i\left[\left\|u_i(t,x)-u_i^*\right\|_2 e^{\varepsilon t} + \overline{a}_i\sum_{j=1}^{n}|d_{ij}|G_j\int_{t-\tau_{ij}(t)}^{t}\left\|u_j(s,x)-u_j^*\right\|_2 e^{\varepsilon(s+\tau)}ds\right]. \quad (13)$$

Calculating the upper right Dini derivate $D^+V(t)$ of $V(t)$ along the solution of (4), by (10) and the assumption of $\tau_{ij}(t),\ i,j=1,2,\cdots,n$, we have

$$\begin{aligned}
D^+V(t) &\le \sum_{i=1}^{n} l_i\left[(\varepsilon-\underline{a}_i b_i)\left\|u_i(t,x)-u_i^*\right\|_2 e^{\varepsilon t} + \overline{a}_i e^{\varepsilon t}\sum_{j=1}^{n}|c_{ij}|F_j\left\|u_j(t,x)-u_j^*\right\|_2\right. \\
&\quad + \overline{a}_i e^{\varepsilon t}\sum_{j=1}^{n}|d_{ij}|G_j\left\|u_j(t-\tau_{ij}(t),x)-u_j^*\right\|_2 + \overline{a}_i\sum_{j=1}^{n}|d_{ij}|G_j\left\|u_j(t,x)-u_j^*\right\|_2 e^{\varepsilon(t+\tau)} \\
&\quad \left. - \overline{a}_i\sum_{j=1}^{n}|d_{ij}|G_j\left\|u_j(t-\tau_{ij}(t),x)-u_j^*\right\|_2 e^{\varepsilon(t-\tau_{ij}(t)+\tau)}(1-\tau_{ij}'(t))\right] \\
&\le e^{\varepsilon t}\sum_{i=1}^{n} l_i\left[(\varepsilon-\underline{a}_i b_i)\left\|u_i(t,x)-u_i^*\right\|_2 + \overline{a}_i\sum_{j=1}^{n}|c_{ij}|F_j\left\|u_j(t,x)-u_j^*\right\|_2 + \overline{a}_i\sum_{j=1}^{n}|d_{ij}|G_j\left\|u_j(t,x)-u_j^*\right\|_2 e^{\varepsilon\tau}\right] \\
&\le e^{\varepsilon t}\sum_{i=1}^{n}\left[l_i(\varepsilon-\underline{a}_i b_i)+F_i\sum_{j=1}^{n} l_j\overline{a}_j|c_{ji}| + G_i e^{\varepsilon\tau}\sum_{j=1}^{n} l_j\overline{a}_j|d_{ji}|\right]\left\|u_i(t,x)-u_i^*\right\|_2 \\
&\le 0,
\end{aligned}$$

hence

$$V(t)\le V(0),\qquad t\ge 0. \quad (14)$$

From (13), we have

$$V(t) \geq \sum_{i=1}^{n} l_i \left\| u_i(t,x) - u_i^* \right\|_2 \geq \min_{1\leq i\leq n} \sum_{i=1}^{n} \left\| u_i(t,x) - u_i^* \right\|_2 ,$$

