Some Properties of Graph $\Re(2^N)$

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Abstract

Let N be a positive integer, $N \geq 3$, and $R(2^N)$ be the set of all non isomorphic 2-regular graphs of order N. The graph of realizations, $\Re(2^N)$, can be defined as a graph whose vertex set is $R(2^N)$ two vertices are adjacent if one can be obtained from the other by a switching. It is known that the graph $\Re(2^N)$ is connected. We prove in this paper that the graph $\Re(2^N)$ is bipartite and it has no Eulerian trail for $N \geq 12$.

Key words:
Realization, Bipartite, 2-regular graph, Eulerian trail.

1.Introduction

Let G be a graph of order n and let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be a vertex set of G. The sequence $\boldsymbol{d} = (d(v_1), d(v_2), \ldots, d(v_n))$ is called degree sequence of G. A sequence $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is a graphic degree sequence if it is a degree sequence of some graph G. In this case, G is called a realization of $\boldsymbol{d}$. A graph G is said to be r-regular graph if all of its vertices have degree r. The degree sequence of an r-regular graph with N vertices will be denoted by $\boldsymbol{d} = r^N$.

Suppose that a company has N employees and the company wants to organize these employees to their shops with the following condition: each shop must have at least 3 employees. If they want to have the most efficiency choice to group 2 shops to be one or splits 1 shop to be 2 shops. For this problem, we can use graph $\Re(2^N)$ (realization of 2-regular graph with N vertices) to solve it. Further, we can use this idea to extend for solving at least k employees problem.

The number of partitions of a positive integer is an interesting problem in Number Theory and they have variety of applications in many fields of Mathematics as well as applied Mathematics. In particular, it has closely related to the graph $\Re(2^N)$. This graph is a particular case of the graph $\Re(\boldsymbol{d})$ which was introduced by Eggleton and Holton in 1979 [1].

A switching on a graph G is a replacement of any two independent edges ab and cd of G by the edges ac and bd that are not edges in G. Therefore we can consider a switching as a graph transformation that transforms a graph G to another graph H such that G and H have the same degree sequence. The following theorem was shown by Hakimi [2] and Havel [3].

Theorem 1.1 *Let* $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ *be a graphic degree sequence. If* G_1 *and* G_2 *are any two realizations of* $\boldsymbol{d}$*, then one can be obtained from the other by a finite sequence of switchings.*

In 1979, Eggleton and Holton [1] defined the graph $\Re(\boldsymbol{d})$ of realizations of $\boldsymbol{d}$ whose vertices are the graphs with degree sequence $\boldsymbol{d}$ and two vertices are being adjacent in the graph $\Re(\boldsymbol{d})$ if one can be obtained from the other by a switching. In particular, they obtained the following theorem.

Theorem 1.2 *The graph* $\Re(\boldsymbol{d})$ *is connected.*

For example, $\boldsymbol{d} = 2^{13}$, there are 10 nonisomorphic 2-regular graphs of order 13. That is, the graph $\Re(2^{13})$ is a graph of order 10. Let $v_1 = C_{13}$, $v_2 = C_3 \cup C_{10}$, $v_3 = C_4 \cup C_9$, $v_4 = C_5 \cup C_8$, $v_5 = C_6 \cup C_7$, $v_6 = 2C_3 \cup C_7$, $v_7 = C_3 \cup C_4 \cup C_6$, $v_8 = C_3 \cup 2C_5$, $v_9 = 2C_4 \cup C_5$ and $v_{10} = 3C_3 \cup C_4$. Thus set of these graphs is the vertex set of the graph $\Re(2^{13})$. Consider C_{13} and a single switching, we obtain $C_3 \cup C_{10}$, $C_4 \cup C_9$, $C_5 \cup C_8$ and $C_6 \cup C_7$. Thus v_1 is adjacent to v_2, v_3, v_4 and v_5. Thus, the graph $\Re(2^{13})$ is the graph as shown by Fig 1.

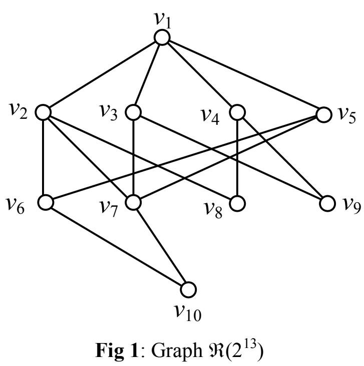

Fig 1: Graph $\Re(2^{13})$

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2. Basic Results

Let σ be a switching on a graph G. Then the graph obtained from G by σ will be denoted by G^{σ}. It is easy to see that the graphs G and G^{σ} have the same degree sequence. Thus we may say that the switching is a degree-preserving operation.

