Portfolio Selection and Risk Management with Markov Chains

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Summary
This paper proposes markovian models in portfolio theory and risk management. At first, we describe discrete time optimal allocation models. Then, we examine the investor's optimal choices either when the returns are uniquely determined by their mean and variance or when they are modeled by a Markov chain. We subject these models to back-testing on out-of-sample data, in order to assess their forecasting ability. Finally, we propose some models to compute VaR and CVaR when the returns are modeled by a Markov chain.
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Key words:
Markov chain, Portfolio theory, VaR and CVaR models.

1. Introduction

In this paper, we pursue two objectives: the first one is to propose different markovian models that may be used to determine optimal portfolio strategies and to value opportunely the risk of a given portfolio; the second one is to compare portfolio selection strategies obtained either by modelling the return distributions by a Markov chain or by using a mean-variance analysis
Semi-Markov processes and hidden Markov models have been widely studied in financial literature to capture the dynamics of asset prices (see among others, Limnios and Oprisan (2001), Elliott and Van der Hoek (1997), Bahr and Hamori (2004)). Many of these approaches are parametric. Here, we propose a non-parametric markovian approach to model asset returns. Following the methodology proposed by Christoffersen (1998), it is possible to test the null hypothesis that the intervals of the distributional support of a given portfolio are independent against the hypothesis that the intervals follow a Markov chain. Several empirical works, carried out by considering both different distributional hypotheses for many return portfolios (Gaussian, Stable Paretian, Student's t, and semi-parametric) and different percentiles θ, have shown that we cannot reject the markovian hypothesis. Therefore, a Markov chain could be a good model to describe the evolution of the distributional support of a given portfolio (see also Lamantia et al. (2006), Iaquinta et al (2006)). This is the main reason according to which we propose an analysis of the impact of the markovian behavior of asset returns on the investor's choices. In what follows, we assume that the interval dependence of portfolios can be characterized by a Markov chain so that we can describe different portfolio selections, VaR and CVaR models. As a matter of fact, given a portfolio of gross returns, we divide the support of the portfolio into N intervals each of which is assumed to be a state of a Markov chain. Following this procedure, we build up the transition matrix. Then, we maximize the expected logarithmic utility function by assuming that in each interval the return is given by the geometric average of the interval extremes. It is worth mentioning that the problem becomes computationally too complex when we consider large portfolios. Furthermore, the transition matrix depends on the portfolio composition and the algorithm provides only a local optimum. In order to obtain a global optimum for the portfolio selection problem, we use a Simulated Annealing type algorithm (see among others Aarts and Korst (1989)). In our empirical comparison we find that the final wealth associated with the markovian portfolio choice is bigger than the common mean-variance one. The main contribution of this paper is the presentation of a general theory and a unifying framework for: 1) examining the portfolio selection problem when the return portfolio evolves along time following a Markov chain; 2) assessing the presented portfolio selection model and the mean-variance one; 3) studying and understanding VaR and CVaR markovian models. The rest of the paper is organized as follows: in Section 2, we formalize the portfolio selection with Markov chains. Section 3 introduces an empirical comparison between the Markovian portfolio selection model and the mean-variance one. Section 4 presents the approaches to compute VaR and CVaR when the portfolio follows a Markov chain. Finally, we briefly summarize the paper.

2. Portfolio selection with homogeneous Markov chains

In this section, we propose a non-parametric distributional analysis of the optimal portfolio choice problem by describing the behavior of portfolios through a homogeneous Markov chain. Let us consider $n+1$ assets: n of these assets are risky assets with gross returns $z_{t+1}=[z_{1,t+1},\dots,z_{n,t+1}]'$ and the $n+1$-th one is a risk-free asset with gross return $z_{0,t+1}$. Generally, we assume the standard definition of i-th gross return in the temporal interval $[t,\ t+1]$, $z_{i,t+1}=\frac{P_{i,t+1}+d_{i,[t,t+1]}}{P_{i,t}}$, where $P_{i,t}$ is the price

