A Comparison among Portfolio Selection Strategies with Subordinated Lévy Processes

Alessandro Staino†, Sergio Ortobelli††, and Ivar Massabò†††

University of Bergamo and University of Calabria, Italy

Summary
In this paper we describe portfolio selection models using Lévy processes. The contribution consists in comparing some portfolio selection strategies under different distributional assumptions. We first implement portfolio models under the hypothesis the log-returns follow a particular process with independent and stationary increments. Then we compare the ex-post final wealth of optimal portfolio selection models with subordinated Lévy processes when limited short sales and transaction costs are allowed.

Key words:
Portfolio theory, Lévy processes, Variance-Gamma distribution, Normal Inverse Gaussian distribution.

1. Introduction

The portfolio selection problem finds its origin with De Finetti and Markowitz' studies (see Markowitz (1987), Pressacco and Serafini (2007) and the references therein) that suggested to describe optimal portfolios as function of their mean and variance. Mean-variance analysis is fully justified under the assumption that the asset returns are elliptically distributed. Elliptical distributions are particular symmetric distributions that generalize the Gaussian one. However, the sample data often displays a certain level of skewness and, moreover, their distributions present fatter tails than the Gaussian one. In order to avoid misspecifications several alternative distributional assumptions have been proposed to describe the asset price evolution. Subordinated Brownian motions are stochastic processes whose distribution at a fixed time is able to own a skewness different from zero and a kurtosis bigger than three. These features make subordinated Brownian motions good substitutes to the normality assumption.
In this paper, we model the returns as a multidimensional time-changed Brownian motion where the subordinator follows either an Inverse Gaussian process or a Gamma process. The dependence structure implied by these distributional hypotheses gives more possibility to joint extreme events. In this framework it is possible to take into account the jumps often observed in the stock prices that could imply large losses for the investors. Under these different distributional hypotheses we discuss static and dynamic portfolio selection models in a mean-risk framework. In particular, we compare these models with the assumption that the log-returns follow a Brownian motion. We evaluate the distributional hypotheses by the point of view of several typologies of investors: investors with exponential utility functions, investors that maximize the mean-Value at Risk ratio, investors that recalibrate periodically their portfolios.
In Section 2, we discuss the different distributional hypotheses and we compare them by the point of view of investors with exponential utility function. In Section 3 we analyze multivariate subordinated Lévy processes and we propose an ex-post comparison of optimal portfolios in a mean-Value at Risk framework. Section 4, deals and compares dynamic portfolio strategies. Finally, we briefly summarize the results in Section 5.

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2. A first empirical comparison among portfolio selection models based on different Lévy processes

In this section, we compare the optimal portfolio composition under different distributional hypotheses. In particular, we consider Lévy processes with semi heavy tails. Thus, this analysis differs from other studies that assume Lévy processes with very heavy tails (see Rachev and Mittnik (2000), Ortobelli et al. (2004)). For sake of completeness, we recall some basic notions on Lévy processes. Lévy processes are all processes with stationary and independent increments with stochastically continuous paths. Typical examples are the Normal Inverse Gaussian process (NIG) and the Variance-Gamma one (VG). Many Lévy processes are often seen as subordinated Brownian motions where the subordinator is a Lévy process whose paths are almost surely non-decreasing. The NIG process and the VG process can be seen as subordinated Lévy processes where the subordinators are respectively the Inverse Gaussian process and the Gamma process.
Inverse Gaussian: An Inverse Gaussian process $X^{(IG)} = \left\{X_t^{(IG)}\right\}_{t\geq 0}$ assumes that any random variable $X_t^{(IG)}$ admits the following density function:

$$f_{IG}(x;ta,b)=$$
$$=\frac{ta}{\sqrt{2\pi}}\exp(tab)x^{-3/2}\exp\left(-\frac{1}{2}(t^2a^2x^{-1}+b^2x)\right)1_{[x>0]},$$

that is defined as Inverse Gaussian distribution $IG(ta,b)$ where *a*, *b* are positive.

Gamma: A Gamma process $X^{(Gamma)}=\left\{X_t^{(Gamma)}\right\}_{t\geq 0}$ admits the Gamma distribution $Gamma(ta,b)$ where *a*, *b* are positive. The density function of the $Gamma(a,b)$ law is given by

$$f_{Gamma}(x;a,b)=\frac{b^a}{\Gamma(a)}x^{a-1}\exp(-xb)1_{[x>0]}.$$

Normal Inverse Gaussian: Subordinating the Brownian motion with an Inverse Gaussian process we obtain a Normal Inverse Gaussian process $NIG(\alpha,\beta,\delta)$, with parameters $\alpha>0$, $\beta\in(-\alpha,\alpha)$ and $\delta>0$, that is

$$X_t^{(NIG)}=\beta\delta^2 I_t+\delta W_{I_t}$$

where I_t is an *IG* process with parameters $a=1$, $b=\delta\sqrt{\alpha^2-\beta^2}$ and W_t is a standard Brownian motion. The density of $X_t^{(NIG)}$ is given by:

$$f_{NIG}(x;\alpha,\beta,t\delta)=$$
$$=\frac{\alpha(t\delta)}{\pi}\exp\left((t\delta)\sqrt{\alpha^2-\beta^2}+\beta x\right)\frac{\mathbf{K}_1(\alpha\sqrt{(t\delta)^2+x^2})}{\sqrt{(t\delta)^2+x^2}}, \quad (1)$$

where $\mathbf{K}_\lambda(x)$ denotes the modified Bessel function of the third kind with index λ (see, among others, Abramowitz and Stegun (1968)).

Variance Gamma: Subordinating the Brownian motion with a Gamma process we obtain a Variance-Gamma process, $VG(\mu,\sigma,\nu)$ with parameters $\sigma>0$, $\nu>0$ and $\mu\in R$, that is

$$X_t^{(VG)}=\mu G_t+\sigma W_{G_t},$$

where G_t is a Gamma process with parameters $a=1/\nu$ and $b=1/\nu$. The Variance-Gamma process can be also defined as the difference between two independent Gamma processes. The density of $X_t^{(VG)}$ is given by:

$$f_{VG}(x;t\mu,\sigma\sqrt{t},\nu/t)=$$
$$=\frac{2e^{\frac{\mu x}{\sigma^2}}\left(\frac{x^2}{2\sigma^2/\nu+\mu^2}\right)^{\frac{t}{2\nu}-\frac{1}{4}}}{\nu^{t/\nu}\sqrt{2\pi}\sigma\Gamma(t/\nu)}\mathbf{K}_{\frac{t}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{x^2(2\sigma^2/\nu+\mu^2)}\right), \quad (2)$$

where $\mathbf{K}_{\frac{t}{\nu}-\frac{1}{2}}(x)$ is the modified Bessel function of the third kind with index $\frac{t}{\nu}-\frac{1}{2}$.