and

$$\begin{aligned} V(0) &= \sum_{i=1}^{n} l_i \left[\left\| u_i(0,x) - u_i^* \right\|_2 + \bar{a}_i \sum_{j=1}^{n} | d_{ij} | G_j \int_{-\tau_{ij}(0)}^{0} \left\| u_j(s,x) - u_j^* \right\|_2 e^{\varepsilon(s+\tau)} ds \right] \\ &\leq \sum_{i=1}^{n} l_i \left[\left\| u_i(0,x) - u_i^* \right\|_2 + \bar{a}_i \sum_{j=1}^{n} | d_{ij} | G_j \int_{-\tau}^{0} \left\| u_j(s,x) - u_j^* \right\|_2 e^{\varepsilon(s+\tau)} ds \right] \\ &\leq \max_{1\leq i\leq n}\{l_i\} \sum_{i=1}^{n} \left[\left\| u_i(0,x) - u_i^* \right\|_2 + \bar{a}_i \tau \sum_{j=1}^{n} | d_{ij} | G_j \left\| u_j(\bar{s},x) - u_j^* \right\|_2 e^{\varepsilon(\bar{s}+\tau)} \right] \quad (-\tau \leq \bar{s} \leq 0) \\ &\leq \max_{1\leq i\leq n}\{l_i\} \left(1 + G_i \tau e^{\varepsilon\tau} \sum_{j=1}^{n} \bar{a}_j | d_{ji} | \right) \sum_{i=1}^{n} \sup_{-\tau\leq s\leq 0} \left\| u_i(s,x) - u_i^* \right\|_2 \\ &= \max_{1\leq i\leq n} \left\{ \max_{1\leq i\leq n}\{l_i\} \left(1 + G_i \tau e^{\varepsilon\tau} \sum_{j=1}^{n} \bar{a}_j | d_{ji} | \right) \right\} \left\| \phi - u^* \right\| . \end{aligned}$$

Let

$$M = \frac{\max_{1\leq i\leq n} \left\{ \max_{1\leq i\leq n}\{l_i\} \left(1 + G_i \tau e^{\varepsilon\tau} \sum_{j=1}^{n} \bar{a}_j | d_{ji} | \right) \right\}}{\min_{1\leq i\leq n}\{l_i\}} ,$$

then $M \geq 1$, and

$$\sum_{i=1}^{n} \left\| u_i(t,x) - u_i^* \right\|_2 \leq M e^{-\varepsilon t} \left\| \phi - u^* \right\| .$$

The proof is completed.

Corollary 1. Under hypothesis (**A1**), (**A2**) and (**A3**), model (3) has one unique equilibrium point, which is globally exponentially stable if any one of the following conditions is true:

(i) $\underline{a}_i b_i > \bar{a}_i \sum_{j=1}^{n} \left[| c_{ij} | F_j + | d_{ij} | G_j \right], \quad i = 1,2,\cdots,n.$

(ii) $\underline{a}_i b_i > F_i \sum_{j=1}^{n} \bar{a}_j | c_{ji} | + G_i \sum_{j=1}^{n} \bar{a}_j | d_{ji} |, \quad i = 1,2,\cdots,n.$

(iii) There exists a positive vector $l = (l_1, l_2, \cdots, l_n)^T > 0$ such that

$$\underline{a}_i b_i l_i > \bar{a}_i \sum_{j=1}^{n} l_j \left[| c_{ij} | F_j + | d_{ij} | G_j \right], \quad i = 1,2,\cdots,n.$$

Proof. In fact, any one of the conditions (i)-(iii) in Corollary 1 can assure $\underline{A}B - \bar{A}(|C|F + |D|G)$ is a nonsingular M-matrix.

In the case $D_{ik} = 0 \, (i = 1,2,\cdots,n, k = 1,2,\cdots m)$, model (3) reduces to the following usual time-vary delayed generalized CGNNs:

$$\begin{aligned} \frac{du_i(t)}{dt} = -a_i(u_i(t)) \Bigg[& b_i(u_i(t)) - \sum_{j=1}^{n} c_{ij} f_j(u_j(t)) \\ & - \sum_{j=1}^{n} d_{ij} g_j(u_j(t - \tau_{ij}(t)) + J_i \Bigg], \end{aligned} \tag{15}$$

for $i = 1,2,\cdots,n$. Here, we have

Corollary 2. Under hypothesis (**A1**), (**A2**) and (**A3**), model (15) has one unique equilibrium point, which is globally exponentially stable if $\underline{A}B-\overline{A}(|C|F+|D|G)$ is a nonsingular M-matrix and $\tau'_{ij}(t)\le 0$.

4. Comparisons and example

Remark 1. For model (3), if $D_{ik}=0$, this model has been studied by many authors, see, for example, Refs.[6,10-12,15-19] and the references cited therein. In [1-12], the authors studied the stability, but the activation function f_j is required to be bounded on R. However, in this paper, it is noted that the models of Refs. [1,6,10-12,15-19] are involved in model (3), we study the global exponential stability of model (3), we only need the activation function f_j to satisfy the assumption (**A3**), not require it to be bounded on R.