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of the *i*-th asset at time *t* and $d_{i,[t,t+1]}$ is the total amount of cash dividends paid by the asset between t and *t+1*. We distinguish the definition of gross return from the definition of return, i.e., $z_i - 1$ (or the alternative definition of return continuously compounded $r_i = \log z_i$). If we denote by x_0 the weight of the risk-less asset and by $x = [x_1, \dots, x_n]'$ the vector of the positions taken in the *n* risky assets, then the return portfolio during the period $[t, t+1]$ is given by

$$z_{(x),t+1} = \sum_{i=1}^{n} x_i z_{i,t+1} + x_0 z_{0,t+1}.$$

Let us assume that the portfolio of gross returns has support on the interval $(\min_k z_{(x),k}; \max_k z_{(x),k})$, where $z_{(x),k}$ is the k -th past observation of the portfolio $z_{(x)}$. At first, we divide the portfolio support $(\min_k z_{(x),k}; \max_k z_{(x),k})$ in N intervals $(a_{(x),i}; a_{(x),i+1})$ where $a_{(x),i} = \left(\frac{\max_k z_{(x),k}}{\min_k z_{(x),k}}\right)^{i/N} \min_k z_{(x),k}$, $i = 0,1,\dots,N$. For simplicity, we assume that the state of the return on the interval $(a_{(x),i}; a_{(x),i+1})$ is given by the geometric average of the extremes $z_{(x)}^{(i)} := \sqrt{a_{(x),i} a_{(x),i+1}}$. Moreover, we add an additional state, $z_{(x)}^{(N+1)} := z_0$, since we assume a fixed riskless return. Secondly, we build the transition matrix $P_t = [p_{i,j;t}]_{1 \le i,j \le N}$ valued at time *t* where the probability $p_{i,j;t}$ points out the probability (valued at time *t*) of a process transition between the state $z_{(x)}^{(i)}$ and the state $z_{(x)}^{(j)}$. Note also that when the portfolio is reduced at only the riskless asset, P_t is a matrix with 1 in the position on the diagonal corresponding to the riskless state and 0 in all the other positions. This fact is consistent with the definition of a constant variable. On the other hand, if we consider a homogeneous Markov chain, the transition matrix is independent of time and it can be simply denoted by P . We also observe that the transition probability matrix associated with the Markov chain is usually sparse. It means that many elements of this matrix are numerically negligible. This property is important because it deeply reduces the computational cost of the algorithm (see Zlatev (1991), Broyden and Vespucci (2004)).

In constructing the approximating Markov chain, we need to choose the length of a time step and the number of states of the process. Generally, for different financial models, the transition matrix has to be differently characterized. In portfolio selection problems, we assume daily steps with the convention that the Markov chain is computed on the returns valued with respect to investor's temporal horizon *T*. For instance, if the investor recalibrates the portfolio every month ($T = 20$ working days), we consider monthly gross returns with daily frequency. Using the same notation as gross returns, this means simply $z_{i,[t,t+20]} = \frac{P_{i,t+20} + d_{i,[t,t+20]}}{P_{i,t}}$. Then, we compute on the portfolio series the relative transition matrix. In this way, we consider the effect of the aggregated risk having an important impact on the investors' choices (see Lamantia et al. (2006), Rachev et al (2007)). We suggest to use a limited number of states in the portfolio selection problem. The reason of this choice is related to the fact that the transition matrix is strictly dependent on the portfolio composition that is the variable of the optimization problem. Consequently, the problem becomes computationally unmanageable when the number of states increases. On the other hand, as we can see in the following analysis, the goodness of the investors' choices is not excessively compromised if a limited number of states is considered. Under these assumptions, the final wealth (after *T* periods (days)) obtained by investing W_0 in the portfolio with composition (x_0, x) is simply given by:

$$S_{(x),t+T} = W_0 \prod_{s=1}^{N+1} \left(z_{(x)}^{(s)} \right)^{\sum_{i=1}^{T} \nu_{(t+i)}^{(s)}}$$

where

$$\nu_{(t+i)}^{(s)} = \begin{cases} 1 & \text{if at } (t+i)\text{-th period the portfolio return is} \\ & \text{in the } s\text{-th state} \\ 0 & \text{otherwise} \end{cases} .$$

Moreover, if at the t -th time step the portfolio is in the m -th state, then the expected value of the logarithm of the final wealth is given by :

$$E_m\left(\log\left(S_{(x),t+T}\right)\right) = \log\left(W_0\right) + \sum_{s=1}^{N+1}\left(\sum_{i=1}^{T} p_{m,s}^{(i)}\right)\log\left(z_{(x)}^{(s)}\right)$$

where $p_{m,s}^{(i)}$ is the element in position (m,s) of the i-th power of the transition matrix, P^i . This is a logical consequence of the Chapman-Kolmogorov equations (see, among others, Seneta (1981)). The expected value of the logarithm of the final wealth is

$$E\left(\log\left(S_{(x),t+T}\right)\right) = \log\left(W_0\right) + \sum_{m=1}^{N+1} p_m \sum_{s=1}^{N+1}\left(\sum_{i=1}^{T} p_{m,s}^{(i)}\right)\log\left(z_{(x)}^{(s)}\right) \quad (1)$$