In portfolio theory, it has been widely used a standard Brownian Motion to model the log-return distribution $X^{(BM)}=\left\{X_t^{(BM)}\right\}_{t\geq 0}$. Under this assumption the log-return at time *t* is normal distributed with mean μt and standard deviation $\sigma\sqrt{t}$ (i.e., $X_t^{(BM)} \sim N\left(\mu t,\sigma^2 t\right)$). In the next subsection we compare optimal portfolio strategies obtained under NIG and VG processes.

2.1 A first empirical comparison

Let us consider the problem to select an optimal portfolio composed by *N* risky assets with log-returns $\mathbf{X}=[X_1,\ldots,X_N]'$ and one risky-free asset with log-return r_f. Let $\mathbf{w}=[w_1,\ldots,w_N]$ be the vector of the weights invested in the risky assets and assume that no short sales are allowed (i.e., $w_i\geq 0$). In the classical mean-variance analysis, investors choose a portfolio that is the convex combination between market portfolio and riskless one. The weights of the market portfolio $\mathbf{w}_M$ are given by the solution of the following optimization problem:

$$\max_{\mathbf{w}}\frac{E(\mathbf{wX})-r_f}{\mathbf{wQw}'}$$
$$\text{s.t.} \quad , \quad (3)$$
$$\sum_{i=1}^N w_i=1;\ w_i\geq 0;\ i=1,...,N$$

where *Q* is the definite positive variance-covariance matrix of the log-return vector ***X***. Now, let us suppose the investors' exponential utility function is:

$$u(x)=a\left(1-\exp\left(-\frac{1}{a}x\right)\right), \quad (4)$$

where "*a*" is their risk tolerance parameter. In order to value the impact of different distributional hypotheses in the portfolio composition, we compute the optimal portfolio that maximizes investor's expected utility when the market portfolio follows a particular subordinated Levy process. That is, we compute the riskless weight λ that maximizes

$$E(u(\lambda r_f+(1-\lambda)\mathbf{w}_M\mathbf{X})), \quad (5)$$

when the market portfolio $\mathbf{w}_M\mathbf{X}$ follows or a Brownian Motion (BM), or a Variance-Gamma process (VG) or a Normal Inverse Gaussian process (NIG). Observe that the analytical value of the expression (5) for the exponential utility function can be easily found using the Laplace transform of the respective distributions (see, among others, Cont and Tankov (2004)).

In this first empirical comparison, we consider daily returns from 04/10/1992 to 01/01/2002 on 10 risky US indexes: DJTM United States Automoniles, DJTM United States Oil & Gas, DJTM United States Basic Resource, Dow Jones Industrials, Dow Jones Utilities, Nasdaq Industrials, NYSE Composite, S&P100, S&P500, S&P900. We assume as riskless asset the Treasury Bill 3-mounth r_f=1.61% a.r. on 01/01/2002. Thus, first we determine the market portfolio solving the optimization problem (3) and then we estimate the parameters of the market portfolios maximizing the likelihood function (MLE) under the three distributional hypotheses.

Table 1: MLE parameters of three months market portfolio returns assuming or a Variance-Gamma process or a Normal Inverse Gaussian process, or a Brownian Motion.

VG	μ =0.0196	σ =0.0637	ν =0.0142
NIG	α =107.2398	β =4.7419	δ =0.4426
BM	μ =0.0196	σ =0.0646	

In Table 1 we report the maximum likelihood estimates (MLE) of the market portfolio parameters supposing that it follows or a Normal Inverse Gaussian process or a Variance-Gamma process, or a Brownian Motion. Since we assume the investor's temporal horizon is three months, the distributional parameters are on three months basis. Secondly, we maximize the expected utility of the final wealth assuming in the utility function (4) three possible risk tolerance parameters: 0.1; 0.15; 0.2.

Table 2: Quotes invested in the riskless asset, maximum expected utility and ex-post final wealth under the assumption that the three months market portfolio follows or a Normal Inverse Gaussian process, or a Brownian Motion, or a Variance-Gamma process.

a=0.10	NIG	BM	VG
Weight on the riskless	0.565	0.5241	0.5579
Expected utility	0.0997	0.098	0.0997
Final wealth	0.926304	0.917861	0.924838
a=0.15	**NIG**	**BM**	**VG**
Weight on the riskless	0.3475	0.2862	0.3369
Expected utility	0.1494	0.1468	0.1494
Final wealth	0.881405	0.868751	0.879217
a=0.20	**NIG**	**BM**	**VG**
Weight on the riskless	0.1301	0.0483	0.1158
Expected utility	0.1982	0.1948	0.1982
Final wealth	0.836528	0.819642	0.833576

Table 2 shows the quote invested in the riskless, the maximum expected utility, and the ex-post final wealth after one year on date 01/01/2003 under the three different distributional hypotheses. From this table we observe that the NIG and VG processes take much more into account the possible losses. As a matter of fact, the quote invested in the riskless is always higher than that one computed for the Brownian Motion. Moreover even the computed maximum expected utility is higher for the NIG and VG processes that implicitly underscores the better performances of the other two processes. These processes are more conservative with respect to the Brownian Motion as confirmed by the ex-post final wealth of Table 2. As a matter of fact, during the 2002, year with very big losses on the US market, we observe a higher final wealth under the NIG and VG processes.

3. Portfolio Selection with Multivariate Subordinated Lévy Processes

In this section, we first discuss the extension of optimal selection problem to multivariate Lévy processes. Then we assess the different distributional approaches and we compare their effects with respect to the portfolio selection problem.