Let G be a graph. We will denote by $k(G)$ for the number of components in the graph G. If G is a realization of 2^N, then there exist integers $m_1, m_2, \ldots, m_k$ such that $N = m_1 + m_2 + \ldots + m_k$, and $m_i \geq 3$, for $i = 1,2,\ldots,k$. Thus $G = \bigcup_{i=1}^{k} C_{m_i}$. Thus in this case, $k(G) = k$ and each component is a cycle length m_i.

Let G be a 2-regular graph of order N. Then we will show in the next theorem that a switching will change the number of components by at most one.

Theorem 2.1 *If $G \in R(2^N)$ and σ is a switching on G, then* $\left|k(G) - k(G^{\sigma})\right| \leq 1$.

Proof. Let G be a graph and a, b, c, d be four distinct vertices of G such that $ab, cd \in E(G)$ and $ac, bd \notin E(G)$. Let $\sigma(a,b;c,d) = \sigma$ be a switching on G by deleting edges ab, cd and adding edges ac, bd, we consider following cases.

Case 1. Suppose that ab is an edge in a component C'_m and cd is an edge in another component C''_n of the graph G. After we apply the switching σ, C'_m and C''_n will become the cycle length $m+n$. Thus, in this case, the number of the components is decreased by one after the switching. Hence, $k(G) = k(G^{\sigma}) + 1$.

Case 2. Suppose that ab and cd are edges in some component C_m. After switching , we see that there are 2 subcases.

Subcase 2.1 The component C_m of G remains the same cycle of length m after the switching. So $G \cong G^{\sigma}$. Thus $k(G) = k(G^{\sigma})$.

Subcase 2.2 The component C_m becomes two components $C_{m'}$ and $C_{m''}$ after the switching, where $m = m' + m''$. Thus, $k(G) = k(G^{\sigma}) - 1$.

Therefore, $\left|k(G) - k(G^{\sigma})\right| \leq 1$.

In Theorem 2.1, we see that if $G_1, G_2 \in R(2^N)$ and σ is a switching such that $G_2 = G_1^{\sigma}$, then either G_2 is isomorphic to G_1 or $\left|k(G_1) - k(G_1^{\sigma})\right| = 1$.

Corollary 2.2 *If $G_1, G_2 \in R(2^N)$ and G_1 is adjacent to G_2, then* $\left|k(G_1) - k(G_2)\right| = 1$.

Proof. Since G_1 is adjacent to G_2, so there is a switching σ such that $G_2 = G_1^{\sigma}$. Because $R(2^N)$ be the set of all non-isomorphic 2-regular graphs of order N. Thus G_1 is not isomorphic to G_2. By Theorem 2.1,

$$\left|k(G_1) - k(G_1^{\sigma})\right| = \left|k(G_1) - k(G_2)\right| = 1.$$

A graph G is a bipartite graph if $V(G)$ can be partitioned into two subsets V_1 and V_2 such that each edge of G joins a vertex in V_1 and a vertex in V_2. The following theorem shows that $\Re(2^N)$ is bipartite.

Theorem 2.3 *The graph $\Re(2^N)$ is bipartite.*

Proof. Let

$$V_1 = \{G \in R(2^N) \mid k(G) \text{ is odd}\}$$

and

$$V_2 = \{G \in R(2^N) \mid k(G) \text{ is even}\}.$$

If $G_1, G_2 \in V_1$, then $k(G_1)$ and $k(G_2)$ are odd. That is $|k(G_1) - k(G_2)| \neq 1$. By corollary 2.2, G_1 is not adjacent to G_2. Similarly, if $G_1, G_2 \in V_2$, then G_1 is not adjacent to G_2. Thus there are no two vertices in V_1 or V_2 are adjacent. By Theorem 1.2, the graph $\Re(2^N)$ is connected, each edge in $\Re(2^N)$ joins a vertex of V_1 and a vertex of V_2. Therefore the graph $\Re(2^N)$ is bipartite.

From now on, we may consider $G \in R(2^N)$ in the form of partition N. Since $G = \bigcup_{i=1}^{k} C_{m_i}$, where $m_i \geq 3$ for each $i = 1,2,\ldots,k$, there exists a unique sequence of integers $m_1, m_2, \ldots, m_k$ such that $N = m_1 + m_2 + \ldots + m_k$. Let $B = \{m_1, m_2, \ldots, m_k\}$ be a multiset. For determining the degree of each vertex, we will rearrange the elements in B in this way.

