Almost Hamiltonian Cubic Graphs

Narong Punnim, Varaporn Saenpholphat, and Sermsri Thaithae

Department of Mathematics, Srinakharinwirot University, Sukhumvit 23, Bangkok 10110, Thailand

Abstract

A Hamiltonian walk in a connected graph G of order n is a closed spanning walk of minimum length in G. For a connected graph G, let h(G) be the length of a Hamiltonian walk in G and call it the Hamiltonian number of G. Let i be a non-negative integer. A connected graph G of order n is called an i-Hamiltonian if h(G) = n+i. Thus a 0-Hamiltonian graph is Hamiltonian. A 1-Hamiltonian graph is called an almost Hamiltonian graph.

We prove in this paper that for an almost Hamiltonian graph G, let h(G) be the length of a Hamiltonian walk in G and call it the Hamiltonian number of G. It was shown in [5] that the Hamiltonian number of a connected graph G is a measure of how far a given graph is from being Hamiltonian.

In [5] an alternative way to define the length h(G) of a Hamiltonian walk in a connected graph G was presented. A Hamiltonian graph G contains a spanning cycle C : v1v2, … , vn, vn+1 = v1, where then vivi+1 ∈ E(G) for 1 ≤ i ≤ n. Thus Hamiltonian graphs of order n ≥ 3 are those graphs for which there is a cyclic ordering C : v1v2, … , vn, vn+1 = v1 of V(G) such that ∑n

Therefore, d(s) ≥ n for each cyclic ordering s of the elements of V(G). The Hamiltonian number h(G) of G is defined in [5] by h(G) = min{d(s)}, where the minimum is taken over all cyclic orderings s of elements of V(G). It was shown in [5] that the Hamiltonian number of a connected graph G is, in fact, the length of a Hamiltonian walk in G.

1. Introduction

While certainly not every connected graph of order at least 3 contains a Hamiltonian cycle, every connected graph does contain a closed spanning walk (in which all vertices are encountered, possibly more than once). If G is a connected graph of size m, there is always a closed spanning walk of length at most 2m. In [6, 7] Goodman and Hedetniemi introduced the concept of a Hamiltonian walk in a connected graph G, defined as a closed spanning walk of minimum length in G. They denoted the length of a Hamiltonian walk in G by h(G). Therefore, for a connected graph G of order n ≥ 3, it follows that h(G) = n if and only if G is Hamiltonian. Hamiltonian walks were studied further by T. Asano, T. Nishizeki, and T. Watanabe [2, 3], J. C. Bermond [4], and P. Vacek [9]. Thus h may be considered as a measure of how far a given graph is from being Hamiltonian.

To illustrate these concepts, consider the graph G = K2,3 of Figure 1. For the cyclic orderings s1 : v1, v2, v3, v4, v5, v1 and s2 : v1, v3, v2, v4, v5, v1 of V(G), we see that d(s1) = 8 and d(s2) = 6. Since G is a non-Hamiltonian graph of order 5 and d(s2) = 6, it follows that h(G) = 6.

Let i be a non-negative integer. A connected graph G of order n is called an i-Hamiltonian if h(G) = n + i. Thus a 0-Hamiltonian graph is Hamiltonian. An almost Hamiltonian graph is a graph G of order n and h(G) = n + 1. Thus K2,3 is an example of an almost Hamiltonian graph.

The following results are known (see [5, 7]).

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Theorem A For every connected graph $G$ of order $n \geq 2$, $n \leq h(G) \leq 2n - 2$.

Moreover, 
1. $h(G) = 2n - 2$ if and only if $G$ is a tree and 
2. for every pair $n, k$ of integers with $3 \leq n \leq k \leq 2n - 2$, there exists a connected graph $G$ of order $n$ having $h(G) = k$.

Thus for a connected graph $G$ of order $n$, $G$ is an $i$-Hamiltonian graph for some $i = 0, 1, 2, \ldots, n-2$. Moreover, for integers $n$ and $i$ with $n \geq 3$ and $0 \leq i \leq n-2$, there is an $i$-Hamiltonian graph $G$ of order $n$.