The multivariate Lévy processes distributions are obtained as a logical extension of univariate ones. So, for example, the *d*-dimensional Multivariate Normal Inverse Gaussian (MNIG) process with parameters $\delta,\alpha > 0$, $\boldsymbol{\beta},\boldsymbol{\mu} \in R^d$ and $\mathbf{Q} \in R^{d^2}$ valued at time *t* can be constructed from:

$$\mathbf{X}_t = \boldsymbol{\mu}t + I_t\mathbf{Q}\boldsymbol{\beta} + \sqrt{I_t}\mathbf{Q}^{1/2}\mathbf{Y},$$

where the intrinsic time Inverse Gaussian process I_t is distributed as $IG\left(\delta t, \sqrt{\alpha^2 - \boldsymbol{\beta}'\mathbf{Q}\boldsymbol{\beta}}\right)$, $\mathbf{Y}$ is a standard *d*-dimensional Gaussian $N_d\left(\mathbf{0},\mathbf{I}\right)$ independent of I_t and then conditional distribution of vector $\mathbf{X}_t / I_t$ is $N_d\left(\boldsymbol{\mu}t + I_t\mathbf{Q}\boldsymbol{\beta}, I_t\mathbf{Q}\right)$ (see Barndorff-Nielsen (1977)). Thus the *d*-dimensional vector $\mathbf{X}_t$ admits density probability function:

$$f_{\mathbf{X}_t}(\mathbf{x}) = \int f_{\mathbf{X}_t/I_t}(\mathbf{x}/z) f_{I_t}(z)dz = \frac{\delta t}{2^{\frac{d-1}{2}}}\left(\frac{\alpha}{\pi q(\mathbf{x})}\right)^{\frac{d+1}{2}} \times$$
$$\times K_{\frac{d+1}{2}}\left(\alpha q(\mathbf{x})\right)\exp\left(p(\mathbf{x})\right)$$

where $q(\mathbf{x}) = \sqrt{\left(\delta t\right)^2 + \left((\mathbf{x} - \boldsymbol{\mu}t)'\mathbf{Q}^{-1}(\mathbf{x} - \boldsymbol{\mu}t)\right)}$ and $p(\mathbf{x}) = \left(\boldsymbol{\beta}'(\mathbf{x} - \boldsymbol{\mu}t) + \delta t\sqrt{\alpha^2 - \boldsymbol{\beta}'\mathbf{Q}\boldsymbol{\beta}}\right)$.

Similarly, we can define the multivariate Variance-Gamma process. However, generally there exist many problems in the maximum likelihood estimation of multivariate Lévy process parameters, in particular when we assume a large number of assets (see, among others, Hanssen and Øigård (2001), Bølviken and Benth (2000)).

For this reason we estimate the parameters of marginal distributions separately by the correlation matrix. Doing so we assume that every couple of subordinated components follows a joint bivariate subordinated process. Suppose that in the market the vector of risky assets has log-returns $\mathbf{X}_t = [X_t^{(1)},..., X_t^{(N)}]'$ distributed as:

$$\mathbf{X}_t = \boldsymbol{\mu} Z_t + \mathbf{Q}^{1/2} W_{Z_t}^{(N)}, \qquad (6)$$

where $\boldsymbol{\mu} = [\mu_1, \mu_2, \ldots, \mu_N]'$, Z_t is the positive Lévy process subordinator, $\mathbf{Q} = \left[\sigma_{ij}^2\right]$ is a fixed definite positive variance-covariance matrix (i.e., $\sigma_{ij}^2 = \sigma_{ii}\sigma_{jj}\rho_{ij}$ where ρ_{ij} is the correlation between the conditional *i*-th component of $\mathbf{X}_t / Z_t$ and its conditional *j*-th component) and $W_t^{(N)}$ is a *N*-dimensional standard Brownian motion (i.e., $\mathbf{Q}^{1/2} W_{Z_t}^{(N)} = \sqrt{Z_t}\mathbf{Q}^{1/2}\mathbf{Y}$ where $\mathbf{Y}$ is a standard *N*-dimensional Gaussian independent of Z_t). Under the above distributional hypotheses we approximate the log-return of the portfolio with the portfolio of log-returns, that is the convex combination of the log-returns:

$$X_t^{(w)} = \mathbf{w}\mathbf{X}_t = (\mathbf{w}\boldsymbol{\mu}) Z_t + \sqrt{\mathbf{w}\mathbf{Q}\mathbf{w}'} W_{Z_t}, \qquad (7)$$

where W_t is a 1-dimensional standard Brownian motion. At this point we will assume, as for the univariate case, that the subordinator Z_t is modeled either as an Inverse Gaussian process $Z_1 \sim \mathbf{IG}(1,b)$ or as a Gamma process $Z_1 \sim \mathbf{G}\left(\frac{1}{v},\frac{1}{v}\right)$.

NIG Processes. When Z_t follows an Inverse Gaussian process $\mathbf{IG}(1,b)$, then, the *i*-th log-return follows a NIG process $\mathbf{NIG}(\alpha_i, \beta_i, \delta_i)$ where the parameters are given by: $\alpha_i = \sqrt{(b/\delta_i)^2 + \beta_i^2}$, $\beta_i = \mu_i / \delta_i^2$ and $\delta_i = \sigma_{ii}$. Thus, the portfolio (7) follows a $\mathbf{NIG}(\alpha_w, \beta_w, \delta_w)$ process whose parameters are:

$$\alpha_w = \sqrt{\left(\frac{b}{\delta_w}\right)^2 + \beta_w^2}, \quad \beta_w = \frac{\mathbf{w}\boldsymbol{\mu}}{\mathbf{w}\mathbf{Q}\mathbf{w}'}, \quad \delta_w = \sqrt{\mathbf{w}\mathbf{Q}\mathbf{w}'}.$$

Mean, variance, and Fisher-Pearson skewness and kurtosis parameters of the portfolio $X_t^{(w)}$ are, respectively,

$$E\left(X_t^{(w)}\right) = t\delta_w\beta_w / \sqrt{\alpha_w^2 - \beta_w^2},$$

$$\mathbf{Variance}\left(X_t^{(w)}\right) = t\alpha_w^2\delta_w\left(\alpha_w^2 - \beta_w^2\right)^{-3/2},$$

$$\mathbf{Sk}\left(X_t^{(w)}\right) = \frac{E\left(\left(X_t^{(w)} - E(X_t^{(w)})\right)^3\right)}{E\left(\left(X_t^{(w)} - E(X_t^{(w)})\right)^2\right)^{3/2}} = 3\beta\alpha_w^{-1}(t\delta_w)^{-1/2}\left(\alpha_w^2 - \beta_w^2\right)^{-1/4}$$

$$\mathbf{Ku}\left(X_t^{(w)}\right) = \frac{E\left(\left(X_t^{(w)} - E(X_t^{(w)})\right)^4\right)}{E\left(\left(X_t^{(w)} - E(X_t^{(w)})\right)^2\right)^2} = 3\left(1 + \frac{\alpha_w^2 + 4\beta_w^2}{\delta_w t\alpha_w^2\sqrt{\alpha_w^2 - \beta_w^2}}\right).$$