where p_m is the probability of being in the state *m*. When no short sales are allowed ($x_i \ge 0$), an investor with logarithmic utility function and temporal horizon *T* tries to solve the following optimization problem

$$\begin{aligned} &\max_x E\left(\log\left(S_{(x),t+T}\right)\right) \\ &\text{subject to} \\ &x_0 + \sum_{i=1}^{n} x_i = 1; \quad x_i \ge 0; \quad i = 0,1,\dots,n \end{aligned} \quad (2)$$

in order to maximize his expected utility. In the case we consider investors with different utility functions, we can get the portfolio distribution in a reasonable time using the algorithm proposed by Iaquinta and Ortobelli (2006). In this case, even if the computational complexity to compute $E(u(x'z))$ is of the same order, we are not able to express in a close formula the expected utility as for logarithmic utility function (1). Since each row of the transition matrix gives the corresponding distribution of the relative state with respect to the other states, when we solve the above optimization problem, we determine those portfolios whose sample path dominates the other ones. The maximization of the expected log-utility function has to give a unique global maximum as a consequence of the monotony of the integral, but the problem (2) generally admits many local maxima. Consequently, the optimization problem appears computationally complex. This fact is a consequence of the discretization process that we adopt when we build up the approximating transition matrix. As a matter of fact, the methodology presented here is non parametric so that, practically, we approximate the distribution of any portfolio. On the other hand, the sensitivity of the expected utility with respect to the portfolio composition is a well known problem in portfolio theory (see, among others, Ziemba and Mulvey (1999), Bertocchi et al. (2005)). In order to approximate the optimal solution in problem (2), we consider two procedures:

1) finding the local optimum near a potential optimal point;
2) using a simulated annealing-type procedure in order to obtain the global maximum.

Procedure 1

The first methodology gives us only a local solution, and what we consider to be the optimum may not be the global optimum. In this case, we can also use a large portfolio and many states for the transition matrix since this procedure is computationally not very complex. Then, we compare the performance of a markovian portfolio strategy with the performance of a portfolio obtained maximizing the expected utility in a mean-variance framework. It is worth mentioning that we choose the optimal portfolio provided by the mean-variance analysis as the starting point for the markovian analysis.

Procedure 2

The first methodology is applicable in many real cases when large portfolios are considered since it is computationally convenient but it is incomplete in the sense that it could potentially return a local optimum and not a global optimum for any instance of the optimization problem. This is the reason why we propose to use a Simulated Annealing type algorithm (see Kirkpatrick et al. (1983)). The problem, in this case, is that the computational complexity increases sensibly and, consequently, we suggest this approach only when the number of assets and states is small. However, we try to simplify the computation using one of the most recent version of simulated annealing-type programs (see Bartholomew Biggs (2004) and LGO software package; Pinter (1996); Ben Hamida and Cont (2005)).

3. A first empirical comparison between portfolio selection strategies

In this section, we evaluate the impact of the previous two procedures on the portfolio choice by proposing a comparison between the markovian model and the mean-variance one. We propose two different comparisons: the first one with procedure 1 and the second one with procedure 2. In the first empirical comparison, we consider the optimal allocation among 24 assets, 23 of which are risky and the 24-th is risk-free with annual rate 6%. In the second empirical comparison, we draw our attention on 10 assets in the US market: 9 of these assets are risky and the 10-th is the 3 months Treasury Bill. In order to compare ex-post the mean-variance model with the markovian one, we use the same algorithm proposed by Giacometti and Ortobelli (2004). At each step we recalibrate the portfolio and distinguish the two analysis:

1) in the markovian case, we solve the optimization problem (2) and then we capitalize the wealth with the ex-post observed returns;
2) in the mean-variance framework, we first maximize the Sharpe ratio in order to find the market portfolio. The convex combination of the market portfolio and of the riskless asset provides the analytical formulation of the efficient frontier. Then, we select the portfolio on the efficient frontier maximizing the expected log-utility function. Finally, we capitalize the wealth with the ex-post observed returns.