Let $P(k, m)$ be a generalized Petersen graph such that $V(P(k, m)) = \{u_i, v_i : i = 0, 1, \ldots, k-1\}$ and $E(P(k, m)) = \{u_i u_{i+1}, v_{i+l}, u_{i+l} : i = 0, 1, 2, \ldots, k-1\}$ where addition is taken modulo $k$ and $m \leq k/2$. In [1] Alspach completed the determination of the parameters $k, m$ for which $P(k, m)$ is Hamiltonian as stated in the following theorem.

Theorem B The generalized Petersen graph $P(k, m)$ is non-Hamiltonian if and only if $m = 2$ and $k \equiv 5(\text{mod } 6)$.

2. Almost Hamiltonian cubic graphs

We have seen that a generalized Petersen graph is not a Hamiltonian graph if and only if $m = 2$ and $k \equiv 5(\text{mod } 6)$. Thus in this case $h(P(k, m)) \geq 2k + 1$. We will show in the next theorem that $P(k, m)$ is an almost Hamiltonian graph if and only if $m = 2$ and $k \equiv 5(\text{mod } 6)$.

Theorem 2.1 Let $P(k, m)$ be a generalized Petersen graph. Then 
\[ h(P(k, m)) = \begin{cases} 2k + 1 & \text{if } m = 2 \text{ and } k \equiv 5(\text{mod } 6), \\ 2k & \text{otherwise}. \end{cases} \]

Proof. Let $m = 2$ and $k \equiv 5(\text{mod } 6)$. By Theorem B, it is sufficient to show that $h(P(k, 2)) = 2k+1$. Consider a closed spanning walk $W : v_0, v_2, \ldots, v_{k-1}, v_1, v_3, \ldots, v_{k-2}, u_{k-2}, u_{k-3}, u_{k-4}, \ldots, u_1, u_0, u_{k-1}, u_0, v_0$ of $P(k, 2)$. It is clear that $W$ has length $2k + 1$. Thus $h((P(k, m)) = 2k + 1$.

It was shown in [8] that all connected cubic graphs of order $n$, where $4 \leq n \leq 8$, are Hamiltonian. It was also shown in [8] that the Petersen graph $P(5, 2)$ and the Tietze graph (denoted by $T_{12}$) are the only 2-connected cubic graph of order 10 and 12, respectively, that are not Hamiltonian. They are, in fact, almost Hamiltonian cubic graphs of respective order. Note that the $T_{12}$ is obtained from $P(5, 2)$ by replacing one vertex of $P(5, 2)$ to a triangle, matching the vertices of the triangle to the former neighbors of the replaced vertex. Thus $T_{12}$ contains a triangle. Let $G$ be a cubic graph and $v \in V(G)$. We denote $G * v$ to be the graph obtained from $G$ by replacing $v$ to a triangle, matching the vertices of the triangle to the former neighbors of $v$. Thus $G * v$ is also a cubic graph containing a triangle.

Theorem 2.2 Let $G$ be a cubic graph of order $n \geq 4$ and $v \in V(G)$. Then $G$ is Hamiltonian if and only if $G * v$ is Hamiltonian.

Proof. Let $G$ be a cubic graph and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $v = v_1$. Thus $G * v$ is the graph with $V(G * v) = (V(G) - v1) \cup \{x_1, y_1, z_1\}$, $\{x_1, y_1, z_1\}$ induced a triangle in $G * v$. Moreover, $G * v$ is also a Hamiltonian graph of $G * v$.

Conversely, suppose that $G * v$ is Hamiltonian and let $C_v : v_1, v_2, \ldots, v_{n+1}$ be a Hamiltonian cycle of $G * v$. Let $C_v : u_1, u_2, \ldots, u_{n+2}, u_1$ be a Hamiltonian cycle of $G * v$. If $x_1$ is a neighbor of $y_1$ and $z_1$ in $C_v$, then $d_G*(x_1) \geq 4$. Thus $x_1$ is a neighbor of $y_1$ or $z_1$ in $C_v$. It is also true for $y_1$ and $z_1$. Thus $x_1, y_1, z_1$ must appear as consecutive vertices in $C_v$. Deleting the three vertices and replacing by $v_1$, we obtain a Hamiltonian cycle of $G$.

Let $G$ be a cubic graph of order $n$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $G' = G * v_1$ and for $1 \leq i \leq n-1$, put $G^{gi} = G * v_{gi}$. Thus from Theorem 2.2 we have the following corollary.

Corollary 2.3 Let $G$ be a cubic graph of order $n$. Then $G$ is Hamiltonian if and only if $G^i$ is Hamiltonian for all $1 \leq i \leq n$.