In order to estimate all these parameters, we estimate the parameters $(\alpha_i, \beta_i, \delta_i)$ for each asset maximizing the log-likelihood function

$$L(\alpha_i, \beta_i, \delta_i) = \sum_{k=1}^{n} \log\left(f_{NIG}(y_k; \alpha_i, \beta_i, \delta_i)\right), \; i=1,\ldots,N,$$

where f_{NIG} is the density of NIG process given by (1), y_k is the *k*-th observation of *i*-th asset, and *n* is the sample size. Given the set of parameters $\{(\alpha_i, \beta_i, \delta_i)\}_{i=1}^N$, we compute the values $b_i = \delta_i\sqrt{\alpha_i^2 - \beta_i^2}$ and we take its empirical mean as estimate of the parameter $\hat{b} = \frac{1}{N}\sum_{i=1}^N b_i$. Given $\hat{b}$, we estimate again $(\alpha_i, \beta_i, \delta_i)$ for each asset maximizing the log-likelihood function $L(\alpha_i, \beta_i, \delta_i)$ subject to $\delta_i\sqrt{\alpha_i^2 - \beta_i^2} = \hat{b}$. Thus, we consider a

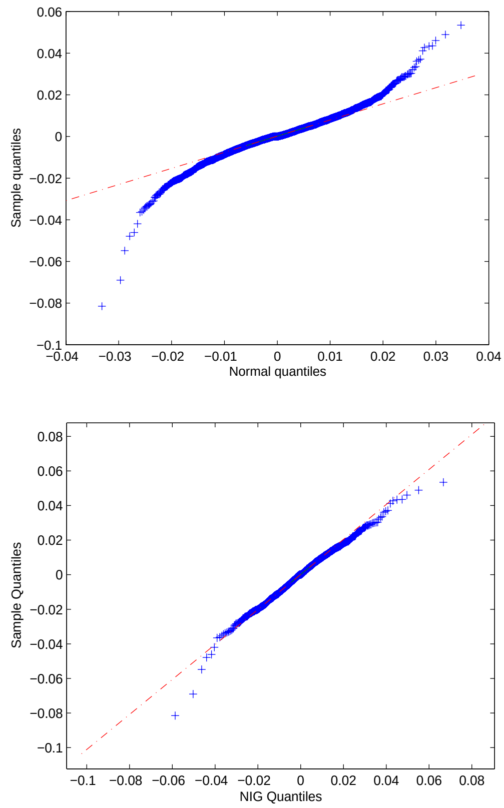

Fig. 1. *QQ-plots versus NIG and Gaussian approximation of Down Jones Composite 65.*

multivariate NIG process where we have not a unique value α for all the components of the vector (in this sense we get a generalization of the classic MNIG process). Since $\delta_i = \sigma_{ii}$ then we have to estimate the correlation matrix of the conditional Gaussian vector $\mathbf{X}_t / Z_t$. Observe that the joint density function of the *i*-th and *j*-th assets is given by

$$f_{ij}(y^i, y^j; \beta_i, \delta_i, \beta_j, \delta_j, \rho_{ij}) = \int_0^\infty f_{\mathbf{N_2}}\left(y^i, y^j; \eta, \Sigma_{NIG}\right) f_{\mathbf{IG}}(u;1,b)\,du,$$

where $f_{\mathbf{IG}}$ is the density function of the Inverse Gaussian distribution with parameters $a=1$, $\hat{b}$ and $f_{\mathbf{N_2}}$ is the joint density function of the 2-dimensional Gaussian distribution with mean $\eta=\left(\beta_i\delta_i^2u,\beta_j\delta_j^2u\right)$ and covariance matrix

$$\Sigma_{NIG}=\begin{bmatrix}\delta_i^2u & \delta_i\delta_j\rho_{ij}u\\ \delta_i\delta_j\rho_{ij}u & \delta_j^2u\end{bmatrix}.$$

Therefore, for each couple (i,j) of assets we estimate ρ_{ij} maximizing the log-likelihood function

$$L(\rho_{ij})=\sum_{k=1}^{n}\log\left(f_{ij}(y_k^i,y_k^j;\beta_i,\delta_i,\beta_j,\delta_j,\rho_{ij})\right).$$

Next we estimate all parameters of the NIG model using the daily returns of five indexes (Down Jones Composite 65 (DJC65), Down Jones Industrials (DJI), Down Jones Utilities (DJU), S&P 500 Composite and S&P 100) observed during the period 04/10/1992 - 12/31/2005.

Table 3: Maximum Likelihood Estimates on daily basis under the NIG model.

	α	β	δ	b	D_n
DJC65	86.4431	3.61882	0.0077	0.66503	0.0337
DJI	81.8854	3.3581	0.008128	0.66503	0.0325
DJU	80.5076	1.85824	0.008263	0.66503	0.0424
S&P500	81.2785	3.1116	0.008188	0.66503	0.032
S&P100	77.6941	2.77099	0.008565	0.66503	0.0338
	DJC65	**DJI**	**DJU**	**S&P500**	**&P100**
DJC65	1	0.9416	0.6061	0.9077	0.8907
DJI	0.9416	1	0.4858	0.9366	0.9373
DJU	0.6061	0.4858	1	0.5136	0.4804
S&P500	0.9077	0.9366	0.5136	1	0.9864
S&P100	0.8907	0.9373	0.4804	0.9864	1

The lower part of Table 3 shows the estimate of the correlation matrix. The upper part of Table 3 reports the estimates of the parameters $(\alpha_i,\beta_i,\delta_i)$ and the common parameter b. While the last column exhibits the Kolomogorov-Smirnov distance $D_n=\sup_{-\infty<x<\infty}|F_{\mathbf{NIG}}(x)-F_n(x)|$.

From Kolmogorov-Smirnov test we deduce that the NIG distributional assumption presents a relevant improvement with respect to the normality that is $D_n \ll 0.4$ for all assets. This result is confirmed by Figure 1 that displays QQplots of Down Jones Composite 65 sample data versus NIG and Gaussian distributions. From this analysis we deduce that the NIG distribution describes better the sample data than the Gaussian one in particular on the tails.