For $n_i \in B$, $i = 1,2,\ldots,k$. Let $A = \{n_1, n_2, \ldots, n_r\}$, $n_i < n_j$ for $1 \leq i < j \leq r \leq k$ and $\Omega_G = \{n_1, n_2, \ldots, n_r, n_{r+1}, \ldots, n_k\}$, $n_i \leq n_{i+1}$ for each i ; $r+1 \leq i < k$. We will call Ω_G to be a partition of N at level k of G. From now, we will use

Ω_G (or simply Ω, if the graph G is clear from the context) to represent the graph G.

Let G be any vertex at level k of the graph $\Re(2^N)$. For $1 < k < \left\lfloor \frac{N}{3} \right\rfloor$, we have $d(G) = |N_1(G)| + |N_2(G)|$, where $N_1(G)$ and $N_2(G)$ are neighbors of G at level $k-1$ and $k+1$, respectively. For $k = 1$, we have $d(G) = |N_2(G)|$ and for $k = \left\lfloor \frac{N}{3} \right\rfloor$, we have $d(G) = |N_1(G)|$. For example the graph $\Re(2^{13})$ in Fig. 1, Let $G = v_7$, we have $d(G) = |N_1(G)| + |N_2(G)| = 3 + 1 = 4$.

Suppose that $\Omega = \{3, 4, 12, 3, 3, 4\}$ is a partition of 29 at level 6 of $G \in R(2^{29})$. We see that Ω is obtained from {3, 4, 7, 12, 3}, {3, 4, 16, 3, 3}, {3, 4, 15, 3, 4}, {3, 4, 6, 12, 4} and {3, 8, 12, 3, 3} by a switching at level 5. Moreover, we see that $|N_1(G)|$ is a 2-combination of {3, 4, 12} adding it with a 2-combination of {3, 3, 3} or {4, 4}. Thus $|N_1(G)| = \binom{3}{2} + 2 = 5$. In general, let Ω be a partition of N at level $k \geq 2$ of $G \in R(2^N)$ and A be a set of all distinct elements in Ω. Suppose that $|A| = r \leq k$, let β be the number of distinct elements in Ω which occur in A at least 2 times. Then $|N_1(G)| = \binom{r}{2} + \beta$.

It is easy to see that if $G = \bigcup_{i=1}^{k} C_3$, $\bigcup_{i=1}^{k} C_4$ or $\bigcup_{i=1}^{k} C_5$, then $d(G) = |N_1(G)| = 1$.

In order to calculate $|N_2(G)|$, we first observe that for each $n_i \in A$ for $i = 1,2,\ldots,r$ and $n_i \geq 6$, we can replace n_i by sub-partitions $\{3, n_i - 3\}$, $\{4, n_i - 4\}$, . . . , $\{k_i, n_i - k_i\}$, where $k_i = \left\lfloor \frac{n_i}{2} \right\rfloor$. We will denote the number of sub-partitions of n_i by $\psi(n_i)$. Thus

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$$\psi(n_i) = \begin{cases} \left\lfloor \frac{n_i}{2} \right\rfloor - 2 & ; \ n_i \geq 6 \\ 0 & ; \ 3 \leq n_i \leq 5 \end{cases}.$$

Thus $|N_2(G)| = \sum_{i=1}^{r} \psi(n_i)$ where $n_i \in A$.

The following theorems are well known.

Theorem A *Let G be a nontrivial connected graph. Then G is Eulerian if and only if every vertex of G has even degree.*

Theorem B *Let G be a nontrivial connected graph. Then G contains an Eulerian trail if and only if exactly two vertices of G have odd degree.*

Using Fig.2 and Theorem A, it is clear that the graphs are non-Eulerian. Theorem B implies that the graphs $\Re(2^6)$ to $\Re(2^{11})$ have Eulerian trails. The next theorem guarantees that the graph $\Re(2^N)$ has no Eulerian trail for $N \geq 12$.