Now consider when $G$ is not Hamiltonian cubic graph. Thus $G * v$ is not Hamiltonian by Theorem 2.2. Let $K'_4$ be the graph obtained from $K_4$ and a subdivision to an edge of $K_4$ (see Figure 2).

Let $G$ be a graph obtained from three copies of $K'_4$ and connecting three vertices of degree two to a new vertex $v$. 
Thus $G$ is cubic of order 16 with $h(G) = 21$ but $h(G) = 24$. That is $G$ is a 5-Hamiltonian while $G * v$ is 6-
Hamiltonian. Note that $G * w$ is 5-Hamiltonian, for each vertex $w$ of $G$ difference from $v$.

**Theorem 2.4** For an even integer $n \geq 10$, there exists an
almost Hamiltonian cubic graph of order $n$.

**Proof.** The Petersen graph $P(5,2)$ is the unique almost
Hamiltonian cubic graph of order 10 and the Tietze graph
$T_{12}$ is also the unique almost Hamiltonian cubic graph of
order 12 and $T_{12} = G * v$, where $G = P(5,2)$ and $v \in V(P(5,2))$.
Let $u_i, v_i, w_i$ be the induced triangle of $T_{12}$. Let $T_{14} = T_{12} * v_i$. Thus for an integer $i \geq 1$, let $u_i, v_i, w_i$ be the
induced triangle of $T_{12} * v_i$ and $T_{12} * v_i = T_{12} * v_{i-1} * v_i$. By
assuming that the graph $T_{12} * v_i$ is almost hamiltonian, we
have that $h(T_{12} * v_i) \leq 12 + 2i + 1$. By Theorem 2.2 we
have that $T_{12} * v_i$ is not Hamiltonian. Therefore $h(T_{12} * v_i) =
12 + 2i + 1$ and $T_{12} * v_i$ is almost Hamiltonian.

A Hamiltonian graph is necessary 2-connected. The same result is also hold in the class of almost Hamiltonian
cubic graphs. The following results can be considered as a
characterization of cubic graphs for being almost Hamiltonian.

**Theorem 2.5** Let $G$ be a connected cubic graph of order
$n \geq 10$. If $G$ is almost Hamiltonian, then $G$ is 2-
connected.

**Proof.** Suppose $G$ is not 2-connected and $v$ is a cut vertex
of $G$. Since $G$ is cubic, there exists a vertex $u$ such that $u$ is
also a cut vertex of $G$ and $u$ is adjacent to $v$. Furthermore,$uv$ is a cut edge of $G$. Let $G - e = G_1 \cup G_2$. It follows that
$h(G) \geq h(G_1) + h(G_2) + 2 \geq n + 2$. The proof is complete.

**Theorem 2.6** Let $G$ be a connected non-Hamiltonian
cubic graph of order $n \geq 10$. Then $G$ is an almost
Hamiltonian graph if and only if for every Hamiltonian
walk $W$ of $G$, $W$ contains a cycle of order $n - 1$.

**Proof.** Suppose $h(G) = n + 1$. Let $v_1, v_2, \ldots, v_{n+2} = v_1$ be a
Hamiltonian walk of length $n + 1$. Thus there exist $v_i$ and
$v_j$ with $1 \leq i < j \leq n$ and $v_j = v_1$ and all other vertices are
distinct. Without loss of generality we may assume that $i = 1$.
If $j \geq 4$, then $d(v_j) \geq 4$. Thus $j = 3$ and $v_3, v_4, \ldots, v_{n+2}
= v_1$ is a cycle in $G$ of length $n - 1$. Suppose $G$ contains a
cycle $v_1, v_2, \ldots, v_6 = v_1$ of length $n - 1$. Let $v \in V(G) -
\{v_1 = v_n, v_2, v_3, \ldots, v_{n-1}\}$. Thus there exists an integer $k
with $1 \leq k \leq n - 1$ such that $v_k$ is adjacent to $v$. We now
form a Hamiltonian walk $v_1, v_2, \ldots, v_k, v, v_4, \ldots, v_n = v_1$
and this walk has length $n + 1$. Therefore $h(G) = n + 1$.