Empirical comparison with Procedure 1

In this comparison, we use monthly gross returns (20 working days) with daily frequency taken from 23 international risky indexes valued in USD and quoted from January 1993 to January 1998. By assuming that short selling is not allowed, we examine optimal allocation among the riskless return 6% p.a. and 23 index returns: DAX 30, DAX 100 Performance, CAC 40, FTSE all share, FTSE 100, FTSE actuaries 350, Reuters Commodities, Nikkei 225 Simple average, Nikkei 300 weighted stock average, Nikkei 300 simple stock average, Nikkei 500, Nikkei 225 stock average, Nikkei 300, Brent Crude Physical, Brent current month, Corn No.2 Yellow cents, Coffee Brazilian, Dow Jones Futures, Dow Jones Commodities, Dow Jones Industrials, Fuel Oil No.2, Goldman Sachs Commodity, S&P 500. In order to assess the reliability of the models proposed here, we split the historical data into two parts:

1) the first part is used to estimate the transition matrix (in the markovian case) or the mean and the variance covariance matrix (in the mean-variance analysis);

2) the second part of the data-set is used to calculate the ex-post sample path of the final wealth, using the optimal portfolios computed at the beginning of each period for the two different models. It is worth mentioning that we fix the memory of the underlying process and use the last 500 observations (a little bit less than two years of daily observations, since we have observed a sufficient stability of the results with this number of observations).

We consider an initial wealth $W_0 = 1$ and in the ex-post analysis we calibrate the portfolio 27 times. After k periods, the main steps to compute the ex-post final wealth in the mean-variance context are the following:

Step 1 At the k-th period ($k = 0,1,\dots,26$) we determine the market portfolio $x_M^{(k)}$ that maximizes the Sharpe ratio, i.e., it is the solution of the following optimization problem:

$$\max_{x^{(k)}} \frac{E\left(\left(x^{(k)}\right)' z - z_0\right)}{\sigma_{\left(x^{(k)}\right)' z - z_0}}$$

s.t.

$$\left(x^{(k)}\right)' e = 1,$$

$$x_i^{(k)} \geq 0; \quad i = 1,\dots,n.$$

Step 2 We maximize the expected log-utility of the convex combination of the riskless and of the market portfolio, i.e.:

$$\max_{\lambda^{(k)}} \sum_{i=20k}^{499+20k} \log\left(\lambda^{(k)} z_0^{(i)} + \left(1-\lambda^{(k)}\right)\left(x_M^{(k)}\right)' z^{(i)}\right)$$

s.t.

$$0 \leq \lambda^{(k)} \leq 1,$$

where $z_0^{(i)}$ is the i -th observation of the riskless (that in this first empirical analysis is assumed to be constant at 6% p.a.). Then, after k periods, the optimal investment in the riskless is $\lambda^{(k)}$ while, in the risky assets, it is given by $\left(1-\lambda^{(k)}\right) x_M^{(k)}$.

Step 3 The ex-post final wealth at the k-th period is given by:

$$W_k = W_{k-1}\left(\lambda^{(k)} z_0^{(20+k)} + \left(1-\lambda^{(k)}\right)\left(x_M^{(k)}\right)' z^{(20+k)}\right).$$

On the other hand, we consider a Markov chain with 26 states; 25 of them to describe the behavior of the return portfolio and the last one for the riskless return. For every portfolio $z_{(x)}$, we compute the 26×26 transition matrix and its first 20 powers $P, P^2, \dots, P^{20}$. The logarithm of the expected return after one month (i.e., 20 working days) is given by:

$$E\left(\log\left(S_{(x),t+20}\right)\right) = \sum_{m=1}^{N} p_m \sum_{s=1}^{N}\left(\sum_{i=1}^{20} p_{m,s}^{(i)}\right)\log\left(z_{(x)}^{(s)}\right).$$

Therefore, starting from the mean-variance optimal solution, after k periods we obtain the first local maximum solution of the optimization problem (2). Then, we repeat step 3 in order to compute the ex-post final wealth at the k-th period.

From this first comparison it appears clear (see Figure 1) that the final wealth increases significantly with the markovian approach, in particular when we do not consider transaction costs. As a matter of fact, the markovian approach presents a final wealth that is greater (75% of the initial one) than the classic mean-variance one even during the crisis of the Asian market (September 1997-January 1998). A possible explanation is that the proposed markovian approach requires more time to capture the new volatility regime identified by crisis times, because of the finite memory property of Markov chains.