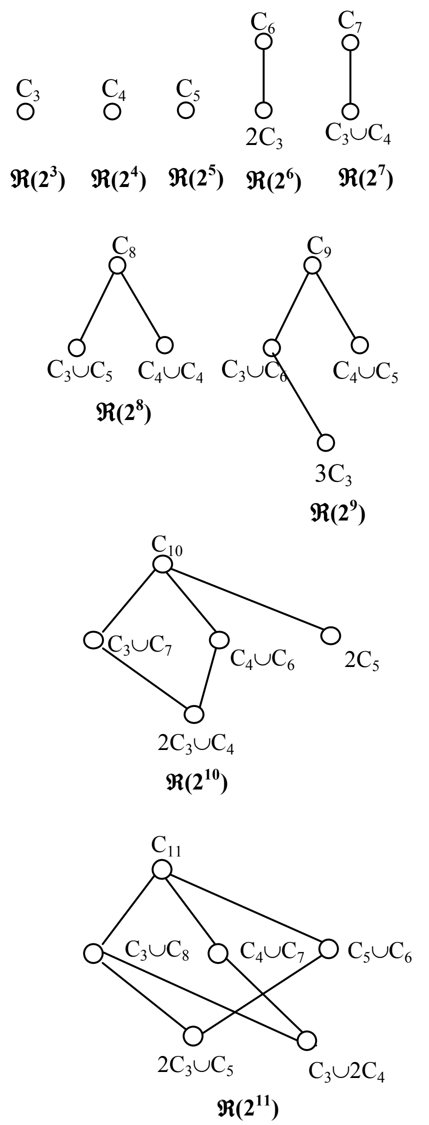

Fig 2 The graphs $\Re(2^3)$ to $\Re(2^{11})$

Theorem 2.4 *There is no Eulerian trail in the graph* $\Re(2^N)$ *for* $N \geq 12$.

Proof. Let N be a positive integer, $N \geq 12$. In order to prove this theorem, it is suffice to show that there exist G_1, G_2 and $G_3 \in R(2^N)$ such that the degrees of G_1, G_2 and G_3 are odd. Let k be an integer, $k \geq 3$.

Case 1 $N = 4k$.

Let $\Omega_1 = \{3, N-3\}$, $\Omega_2 = \{4, N-4\}$ be partitions of N at level 2 and $\Omega_3 = \{4,4,...,4\}$ be a partition N at level k of G_1, G_2 and G_3, respectively. Since $N \geq 12$, so $3 < N-3$ and $4 < N-4$. That is $|N_1(G_1)| = |N_1(G_2)| = 1$. Since $\left\lfloor \frac{N-3}{2} \right\rfloor = \left\lfloor \frac{4k-3}{2} \right\rfloor = 2k-2$ and $\left\lfloor \frac{N-4}{2} \right\rfloor = \left\lfloor \frac{4k-4}{2} \right\rfloor = 2k-2$, it follows that

$|N_2(G_1)| = \psi(3) + \psi(N-3) = \left\lfloor \frac{N-3}{2} \right\rfloor - 2 = 2k-4$ and

$|N_2(G_2)| = \psi(4) + \psi(N-4) = \left\lfloor \frac{N-4}{2} \right\rfloor - 2 = 2k-4$.

Thus, $d(G_1) = |N_1(G_1)| + |N_2(G_1)| = 2k-3$ and $d(G_2) = |N_1(G_2)| + |N_2(G_2)| = 2k-3$ are odd. Since $\Omega_3 = \{4,4,\ldots,4\}$, so $d(G_3) = |N_1(G_3)| = 1$ is odd.

Case 2. $N = 4k+1$

Let $\Omega_1 = \{4, N-4\}$, $\Omega_2 = \{5, N-5\}$ and $\Omega_3 = \{6, N-6\}$ be partitions of N at level 2 of G_1, G_2 and G_3 respectively. Since $N \geq 13$, so $4 < N-4$, $5 < N-5$ and $6 < N-6$, it follows that $|N_1(G_1)| = |N_1(G_2)| = |N_1(G_3)| = 1$. Since $\left\lfloor \frac{N-4}{2} \right\rfloor = \left\lfloor \frac{4k-3}{2} \right\rfloor = 2k-2$, $\left\lfloor \frac{N-5}{2} \right\rfloor = \left\lfloor \frac{4k-4}{2} \right\rfloor = 2k-2$ and $\left\lfloor \frac{N-6}{2} \right\rfloor = \left\lfloor \frac{4k-5}{2} \right\rfloor = 2k-3$. Therefore,

$|N_2(G_1)| = \psi(4) + \psi(N-4) = \left\lfloor \frac{N-4}{2} \right\rfloor - 2 = 2k-4$ and

$|N_2(G_2)| = \psi(5) + \psi(N-5) = \left\lfloor \frac{N-5}{2} \right\rfloor - 2 = 2k-4$.

Since $|N_2(G_3)| = \psi(6) + \psi(N-6) = 1 + \left\lfloor \frac{N-6}{2} \right\rfloor - 2 = 2k-4$, it follows that

$$d(G_1) = |N_1(G_1)| + |N_2(G_1)| = 2k-3,$$
$$d(G_2) = |N_1(G_2)| + |N_2(G_2)| = 2k-3 \text{ and}$$
$$d(G_3) = |N_1(G_3)| + |N_2(G_3)| = 2k-3 \text{ are odd.}$$

Case 3. $N = 4k+2$, we will consider into 2 subcases.