Let $G$ be a Hamiltonian cubic graph. We have shown in
Theorem 2.2 that for every $v \in V(G)$, $G * v$ is
Hamiltonian and vise versa. We have also mentioned that
there is a 5-Hamiltonian graph $G$ and $v \in V(G)$ such that
$G * v$ is 6-Hamiltonian.

Let $G$ be a connected cubic graph of order $n$ with $V(G) =
\{v_1, v_2, \ldots, v_n\}$. Let $G^* = G^*$. Figure 3 shows the graphs
$P(5,2)$ and $P^*(5,2)$.

![Figure 3: Graphs P(5,2) and P*(5,2)](image)

**Theorem 2.7** If $G$ is an almost Hamiltonian cubic graph of
order $n$, then $h(G) = 3n + 2$.

**Proof.** By Theorem 2.2, it follows that $h(G) \geq 3n + 1$. Assume,
the contrary, that $h(G) = 3n + 1$. By Theorem 2.6, let $C = x_1, x_2, \ldots, x_{3n-1}, x_1$ be a cycle of length $3n - 1$ of $G^*$,
where $V(G^*) = \{x_1, x_2, \ldots, x_{3n}\}$. Without loss of generality
we may assume that $x_{3n}$ is adjacent to $x_1$. Since $G^*$ is non-
Hamiltonian, $x_{3n}x_2, x_{3n}x_{3n-1} \notin E(G^*)$. Since $G^*$ is cubic,
there exist $i, j$ with $1 < i < j < 3n - 1$ such that $\{x_{3n}, x_i, x_j\}$
induced a triangle in $G^*$. Since $G^*$ is cubic, $j = i + 1$ and
$G^*$ is Hamiltonian. This is a contradiction.

![Figure 4: Part of G^*](image)

In order to show that $h(G^*) = 3n + 2$, we will construct
a Hamiltonian walk of $G^*$ of length $3n + 2$. In the proof of
Theorem 2.6, let

$$W : v_1, v_2, \ldots, v_k, v, v_4, \ldots, v_{n+1} = v_1$$

be a Hamiltonian walk of $G$.

For each $i, 1 \leq i \leq n$, we replace vertices $v_i$ and $v$ in $W$
by triangles $x_i, y_i, z_i$ and $x, y, z$ respectively, and then
arrange them in such a way that $x_i$ is adjacent to $y_{i+1}$, for all $i
= 1, 2, \ldots, n - 1$. Without loss of generality we may
assume that $z_k$ is adjacent to $x$ as shown in Figure 4. Thus
the Hamiltonian walk

$$W : z_1, x_1, y_1, z_2, x_2, \ldots, y_k, z_k, x_k, y_k, z_1, x_1, y_1, z_2, x_2, y_2, \ldots, y_{n-1}, z_{n-1}, x_{n-1}, y_{n-1}, z_n, z_1$$
has length $3n + 2$.

The following result can be obtained as a direct consequence of Theorem 2.7.

**Corollary 2.8** $h(P^*(k, 2)) = 6k + 2$, for every positive integer $k$ with $k \equiv 5 \pmod{6}$.

### 3. Conclusion

A Hamiltonian walk in a connected graph $G$ of order $n$ is a closed opening walk of minimum length in $G$. Let $h(G)$ be the length of a Hamiltonian walk in $G$. The graph parameter $h$ is called the Hamiltonian number of $G$. Thus $h(G)$ may be considered as a measure of how far the graph $G$ is from being Hamiltonian. A connected graph $G$ of order $n$ is called an $i$-Hamiltonian if $h(G) = n + i$. Thus a 0-Hamiltonian graph is Hamiltonian. A 1-Hamiltonian graph is called an almost Hamiltonian graph. Some characterizations of almost Hamiltonian cubic graphs are obtained in this paper. In other words, we proved that a cubic graph $G$ of order $n$ is almost Hamiltonian if and only if $G$ is 2-connected containing a cycle of length $n - 1$. In particular, we proved that the generalized Petersen graph $P(k, m)$ is almost Hamiltonian if and only if $m = 2$ and $k \equiv 5 \pmod{6}$. Let $G$ be a cubic graph of order $n$. we denote $G^*$ the graph obtained from $G$ by replacing each vertex of $G$ to a triangle, matching the vertices of the triangle to the former neighbors. We proved that $G$ is Hamiltonian if and only if $G^*$ is Hamiltonian and if $G$ is almost Hamiltonian, then $G^*$ is 2-Hamiltonian.

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### References