The validity of these results is confirmed by the fact that we considered for our analysis the principal international indexes. Moreover, the optimal allocation of the portfolio corresponding to different time periods (available from the authors upon request) shows that, mainly, American indexes, and in particular Dow Jones Industrials and S&P 500, are chosen. Intuition suggests that in the period analyzed the US market presented a greater efficiency than the other ones. Furthermore, the flexibility of the markovian approach during the crisis of the Asian markets should encourage fund managers to pursue international diversification using this non parametric approach. The previous results are substantially confirmed even when we consider the impact of constant transaction costs of 0.5% (using the same algorithm proposed in Giacometti and Ortobelli (2004)). In this case, we observe that transaction costs have a higher impact on the markovian approach that, after two years, presents a greater final wealth (68% of the initial wealth) than the classic mean-variance one (see Figure 2). On the other hand, it is well known that the sensitivity of the expected utility to the little changes in the weights represents a big problem in the portfolio theory. As a matter of fact, optimal mean-variance portfolio weights are often quite extreme and change frequently. In order to value the volatility of the optimal portfolio compositions, we have to look at the portfolio weights under both the mean-variance and the markovian set up. In particular, we have computed the average of the absolute differences between optimal portfolio weights obtained during the period of analysis. This simple computation is given by:

$$\mathtt{Portdiff}_1 = \frac{1}{26}\sum_{k=1}^{26}\left(\left|\lambda^{(k)} - \lambda^{(k-1)}\right| + \right.$$
$$\left. + \sum_{j=1}^{23}\left|\left(1-\lambda^{(k)}\right)x_{M,j}^{(k)} - \left(1-\lambda^{(k-1)}\right)x_{M,j}^{(k-1)}\right|\right).$$

With the markovian approach, we observe values of $\mathtt{Portdiff}_1$ equal to 46% and to 38% without and with transaction costs, respectively. With the mean-variance approach, the differences in the portfolio compositions are respectively equal to 43% and to 40% (without and with transaction costs). Thus, there is no evidence that weights are either less extreme or less volatile with the markovian approach than the mean-variance one.

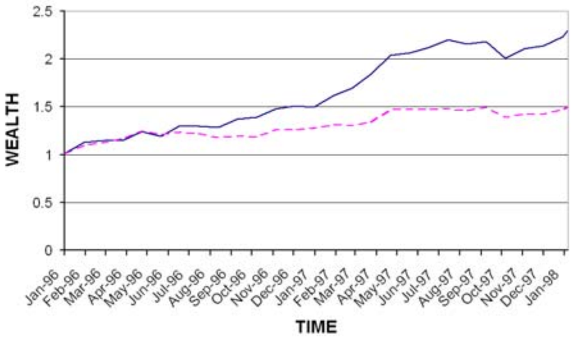

Fig. 1. *Ex-post* sample path of the final wealth (with procedure 1 and without considering transaction costs)

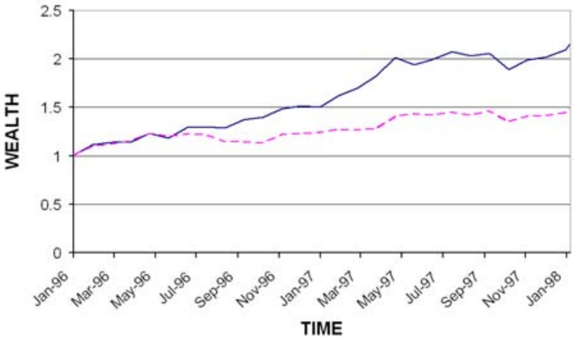

Fig. 2. *Ex-post* sample path of the final wealth (with procedure 1 considering transaction costs)

Empirical comparison with Procedure 2

In this comparison, we use six-month returns (123 working days) with daily frequency taken from 9 risky assets in the Dow Jones Industrial index quoted from January 1995 to April 2005. At first, assuming that short selling is not allowed, we examine the optimal allocation among the Treasury Bill three month return and 9 asset returns: Altria, Boeing, Citigroup, Coca Cola, Intel, Johnson, Microsoft, Procter & Gamble, Pfizer. The main reason of this choice for procedure 2 is that this procedure is computationally more complex than procedure 1 so that the computational complexity of procedure 2 using the same set of assets of procedure 1 would be unmanageable. Furthermore, the choice of this set of assets is related to the fact that our aim is also to show the impact of an international diversification of the assets in portfolio. Since in the previous analysis we have considered a set of international assets, here we consider only assets from the US market in order to make a comparison.