Subcase 3.1 Suppose $k = 3$, we have $N = 14$.

Let $\Omega_1 = \{14\}$, $\Omega_2 = \{5,9\}$ and $\Omega_3 = \{3,4,3,4\}$ be partitions of N representing G_1, G_2 and G_3 respectively. Since $d(G_1) = |N_2(G_1)| = \psi(14) = \left\lfloor \frac{14}{2} \right\rfloor - 2 = 5$ is odd, $|N_1(G_2)| = 1$ and $|N_2(G_2)| = \psi(5) + \psi(9) = \left\lfloor \frac{9}{2} \right\rfloor - 2 = 2$, it follows that $d(G_2) = |N_1(G_2)| + |N_2(G_2)| = 1 + 2 = 3$ is odd. Moreover, $d(G_3) = |N_1(G_3)| = \binom{2}{2} + 2 = 3$ is also odd.

Subcase 3.2 $k \geq 4$ then $N \geq 18$.

Let $\Omega_1 = \{N\}$, $\Omega_2 = \{5, N-5\}$ and $\Omega_3 = \{7, N-7\}$ be partitions of N representing G_1, G_2 and G_3, respectively. Since $d(G_1) = |N_2(G_1)| = \psi(N) = \left\lfloor \frac{4k+2}{2} \right\rfloor - 2 = 2k-1$ is odd, $|N_1(G_2)| = |N_1(G_3)| = 1$, $\left\lfloor \frac{N-5}{2} \right\rfloor = \left\lfloor \frac{4k-3}{2} \right\rfloor = 2k-2$, and $\left\lfloor \frac{N-7}{2} \right\rfloor = \left\lfloor \frac{4k-5}{2} \right\rfloor = 2k-3$, it follows that

$|N_2(G_2)| = \psi(5) + \psi(N-5) = \left\lfloor \frac{N-5}{2} \right\rfloor - 2 = 2k-4$ and

$|N_2(G_3)| = \psi(7) + \psi(N-7) = 1 + \left\lfloor \frac{N-7}{2} \right\rfloor - 2 = 2k-4$.

Thus $d(G_2) = |N_1(G_2)| + |N_2(G_2)| = 2k-3$ and $d(G_3) = |N_1(G_3)| + |N_2(G_3)| = 2k-3$ are odd.

Case 4. $N = 4k+3$

Let $\Omega_1 = \{N\}$, $\Omega_2 = \{3, N-3\}$ and $\Omega_3 = \{3,4, N-7\}$ be partitions of N representing G_1, G_2 and G_3, respectively. Since $N \geq 15$, $3 < N-3$ and $3 < 4 < N-7$. Thus $d(G_1) = |N_2(G_1)| = \psi(N) = \left\lfloor \frac{4k+3}{2} \right\rfloor - 2 = 2k-1$ is odd.

Since $|N_1(G_2)| = 1$, $|N_1(G_3)| = \binom{3}{2} = 3$, $\left\lfloor \frac{N-3}{2} \right\rfloor = \left\lfloor \frac{4k}{2} \right\rfloor = 2k$ and $\left\lfloor \frac{N-7}{2} \right\rfloor = \left\lfloor \frac{4k-4}{2} \right\rfloor = 2k-2$, it follows that

$|N_2(G_2)| = \psi(3) + \psi(N-3) = \left\lfloor \frac{N-3}{2} \right\rfloor - 2 = 2k-2$ and

$|N_2(G_3)| = \psi(3) + \psi(4) + \psi(N-7) = \left\lfloor \frac{N-7}{2} \right\rfloor - 2 = 2k-4$.

Thus $d(G_2) = |N_1(G_2)| + |N_2(G_2)| = 2k-1$ and $d(G_3) = |N_1(G_3)| + |N_2(G_3)| = 2k-1$ are odd.

Thus graph $\Re(2^N)$ has at least 3 vertices whose degree are odd. That is the graph $\Re(2^N)$ has no Eulerian trail for $N \geq 12$.

3. Conclusion

Even though the graph $\Re(2^N)$ is a particular case of graph $\Re(\boldsymbol{d})$ but it has many interest properties. Since the vertex set of $\Re(2^N)$ is the set of all non-isomorphic 2-regular graphs of order N, it has close relationship to partition of positive integer N. This graph can be applied to solve at least 3 employees problem and can extend to solve at least k employees problem. In this paper, we proved that the graph $\Re(2^N)$ is bipartite and has no Eulerian trail for each $N \geq 12$.

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