Remark 2. The system (3) in this paper is the same as model (2.1) in [23], but the results in [23] requires the activation f_j and g_j to be bounded. Also, the exponential stability criteria in [23] are independent of the magnitude of the delays, which are not the same as our results. However, when $\tau_{ij}(t)$ is a constant, the exponential stability criteria in [23] is a corollary of our results.

Example. Consider the networks with time-varying delays and reaction-diffusion term:

$$\frac{\partial}{\partial t}\begin{pmatrix}u_1(t,x)\\ u_2(t,x)\end{pmatrix}=\begin{pmatrix}D_{11}\frac{\partial u_1}{\partial x_1} & D_{12}\frac{\partial u_1}{\partial x_2}\\ D_{21}\frac{\partial u_2}{\partial x_1} & D_{22}\frac{\partial u_2}{\partial x_2}\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x_1}\\ \frac{\partial}{\partial x_2}\end{pmatrix}$$

$$-\begin{pmatrix}2+\sin u_1 & \\ & 2+\cos u_2\end{pmatrix}\left\{\begin{pmatrix}3 & \\ & 3\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}\right.$$

$$-\begin{pmatrix}-0.2 & 0.1\\ 0.3 & 0.1\end{pmatrix}\begin{pmatrix}\frac{1}{2}(|u_1+1|+|u_1-1|\\ \frac{1}{2}(|u_2+1|+|u_2-1|\end{pmatrix}$$

$$-\begin{pmatrix}0.2 & 0.2\\ 0.1 & 0.3\end{pmatrix}\begin{pmatrix}\tanh(u_1(t-\tau_1(t),x)\\ \tanh(u_2(t-\tau_2(t),x)\end{pmatrix}$$

$$\left.+\begin{pmatrix}J_1\\ J_2\end{pmatrix}\right\},$$

$$\frac{\partial u_i}{\partial \tilde{n}}=\left(\frac{\partial u_1}{\partial x_1},\frac{\partial u_2}{\partial x_2}\right)^T=0,\quad t\ge 0,\ x\in\partial\Omega,$$

$$u_i(s,x)=\phi_i(s,x),\quad -\frac{\pi}{2}\le s\le 0.$$

where $\tanh(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$, $D_{ik}=t^2x_k^{2k}\quad(i,k=1,2)$, $\tau_i(t)=\dfrac{\pi}{2}-\arctan t$. It is clear that $0\le\tau_i(t)\le\dfrac{\pi}{2}$, $\tau'_i(t)<0,\ i=1,2$, and

$$\underline{A}=\begin{pmatrix}1 & \\ & 1\end{pmatrix},\ \overline{A}=\begin{pmatrix}3 & \\ & 3\end{pmatrix},\ B=\begin{pmatrix}3 & \\ & 3\end{pmatrix},\ F=\begin{pmatrix}2 & \\ & 2\end{pmatrix},$$

$$G=\begin{pmatrix}1 & \\ & 1\end{pmatrix},\ |C|=\begin{pmatrix}0.2 & 0.1\\ 0.3 & 0.1\end{pmatrix},\ |D|=\begin{pmatrix}0.2 & 0.2\\ 0.1 & 0.3\end{pmatrix}.$$

It follows that

$$\underline{A}B-\overline{A}(|C|F+|D|G)=\begin{pmatrix}1.2 & -1.8\\ -0.9 & 1.5\end{pmatrix}$$

is a nonsingular M -matrix.

Therefor, we know that this network has one unique equilibrium point that is globally exponential stable from Theorem 1.

Remark 3. It is worth noting that f_1, f_2 are unbounded on R. Thus the exponential stability criteria in [23] cannot be applied to here.

5. Conclusions

In this paper, a generalized Cohen-Grossberg neural networks with time-varying delays and reaction-diffusion term have been studied. Some sufficient conditions for the existence and exponential stability of the equilibrium point have been established. These obtained results are new and they complement previously known results. Moreover, an example are given to illustrate the effectiveness of the new results.

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