Similarly as before, also in this empirical analysis we split the historical data into two parts and consider an initial wealth $W_0 = 1$. In the ex-post analysis, we calibrate the portfolio 17 times and the main steps to compute the ex-post final wealth are exactly the same of the previous analysis. We consider a Markov chain with 6 opportune states, that is we divide the portfolio support in the following way: 5 intervals take into account the dynamics of the risky portfolio and the 6-th interval takes into account the riskless rate. For every portfolio $z_{(x)}$, we compute the 6×6 transition matrix and its first 123 powers. Then, with formula (1) we can compute the expected log final wealth after 123 working days. After k periods, we obtain the global maximum solution of the optimization problem (2) by using a simulated annealing-type algorithm with 100 iterations at each period. Then, we compute the ex-post final wealth at the k -th period, as in Step 3 of the previous algorithm. This comparison shows (see Figure 3) that the markovian approach significantly increases the final wealth on these 10 assets until September 11-th 2001. As a matter of fact, the better performance of the markovian approach is maintained even during the high volatility period of the late 1990's (Asian and Russian crises). Before September the 11-th, the markovian approach presented a greater wealth (115% of the initial wealth) than the mean-variance one while, in December 2001, there is a dramatic loss of value of both portfolios. From December 2001 to the end of the observation period (April 2005) both strategies maintained a stationary behavior, presenting almost the same final wealth. Consequently, the markovian approach presents no better performance than the mean-variance one when we consider a portfolio without international diversification. It is worth mentioning that, by using a partial data-set from that one used in procedure 1 (i.e., 2 US assets, 2 European assets, 2 Asian assets), the analysis confirms that the markovian approach performs better than the mean-variance one. On the contrary, as proved in procedure 1, an international diversification of the set of assets makes the markovian approach more suitable since it provides a greater wealth both with and without transaction costs. As in the previous analysis, we want to value the average of the absolute differences between optimal portfolio weights obtained during the period of analysis, i.e.,

$$\mathbf{Portdiff}_2 = \frac{1}{16}\sum_{k=1}^{16}\left(\left|\lambda^{(k)} - \lambda^{(k-1)}\right| + \right.$$
$$\left. + \sum_{j=1}^{9}\left|\left(1-\lambda^{(k)}\right)x_{M,j}^{(k)} - \left(1-\lambda^{(k-1)}\right)x_{M,j}^{(k-1)}\right|\right).$$

With the markovian approach, we observe values of $\mathbf{Portdiff}_2$ equal respectively to 76% and to 70%

without and with constant transaction costs of 0.5%. With the mean-variance approach the differences in the portfolio compositions are respectively equal to 63% and to 58% (without and with transaction costs). Consequently, in this case, the portfolio weights obtained with the markovian approach are more volatile than those obtained with the mean-variance one. In particular, although the markovian approach shows good performances (also when we do not use many states for the transition matrix), it appears to be more effective when we adopt international diversification strategies. On the other hand, the mean-variance approach appears more conservative than the markovian one when we use US stock assets.

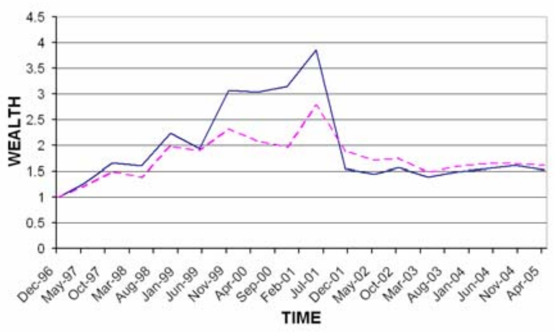

Fig. 3. *Ex-post* sample path of the final wealth (with procedure 2 without considering transaction costs)

4. VaR and CVaR models with Markov Chains

In this section, we propose some alternative models to compute Value at Risk (VaR) and Conditional Value at Risk (CVaR) with a homogeneous Markov chain. In financial literature, the most used VaR model (see Longerstaey and Zangari (1996)) assumes that the conditional distribution of the continuously compounded returns obeys to a Gaussian law. However, it is well known that asset returns are not conditionally normally distributed (see, among others, Rachev and Mittnik (2000)). Next, we show that using our approach it is possible to explain the stylized features of observed portfolio returns. In particular, our approach is able to capture heavy tails better than VaR models commonly used by practitioners. Recall that the VaR is the minimum loss among the worst $(1-\theta)$ cases that could occur in a given temporal horizon. If we denote by τ the investor's temporal horizon, by $W_{t+\tau}-W_t$ the profit/loss realized in the interval $[t,t+\tau]$ and by θ the level of confidence, then VaR is given by the loss such that,

$$\mathbf{VaR}_{\theta,t+\tau}(W_{t+\tau}-W_t)=\inf\left\{q \mid \Pr(W_{t+\tau}-W_t\le q)>1-\theta\right\}.$$