Variance-Gamma Processes. When Z_t follows a Gamma process (i.e., at time t=1 $Z_1 \sim \mathbf{G}\left(\frac{1}{\nu},\frac{1}{\nu}\right)$), then the log-return of the i-th asset follows a Variance-Gamma process with parameters μ_i, σ_{ii} and ν, (i.e., $X_1^{(i)} \sim \mathbf{VG}(\mu_i,\sigma_i,\nu)$). Analogously, the portfolio (7) follows a Variance-Gamma process with parameters $\mu_w=\mathbf{w\mu}$, $\sigma_w=\sqrt{\mathbf{wQw'}}$ and ν. Thus, mean, variance, skewness and kurtosis of the portfolio $X_t^{(w)}$ are given by:

$$E\left(X_t^{(w)}\right)=t\mu_w,$$

$$\mathbf{Variance}\left(X_t^{(w)}\right)=\sigma_w^2t+\nu\mu_w^2t,$$

$$\mathbf{Sk}\left(X_t^{(w)}\right)=\mu_w\nu\left(3\sigma_w^2+2\nu\mu_w^2\right)/\left(\sqrt{t}(\sigma_w^2+\nu\mu_w^2)^{3/2}\right),$$

$$\mathbf{Ku}\left(X_t^{(w)}\right)=3\left(1+2\nu/t-\nu\sigma_w^4/\left(t(\sigma_w^2+\nu\mu_w^2)^2\right)\right).$$

As for the NIG process, in order to estimate all these parameters, we estimate the parameters $(\mu_i,\sigma_{ii},\nu_i)$ for each asset maximizing the log-likelihood function

$$L(\mu_i,\sigma_{ii},\nu_i)=\sum_{k=1}^{n}\log\left(f_{VG}(y_k;\mu_i,\sigma_{ii},\nu_i)\right),\ i=1,\ldots,N,$$

where f_{VG} is the density of Variance-Gamma process given by (2), y_k is the k-th observation of i-th asset, and n is the sample dimension. Given the set of estimates $\{(\hat{\mu}_i,\hat{\sigma}_i,\hat{\nu}_i)\}_{i=1}^N$ we take as estimate of ν its mean $\hat{\nu}=\frac{1}{N}\sum_1^N\nu_i$. Then, for each asset we estimate the parameters $\hat{\mu}_i$ and $\hat{\sigma}_i$ maximizing the log-likelihood $L(\mu_i,\sigma_i)=\sum_{k=1}^{n}\log\left(f_{\mathbf{VG}}(y_k;\mu_i,\sigma_i,\hat{\nu})\right)$. Finally, for each couple (i,j) of assets, we estimate the correlation coefficient ρ_{ij} maximizing the log-likelihood function:

$$L(\rho_{ij})=\sum_{k=1}^{n}\log\left(f_{ij}(y_k^i,y_k^j;\mu_i,\sigma_{ii},\mu_j,\sigma_{jj},\rho_{ij})\right),$$

Table 4: Maximum Likelihood Estimates on daily basis under the VG model.

	μ	σ	ν	D_n	
DJC65	0.000321	0.009261	0.96608	0.0318	
DJI	0.000334	0.009819	0.96608	0.0308	
DJU	0.000192	0.010054	0.96608	0.0401	
S&P500	0.000314	0.009937	0.96608	0.0331	
S&P100	0.00031	0.010399	0.96608	0.032	
	DJC65	**DJI**	**DJU**	**S&P500**	**S&P100**
DJC65	1	0.9434	0.606	0.9079	0.8911
DJI	0.9434	1	0.4824	0.9371	0.9374
DJU	0.606	0.4824	1	0.5068	0.4725
S&P500	0.9079	0.9371	0.5068	1	0.9868
S&P100	0.8911	0.9374	0.4725	0.9868	1

where

$$f_{ij}(y^i, y^j; \mu_i, \sigma_i, \mu_j, \sigma_j, \rho_{ij}) =$$
$$= \int_0^\infty f_{\mathbf{N_2}}\left(y^i, y^j; \theta, \Sigma_{VG} u\right) f_{\mathbf{G}}(u; 1/\nu, 1/\nu) du,$$

$f_{\mathbf{G}}$ is the density function of the Gamma distribution with parameters $a = b = 1/\hat{\nu}$ and $f_{\mathbf{N_2}}$ is the density function of the 2-dimensional normal distribution with mean $\theta = \left(\mu_i u, \mu_j u\right)$ and covariance matrix

$$\Sigma_{VG} = \begin{bmatrix} \sigma_{ii}^2 u & \sigma_{ii}\sigma_{jj}\rho_{ij} u \\ \sigma_{ii}\sigma_{jj}\rho_{ij} u & \sigma_{jj}^2 u \end{bmatrix}.$$

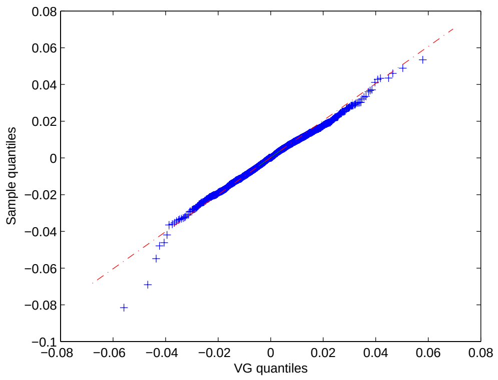

Fig. 2. *QQ-plot Variance-Gamma approximation of Down Jones Composite 65.*

Next we estimate all parameters of the VG model using the same daily returns previously introduced. The upper part of Table 4 exhibits the estimates (on daily basis) of the parameters $(\mu_i, \sigma_{ii}, \nu)$. The last column shows the Kolmogorov-Smirnov distances, which are quite similar to those of the NIG model. The lower part of Table 4 shows the estimate of the correlation matrix. Figure 2 displays QQplot Variance-Gamma distribution of Down Jones Composite 65 sample data that can be compared with analogous Gaussian QQplot of Figure 1. Thus, even the Variance-Gamma process provides a better distributional approximation with respect to the Brownian Motion since it takes into account heavier tails.

3.1 Ex-post comparison among optimal portfolios obtained under different Lévy processes

Consider the problem to select a portfolio among the previous five indexes (Down Jones Composite 65, Down Jones Industrials, Down Jones Utilities, S&P 500 Composite and S&P 100) assuming that the investor has a temporal horizon equal to one month. Suppose the investor decides to invest his money (1000 USD – his initial wealth), in the portfolio that maximizes the mean-Value at Risk ratio (see Favre and Galeano (2002), Biglova et al. (2004)):

$$\frac{E\left(X_{21}^{(w)} - r_f\right)}{\mathbf{VaR}_{0.01}(X_{21}^{(w)} - r_f)},$$

where $r_f = 0.3884\%$ is the 1-month log-return of LIBOR on 12/31/2005, $X_{21}^{(w)}$ is the portfolio of monthly log-returns (i.e., $X_t^{(w)}$ valued at time $t = 21$ days), and the Value at Risk $\mathbf{VaR}_{0.01}$ of continuous random variable $X_{21}^{(w)} - r_f$, is the opposite of the 1% quantile. We assume no short sales are allowed, that is $w_i \geq 0$, $i = 1,\ldots,5$, and $\sum_{i=1}^5 w_i = 1$. Thus, we remark that the problem is well posed, since $\mathrm{VaR}_{0.01}$ of every portfolio is positive. In order to take into account skewness (generally different from zero) and kurtosis we approximate $\mathbf{VaR}_{0.01}\left(X_{21}^{(w)} - r_f\right)$ with the Iaquinta et al.'s approximation (see Iaquinta et al. (2007)). Therefore,

$$\mathbf{VaR}_{0.01}\left(X_{21}^{(w)} - r_f\right) = -\left(E\left(X_{21}^{(w)}\right) + h_w\sqrt{\mathbf{Variance}\left(X_{21}^{(w)}\right)} - r_f\right), \quad (8)$$

where

$$h_w = \left(\frac{\mathbf{Ku}\left(X_{21}^{(w)}\right)-1}{2\mathbf{Sk}\left(X_{21}^{(w)}\right)}\right) - f(X_{21}^{(w)})\frac{1}{2}\left(\left(\frac{\mathbf{Ku}\left(X_{21}^{(w)}\right)-1}{\mathbf{Sk}\left(X_{21}^{(w)}\right)}\right)^2 + 4 + f(X_{21}^{(w)})\frac{4p_{0.99}\sqrt{\left(\mathbf{Ku}\left(X_{21}^{(w)}\right)-1-\left(\mathbf{Sk}\left(X_{21}^{(w)}\right)\right)^2\right)\left(\mathbf{Ku}\left(X_{21}^{(w)}\right)-1\right)}}{\left|\mathbf{Sk}\left(X_{21}^{(w)}\right)\right|}\right)^{1/2},$$

$$f(X_{21}^{(w)}) = \begin{cases} -1 & if \ \mathbf{Sk}\left(X_{21}^{(w)}\right) < 0 \ and \ d\left(X_{21}^{(w)}\right) \geq 0 \\ 1 & otherwise \end{cases},$$

$$d\left(X_{21}^{(w)}\right) = \left(\frac{\mathbf{Ku}\left(X_{21}^{(w)}\right)-1}{\mathbf{Sk}\left(X_{21}^{(w)}\right)}\right)^2 + 4 - \frac{4p_{0.99}\sqrt{\left(\mathbf{Ku}\left(X_{21}^{(w)}\right)-1-\left(\mathbf{Sk}\left(X_{21}^{(w)}\right)\right)^2\right)\left(\mathbf{Ku}\left(X_{21}^{(w)}\right)-1\right)}}{\left|\mathbf{Sk}\left(X_{21}^{(w)}\right)\right|}$$

and $p_{0.99}$ is the 99% quantile of the standard normal distribution. We add in passing that one could estimate $\mathbf{VaR}_{0.01}$ using the Cornish-Fisher expansion (as suggested by Favre and Galeano (2002)), in this case h_w is given by:

$$h_w = p_{0.01} + \frac{1}{6}\left(p_{0.01}^2 - 1\right)\cdot\mathbf{Sk}\left(X_{21}^{(w)}\right) + \frac{1}{24}\left(p_{0.01}^3 - 3p_{0.01}\right)\cdot\left(\mathbf{Ku}\left(X_{21}^{(w)}\right) - 3\right),$$

and $p_{0.01}$ is the 1% quantile of the standard normal distribution. Taking this into account, we solve the optimization problem

$$\max_w \frac{\mathbf{E}\left(X_{21}^{(w)} - r_f\right)}{\mathbf{VaR}_{0.01}(X_{21}^{(w)} - r_f)}$$
$$\text{s.t.} \qquad (9)$$
$$\sum_{i=1}^{5} w_i = 1;\ w_i \ge 0;\ i = 1,...,5$$

under the three possible distributional assumptions:

1) Normal Inverse Gaussian $X_{21}^{(w)} \sim \mathbf{NIG}(\alpha_w, \beta_w, 21\delta_w)$;

2) Variance-Gamma $X_{21}^{(w)} \sim \mathbf{VG}(21\mu_w, \sqrt{21}\sigma_w, v/21)$;

3) Brownian Motion $X_{21}^{(w)} \sim \mathbf{N}(21\mu_w, 21\sigma_w^2)$, where $\mu_w = \mathbf{w\mu}$, $\sigma_w = \sqrt{\mathbf{wQw'}}$.

As solution of problem (9) we obtain the following optimal portfolio weights:

$$\mathbf{w}_{(NIG)} = (0.3265, 0.6735, 0, 0, 0);$$
$$\mathbf{w}_{(VG)} = (0.2307, 0.7693, 0, 0, 0);$$
$$\mathbf{w}_{(BM)} = (0.3189, 0.6811, 0, 0, 0).$$

Table 5: Monthly evolutions of W(NIG), W(VG) and W(BM).

	NIG	VG	BM
01/01/06	1000	1000	1000
01/02/06	**1022.98**	1022.71	1022.95
01/03/06	**1035.14**	1034.03	1035.06
01/04/06	**1041.41**	1040.97	1041.38
01/05/06	**1058.52**	1058.49	1058.52
01/06/06	**1056.93**	1055.09	1056.79
01/07/06	**1061.37**	1057.33	1061.05
01/08/06	**1040.21**	1039.61	1040.16
01/09/06	1067.3	**1068.02**	1067.36
01/10/06	1084.39	**1085.74**	1084.5
01/11/06	1121.98	**1122.18**	1122
01/12/06	1136.25	**1136.73**	1136.29
01/01/07	1153.31	**1156.15**	1153.54

The three optimal portfolios are composed by the same assets. In particular under BM and NIG distributional assumptions the portfolio composition is almost the same. While, the VG model presents a significant difference in the portfolio composition with respect to the other two processes. In order to value the impact of these choices we consider an investor who recalibrates the portfolio every month during the year 2006 such that the percentages in the portfolio composition remain the same under each distributional assumption.

Table 5 reports the ex-post monthly evolutions of $w_{(NIG)}$, $w_{(VG)}$, and $w_{(BM)}$ supposing the investor's initial wealth is 1000 USD. Observe that there are not significant differences among the final wealth obtained with the NIG, VG processes and Brownian Motion one. However, both alternative processes (NIG, VG) present a better performance in different periods of the year, even if during the 2006 the market was growing and the asset prices haven't shown big jumps.

4. A comparison among dynamic portfolio strategies

In this section, we deal the dynamic portfolio selection problem among N+1 assets: N are risky assets and the (N+1)-th is risk free. As in equation (6) we assume that the vector of log-return risky assets follows a Lévy subordinated process $\mathbf{X}_t = \boldsymbol{\mu} Z_t + \mathbf{Q}^{1/2} W_{Z_t}^{(N)}$. In particular, we distinguish between portfolio selection problems with unlimited short sales and portfolio selection considering some institutional constrains (limited short sales, and transaction costs).