Hence, the VaR is the percentile at the level $(1-\theta)$ of the profit/loss distribution in the interval $[t,t+\tau]$. The temporal horizon τ and the level of confidence θ are chosen by the investor. The choice of τ depends upon the frequency the investor wishes to control his/her investment. Unfortunately, $\mathbf{VaR}_\theta$ is not a coherent risk measure and it cannot offer exhaustive information about the expected future losses. In financial literature, we can find different definitions of VaR and CVaR that change slightly with respect to the use of the risk measure. For example, in portfolio theory it is generally used a positive risk measure thus, typically, the above definitions change in the sign of VaR and CVaR functions. Alternatively to VaR, Artzner et al. (1999), and Szegö (2004) proposed the CVaR to evaluate the exposure to market risks. The CVaR is a coherent risk measure, i.e., it is a positively homogeneous, translation invariant, subadditive and monotone risk measure. The CVaR measures the expected value of profit/loss given that the VaR has not been exceeded, that is

$$\mathbf{CVaR}_{\theta,t+\tau}(W_{t+\tau}-W_t)=\frac{1}{1-\theta}\int_0^{1-\theta}\mathbf{VaR}_{q,t+\tau}(W_{t+\tau}-W_t)\,dq$$

and if we assume a continuous distribution for the profit/loss distribution, we obtain that

$$\mathbf{CVaR}_{\theta,t+\tau}(W_{t+\tau}-W_t)=E\left(W_{t+\tau}-W_t \mid W_{t+\tau}-W_t\le q\right).$$

Let us consider n assets with vector of log returns $r=[r_1,\dots,r_n]'$, where $r_i=\log z_i$. The portfolio of log returns is given by

$$r_{(x)}=\sum_{i=1}^n x_i r_i,$$

where $x=[x_1,\dots,x_n]'$ is the vector of the positions taken in the n assets forming the portfolio. Let us assume that the portfolio of log returns has support on the interval $\left(\min_k r_{(x),k},\max_k r_{(x),k}\right)$, where $r_{(x),k}$ is the k-th past observation of the portfolio $r_{(x)}$. Since it makes sense to consider only the left tail of the portfolio distribution and we are generally interested in confidence levels $\theta\ge 95\%$, we assume that an opportune return portfolio left tail corresponding to $\left(\min_k r_{(x),k},\min_k r_{(x),k}+5\frac{\max_k r_{(x),k}-\min_k r_{(x),k}}{12}\right)$ follows a homogeneous Markov chain of $N+1$ states. The choice of the opportune return portfolio left tail is strictly dependent on the level of confidence θ. In particular, we have observed that, for percentiles equal or lower than 5%, it makes sense to consider 5/12 of the portfolio support. Then, we divide the return portfolio left tail in N additive intervals $\left(a_{(x),i},a_{(x),i+1}\right)$ that represent the first N states, where: $a_{(x),i}=\min_k r_{(x),k}+i\,\mathbf{STEP}\quad i=0,\dots,N$, $\mathbf{STEP}=5\frac{\max_k r_{(x),k}-\min_k r_{(x),k}}{12N}$ and $a_{(x),N+1}=\max_k r_{(x),k}$.

For simplicity, we assume that on the interval

$\left(a_{(x),i}, a_{(x),i+1}\right)$ the state of the return is given by the average of the extremes $r_{(x)}^{(i)} := \frac{a_{(x),i}+a_{(x),i+1}}{2}$, $i = 0,\ldots,N$, and that the last state of the portfolio return is $r_{(x)}^{(N+1)} := \frac{\max_k r_{(x),k} + \min_k r_{(x),k}}{2}$. The last interval is the portfolio return right tail and it is identified by

$$\left(\frac{\max_k r_{(x),k} + 7\min_k r_{(x),k}}{12}, \max_k r_{(x),k}\right).$$

Once we have build up the transition matrix $P = \left[p_{i,j}\right]_{1\le i,j\le N+1}$ of the portfolio, we need to consider in which state we have observed the portfolio. Let m be the starting state of the portfolio $r_{(x)}$. Then, in the markovian case, VaR and CVaR with confidence level θ ($\ge 95\%$) are given by

$$\mathbf{VaR}_\theta\left(r_{(x)}\right) = \left\{ r_{(x)}^{(s)} \,\middle|\, \sum_{i=1}^{s-1} p_{m,i} < (1-\theta); \sum_{i=1}^{s} p_{m,i} \ge (1-\theta) \right\}, \quad (3)$$

$$\mathbf{CVaR}_\theta\left(r_{(x)}\right) = \frac{1}{1-\theta} \sum_{i\ :\ r_{(x)}^{(i)} \le \mathbf{VaR}_\theta} r_{(x)}^{(i)} p_{m,i}. \quad (4)$$

In order to understand if the markovian approach approximates the tail behavior of the portfolio returns, we propose a basic empirical comparison with the RiskMetrics EWMA Gaussian approach (see Longerstaey and Zangari (1996)). We consider 100 random portfolios of the previous 23 international indexes from January 1993 to January 1998. For each of them we consider the starting state m and a 50×50 transition matrix. Then, we compute VaR and CVaR with formulas (3) and (4) for different percentiles and we compare these values with the VaR and CVaR values computed with the empirical distribution (denoted with $\mathbf{VaR}_\theta\mathbf{EMP}$ and $\mathbf{CVaR}_\theta\mathbf{EMP}$) and with the RiskMetrics model (denoted with $\mathbf{VaR}_\theta\mathbf{GAUSS}$ and $\mathbf{CVaR}_\theta\mathbf{GAUSS}$). For $\theta = 99\%$, we observe that in the 81% of the cases

$$\left|\mathbf{VaR}_\theta\mathbf{EMP} - \mathbf{VaR}_\theta\mathbf{Markov}\right| <$$
$$< \left|\mathbf{VaR}_\theta\mathbf{EMP} - \mathbf{VaR}_\theta\mathbf{GAUSS}\right|$$

and in the 96% of the cases

$$\left|\mathbf{CVaR}_\theta\mathbf{EMP} - \mathbf{CVaR}_\theta\mathbf{Markov}\right| <$$
$$< \left|\mathbf{CVaR}_\theta\mathbf{EMP} - \mathbf{CVaR}_\theta\mathbf{GAUSS}\right|.$$

On the other hand, we do not have strong differences in the performance when we consider percentiles equal or greater than 97%. According to this simple analysis, we can observe that the markovian hypothesis considers the heavy tails generally presented by the portfolio returns. Moreover, we observe that we can easily predict the losses at a given future date using formulas similar to (3) and (4). In particular, some recent studies (see Iaquinta, and Ortobelli (2006)) have shown that the assumption of a Markovian behavior of daily returns permits to approximate the monthly and yearly return tail distributions much better than the classic RiskMetrics approach. This is confirmed by simple statistical tests (Kolmogorov-Smirnov (K-S) test and Anderson-Darling (A-D) test) valued on the ex-post return distributions forecasted after 60 days of some US indexes (S&P 500, Nasdaq, and Dow Jones Industrial) quoted from January 1995 to January 2006. In particular, Table 1 considers the Kolmogorov-Smirnov (K-S) test

$$K - S = \sup\left|F_E(x) - F(x)\right|$$

and the Anderson-Darling (A-D) statistic test

$$A - D = \sup\frac{\left|F_E(x) - F(x)\right|}{\sqrt{F(x)(1-F(x))}}$$

where $F_E(x)$ is the empirical cumulative distribution and $F(x)$ is the theoretical one.

Table 1: This table summarizes Kolmogorov-Smirnov test (K-S) and Anderson-Darling test (A-D) for S&P500, Nasdaq, and Dow Jones Industrials 60 days return series (quoted from January 1995 to January 2006) whose distributions are forecasted either with the markovian approach or assuming the RiskMetrics model.

60 days returns		**RiskMetrics**	**Markovian**
S&P500	**K-S**	0,0545	0,0692
	A-D	32.011	28.138
Nasdaq	**K-S**	0.0413	0,0396
	A-D	41.357	30.71
DowJones Industrials	**K-S**	0.0498	0.0461
	A-D	41.492	27.018

Table 1 shows that the Markovian approach takes into account much better the aggregated 60 days risk than the RiskMetrics one. Consequently, we have to expect that the losses forecasted with this model are more realistic than those predicted under the most used approaches.

5. Concluding Remarks

This paper proposes alternative models for the portfolio selection and the VaR and CVaR calculation. In the first part, we describe a portfolio selection model that uses a Markov chain to capture the behavior and the evolution of portfolio returns. The ex-post empirical comparison between the mean-variance approach and the markovian one shows that this last approach performs better in the sense that it provides greater increments in the final wealth, especially when the market is growing. In the second part, we present some alternative markovian VaR and CVaR models. Also in this case, a first empirical analysis shows that the markovian approach describes accurately the tail behavior of portfolio returns, in particular when the Gaussian hypothesis of the conditional return distribution determines intervals of confidence whose forecast ability is low. Furthermore, we believe that markovian and semi-markovian approaches to value the expected risk exposure of portfolios can be easily expressed using some recently studied methodologies either based on the approximation of more or less complex diffusion processes and capturing their markovianity with a Markov chain (see, among others, Duan et al (2003)) or using semi-markovian approaches (see Limnios and Oprisan (2001)).

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