4.1 Optimal portfolio strategies when unlimited short sales are allowed

Suppose an investor has a temporal horizon t_T and he recalibrates its portfolio T times at some intermediate date, say $t = t_0, \dots, t_{T-1}$ (where $t_0 = 0$). Since Lévy processes have independent and stationary increments the distribution of the random vector of log-returns on the period $(t_j, t_{j+1}]$ (i.e., $\mathbf{X}_{t_{j+1}} - \mathbf{X}_{t_j}$) is the same of $\mathbf{X}_{t_{j+1}-t_j} = [X_{t_{j+1}-t_j}^{(1)}, ..., X_{t_{j+1}-t_j}^{(N)}]'$. Considering that log-returns represent a good approximation of returns when $t_{j+1} - t_j$ is little enough, we assume that $\max_{j=0,...,T-1}\left(t_{j+1} - t_j\right)$ is less or equal than one month and we use $\mathbf{Y}_{t_j} := \mathbf{X}_{t_{j+1}} - \mathbf{X}_{t_j} = [Y_{1,t_j}, ..., Y_{N,t_j}]'$ to estimate the vector of returns on the period $(t_j, t_{j+1}]$. Suppose the deterministic variable r_{0,t_j} represents the return on the period $(t_j, t_{j+1}]$ of the risky-free asset, x_{i,t_j} the amount invested at time t_j in the i-th risky asset, and x_{0,t_j} the amount invested at time t_j in the risky-free asset. Then the investor's wealth at time t_{k+1} is given by:

$$\mathrm{W}_{t_{k+1}} = \sum_{i=0}^{N} x_{i,t_{k+1}} = \sum_{i=0}^{N} x_{i,t_k}(1 + Y_{i,t_k}) = (1 + r_{0,t_k})\mathrm{W}_{t_k} + \mathbf{x}_{t_k}\mathbf{P}_{t_k}, \quad (10)$$

where $\mathbf{x}_{t_k} = [x_{1,t_k}, \dots, x_{N,t_k}]$, $\mathbf{P}_{t_k} = [P_{1,t_k}, \dots, P_{N,t_k}]'$ is the vector of excess returns $P_{i,t_k} = Y_{i,t_k} - r_{0,t_k}$. Thus, the final wealth is given by:

$$\mathrm{W}_{t_T} = \mathrm{W}_0 \prod_{k=0}^{T-1}(1 + r_{0,t_k}) + \sum_{i=0}^{T-2} \mathbf{x}_{t_i}\mathbf{P}_{t_i} \prod_{k=i+1}^{T-1}(1 + r_{0,t_k}) + \mathbf{x}_{t_{T-1}}\mathbf{P}_{T-1},$$

where the initial wealth $\mathrm{W}_0 = \sum_{i=0}^{N} x_{i,0}$ is known. Assume that the amounts $\mathbf{x}_{t_j} = [x_{1,t_j}, \dots, x_{N,t_j}]$ are deterministic variables, whilst the amount invested in the risky-free asset is the random variable $x_{0,t_j} = W_{t_j} - \mathbf{x}_{t_j}\mathbf{e}$, where $\mathbf{e} = [1,\dots,1]'$. Therefore, if we want to select the optimal portfolio strategies that solve the mean-variance problem:

$$\begin{cases} \min_{\mathbf{x}_{t_0},\dots,\mathbf{x}_{t_{T-1}}} \mathbf{Variance}[W_{t_T}] \\ \mathbf{s.t.} \\ E[W_{t_T}] = m \end{cases},$$

we can use the closed form solutions determined by Ortobelli et al. (2004). These solutions for Lévy subordinated processes are given by:

$$\mathbf{x}'_{t_k} = \frac{m - W_0 B_0}{B_{k+1}\sum_{j=0}^{T-1} E(\mathbf{P}_{t_j})'\mathbf{Q}_{t_j}^{-1}E(\mathbf{P}_{t_j})}\mathbf{Q}_{t_k}^{-1}E(\mathbf{P}_{t_k})$$

$$k = 0,\dots,T-2$$

$$\mathbf{x}'_{t_{T-1}} = \frac{m - W_0 B_0}{\sum_{k=0}^{T-1} E(\mathbf{P}_{t_k})'\mathbf{Q}_{t_k}^{-1}E(\mathbf{P}_{t_k})}\mathbf{Q}_{t_{T-1}}^{-1} E(\mathbf{P}_{t_{T-1}}),$$

where $B_k = \prod_{j=k}^{T-1}(1 + r_{0,t_j})$ and the components of the matrix $\mathbf{Q}_{t_k} = [q_{ij,t_k}]$, ($k$=0,…,$T$-1), are $q_{ij,t_k} = \mathbf{Cov}(X^{(i)}_{t_{k+1}-t_k}, X^{(j)}_{t_{k+1}-t_k})$. The optimal wealth invested in the riskless asset at time $t_0 = 0$ is the deterministic quantity $x_{0,0} = W_0 - \mathbf{x}_{t_0}\mathbf{e}$, while at time t_j it is given by the random variable $x_{0,t_j} = W_{t_j} - \mathbf{x}_{t_j}\mathbf{e}$, where W_{t_j} is formulated in equation (10). Observe that the covariance q_{ij,t_k} among components of the vector $\mathbf{X}_{t_{j+1}-t_j} = \boldsymbol{\mu} Z_{t_{j+1}-t_j} + \mathbf{Q}^{1/2} W^{(N)}_{Z_{t_{j+1}-t_j}}$ is given by

$$q_{ij,t_k} = \sigma_{ij}^2 E(Z_{t_{k+1}-t_k}) + \mu_i\mu_j \mathbf{Variance}[Z_{t_{k+1}-t_k}],$$

where $\sigma_{ij}^2 = \sigma_{ii}\sigma_{jj}\rho_{ij}$ are the components of matrix $\mathbf{Q} = \left[\sigma_{ij}^2\right]$ (see, among others, Cont and Tankov (2004)). So, for example, in the case the vector of log-returns $\mathbf{X}_t$ follows a NIG process then:

$$q_{ij,t_k} = \mathbf{Cov}(X^{(i)}_{t_{k+1}-t_k}, X^{(j)}_{t_{k+1}-t_k}) = \delta_i\delta_j\rho_{ij}E(I_{t_{k+1}-t_k}) +$$
$$+\beta_i\beta_j\delta_i^2\delta_j^2 \mathbf{Variance}[I_{t_{k+1}-t_k}] = \frac{\delta_i\delta_j\rho_{ij}}{b}(t_{k+1}-t_k) + \frac{\beta_i\beta_j\delta_i^2\delta_j^2}{b^3}(t_{k+1}-t_k).$$

Instead, if $\mathbf{X}_t$ follows a Variance-Gamma process then:

$$q_{ij,t_k} = \mathbf{Cov}(X^{(i)}_{t_{k+1}-t_k}, X^{(j)}_{t_{k+1}-t_k}) = \sigma_{ii}\sigma_{jj}\rho_{ij}E(Z_{t_{k+1}-t_k}) + \mu_i\mu_j \mathbf{Variance}[Z_{t_{k+1}-t_k}] =$$
$$= \sigma_{ii}\sigma_{jj}\rho_{ij}(t_{k+1}-t_k) + v\mu_i\mu_j(t_{k+1}-t_k).$$

However, the mean-variance approach does not consider the presence of skewness and kurtosis in the asset returns. For this reason in the following we will discuss an ex-post comparison among portfolio strategies obtained by maximizing the mean-VaR ratio when institutional restriction are present in the market.

4.2 Ex-post comparison among optimal portfolio strategies with transaction costs and no short sales

Let us compare dynamic strategies with constant and proportional transaction costs of $K = 0.05\%$ when short sales are not permitted. Assume an investor has an initial wealth of 1000 USD and he decides to invest this money in the portfolio that maximizes the mean-VaR ratio recalibrating it every month. As for the previous empirical analysis we consider five indexes (Down Jones Composite 65 (DJC65), Down Jones Industrials (DJI), Down Jones Utilities (DJU), S&P 500 Composite and S&P 100) and a monthly riskless asset with return $r_f = 0.3884\%$. Since we want to compare the ex-post sample paths of the investor's wealth under different distributional assumptions, then we follow the same algorithm proposed by Biglova et al. (2004), Ortobelli et al. (2004), Leccadito et al. (2007). That is we first consider an initial wealth $W_0 =$ (1000 USD minus the transaction costs 0.05%) and in the ex-post analysis we calibrate the portfolio 12 times. Thus, once we have chosen a distributional assumption, after k periods, the main steps to compute the ex-post final wealth in the mean-VaR context are the following:

Step 1 At the k-th period ($k = 0,1,\dots,11$), we determine the market portfolio $w^{(k)}$ that maximizes the mean-VaR ratio, i.e., it is the solution of the following optimization problem:

$$\max_{w^{(k)}} \frac{E\left(X_{21}^{(w^{(k)})} - r_f - t.c.(k)\right)}{\mathbf{VaR_{0.01}}(X_{21}^{(w^{(k)})} - r_f) + t.c.(k)},$$
$$\text{s.t.}$$
$$\mathbf{w}^{(k)}\mathbf{e} = 1,$$
$$w_i^{(k)} \geq 0;\ \ i = 1,\dots,N$$

where $\mathbf{VaR_{0.01}}(X_{21}^{(w^{(1)})} - r_f)$ is given by equation (8), the transaction costs are given by:

$$t.c.(k) = \begin{cases} K\sum_{i=1}^{N}\left| w_i^{(k)} - \frac{w_i^{(k-1)}(1+r_i^{(k-1)})}{\sum_{i=1}^{N} w_i^{(k-1)}(1+r_i^{(k-1)})}\right| & if\ \ k > 1 \\ K = 0.05\% & if\ \ k = 0 \end{cases}$$

and $r_i^{(k-1)}$ is the observed i-th monthly return valued on the period $[t_{k-1}, t_k]$.

Step 2 We value the ex-post final wealth at the k-th period is given by:

$$W_k = W_{k-1}\left(\sum_{i=1}^{N} w_i^{(k-1)}(1 + r_i^{(k-1)}) - t.c.(k)\right).$$

Step 3 We repeat steps 1 and 2 for each distributional hypotheses.

Table 6: Optimal portfolio composition $\mathbf{w}^{(0)}$ when we consider constant and proportional transaction costs under the three distributional hypotheses (NIG, VG, MB).

	DJC65	DJI	DJU	S&P500	S&P100
NIG	0.1967	0.8033	0	0	0
VG	0.0673	0.9327	0	0	0
BM	0.1386	0.8614	0	0	0

The use of the transaction costs in the portfolio optimization has implied that there are not transaction costs at the times of the calibration. That is, the investor chooses his first portfolio $\mathbf{w}^{(0)}$ and the percentage of the final wealth invested in each asset is equal to the ex-post percentage of the first portfolio during each recalibration. In Table 6, we report the weights of the portfolio $\mathbf{w}^{(0)}$ under the three different distributional assumptions (NIG, VG, MB).

Table 7: Monthly evolutions of the final wealth process when we consider constant and proportional transaction costs under the three distributional hypotheses (NIG, VG, MB).

	NIG	VG	BM
01/01/06	1000	1000	1000
01/02/06	**1022.12**	1021.75	1021.95
01/03/06	**1033.13**	1031.63	1032.46
01/04/06	**1040.29**	1039.69	1040.02
01/05/06	**1057.94**	1057.89	1057.92
01/06/06	**1053.91**	1051.42	1052.79
01/07/06	**1055.44**	1049.97	1052.98
01/08/06	**1038.72**	1037.97	1038.38
01/09/06	1067.57	**1068.59**	1068.03
01/10/06	1085.51	**1087.39**	1086.35
01/11/06	1121.48	**1121.83**	1121.64
01/12/06	1136.12	**1136.84**	1136.45
01/01/07	1155.82	**1159.71**	1157.57

The assets that appears in the optimal portfolio compositions are the same (Down Jones Composite 65, Down Jones Industrials) and we observe a greater difference with respect to the previous comparison when the transaction costs are not considered. Table 7 exhibits the ex-post final wealth sample paths under the three distributional assumptions. As for the previous comparison we observe a better performance of the VG and NIG processes in different periods of the year.

5. Conclusions

In this paper, we study the problem to select optimal portfolios when the assets follow a subordinated Brownian Motion. We discuss the portfolio optimization problem by the point of view of investors with exponential utility function and investors that maximize the mean-Value at Risk ratio. Therefore, we propose two models that take into account the heavier behavior of log-return distribution tails. The empirical comparison shows a greater performance of two alternative subordinated Lévy processes.

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Alessandro Staino Ph.D. student at University of Bergamo (Italy) in "Computational methods for financial and economic forecasting and decisions" He is currently visiting at University of Calabria (Italy). His research interests include Lévy processes, mathematical finance and computational finance.

Sergio Ortobelli Lozza Associate Professor at University of Bergamo (Italy). Currently he works in "Lorenzo Mascheroni" Department MSIA. His research interests include mathematical finance, stochastic processes, portfolio theory, and computational finance.

Ivar Massabò Professor at Manage-ment Science Department, University of Calabria (Italy). While his first researches were mainly addressed on topology and advanced real analysis, currently he works on mathematical finance, risk management theory, option pricing modeling, and actuarial mathematics.