A Study of Connectivity Index of Graph Relevant to Ad Hoc Networks

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Summary

Connectivity is one of the most fundamental aspects of MANETs. The fundamental application of a network is to facilitate the exchange of data among its nodes. This paper introduces connectivity Index (CI) as one of the parameter to study MANETs

Key words:

Connectivity Index, graph, MANETs, graph, connectivity, Randić index, network topology.

1. Introduction

Mobile Ad-hoc Networks (MANETs) are decentralized, self-organizing networks capable of forming a communication network without relying on any fixed infrastructure. Each node in an ad-hoc network is equipped with a radio transmitter and receiver, which allow it to communicate with other nodes over wireless channels. All nodes can function, if needed, as relay stations for data packets to be routed to their final destination. In other words, ad-hoc networks allow for multi-hop transmission of data between nodes outside the direct radio reach of each other. A special kind of ad-hoc network is the sensor network where the nodes forming the network do not or rarely moves.

Salient Features of mobile ad-hoc networks [1] are, 1) Use of ad-hoc networks can increase mobility and flexibility, as ad-hoc networks can be brought up and brought down in a very short time. 2) Ad-hoc networks can be more economical in some cases, as they eliminate fixed infrastructure costs. 3) Ad-hoc networks can be more robust than conventional wireless networks because of their non-hierarchical distributed control and management mechanisms. 4) Because of multi-hop support in ad-hoc networks, communication beyond the Line of Sight (LOS) is possible at high frequencies. 5. Multi-hop ad-hoc networks can reduce the power consumption of wireless devices, because more transmission power is required for sending a signal over any distance in one long hop than in multiple shorter hops. The gain in transmission power consumption is proportional to the number of hops made [2]. Because of short communication links (multi-hop node-to-node communication instead of long-distance node to central base station communication), radio emission levels can be kept low. This reduces interference levels, increases spectrum reuse efficiency, and makes it possible to use unlicensed unregulated frequency.

Mobile ad-hoc networks have been widely used for tactical military communication systems. The United States Defense Advanced Research Project Agency (DARPA) has sponsored projects such as the Near-Term Digital Radio (NTDR) system to control infantry, armor, and artillery units in battle field scenarios where no communication infrastructure exists.

2. Related Work

Graph Theory concepts play an important role in analyzing these fundamental issues like connectivity, scalability, routing and topology control as well as the problems of the ad-hoc network can be easily expressed mathematically, because a network can be modeled as a graph. That is, there exists bijection between network topology and graph. Graphs can be represented as matrices. The study of matrix is analogues to study of the network that can be automated through algorithms. Connectivity Index (CI) is one such concept that can be used to analyze the MANET. A lot of work has been done related to CI in analyzing the characteristics of the molecules of the various chemical compounds [3-15]. In this paper an attempt is made to utilize this concept particularly on random graphs, which is used as a tool to simulate MANETs.

This paper emphasizes on the study of CI on random graphs to simulate MANETs and is organized as follows: in part 3, a brief introduction of graph theory is made, since some of the important terminologies of the graph theory are often used in other sections, which will be easy to understand, once the definitions are known. Part 4 presents the definition of CI and two important lemmas derived considering Erdős and Rényi Model. We claim that this is a novel approach to study the connectivity of

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the ad-hoc networks and an attempt is made to explore some of the connectivity issues of ad-hoc networks. Part 5 presents a simulation study of the variation of CI of MANET using MATLAB and also the variation with respect to spectral parameters of the MANET. Conclusions are given in Part 6.

3. Basic Definitions of Graph Theory

The study of graphs is known as graph theory. The properties that holds good for a graph will also hold good for a network. A graph *G* is a triplet consists of Vertex set $V(G)$ Edge set $E(G)$ and a relation that associates with each edge, two vertices. An edge between two nodes *i* and *j* is represented as (i, j) and $E(G) \subseteq \{(i, j) \mid \forall\, i, j \in V \text{ and } (i, j) = (j, i)\}$. Graph *G* is denoted by $G(V, E)$. The two variants of the graph are number of vertices and number of edges. An edge is an association between two vertices. With respect to the network, a vertex is a node and an edge is a link between two nodes. Two vertices are said to be *adjacent* to each other, if there exist an edge between them. Two edges are said to be *adjacent* to each other, if one of the end vertex of the edges is common. If each edge of a graph is associated with some specific value (weight), such graph is said to be *weighted*. If in a graph, for each edge (i, j) is different than (j, i) such a graph is said to be *directed graph*. A *simple graph G* is an un-weighted, un-directed graph containing no *self loops* (Edge in which it's end points is same) and no *parallel edges* (Two edges are said to be parallel, if their end points are common). Through out this paper a graph means a simple graph. The number of edges associated with the vertex of a graph is called degree of a vertex denoted by $(d_G(V)\ or\ d(V))$. The *maximum degree* $\Delta(G)$ of a graph *G* is the largest degree over all vertices; the *minimum degree* $\delta(G)$, the smallest. A graph G is *regular* if and only if $\delta(G) = \Delta(G)$. A graph *G* is said to be *connected*, if for every pair of vertices *u, v* belongs to *G*, there exist a path, otherwise the graph is *disconnected*. A dis-connected graph has a number of components; each component of a graph is a connected graph. In the next section, the different random graph theory models used to model an ad-hoc network are described.

4. Connectivity Index (CI) of MANET

Connectivity Index or Randić index denoted by proposed by Randić in 1975. χ(G) is defined as $\sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u) d_G(v)}}$ where $d_G(x)$ denotes the degree of vertex [4]. This concept can be applied to Random Graphs., which will be useful in analyzing the connectivity, scalability, and routing issues of the ad-hoc networks. One can also study the signal interference versus connectivity index. Randić himself demonstrated the Randić index χ was well correlated with a variety of physico- chemical properties of alkanes, such as boiling points, surface area and solubility in water [30]. The following two lemmas are stated and proved by considering Erdős and Rényi Model random graph model. A brief introduction about this model is also presented.

In this section, let *N* be the number of labeled nodes, *n* be the number of edges and the edge can be formed between two nodes with a probability *p* of a *random graph*.

Erdős *and Rényi Model*: Define a random graph as *N* labeled nodes connected by *n* edges, which are chosen randomly from the $\frac{N(N-1)}{2}$ possible edges. In total there are $\binom{N(N-1)}{n}$ graphs with *N* nodes and *n* edges, forming a probability space in which every realization is equi-probable. The number of edges E_l in the random graph is then a random variable with the expectation. $E[E_l] = p\frac{N(N-1)}{2}$. The random graph of Erdős and Rényi [3] is one of the best studied models of a network [29] because of its simplicity.

Consider Erdős and Rényi Model. We know that there will be an edge between two nodes with a probability p. We redefined the CI for random graphs with p as the probability of forming a link between two nodes. We know that the CI for general graphs is defined as $\chi(G, p) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u) d_G(v)}}$, But for random graphs, $d_G(node)$ depends on probability *p* also.

We will prove the following two Lemmas.

1. The connectivity index of *k*-regular Erdős and Rényi random graph is

$$\binom{n}{2} \frac{1}{\binom{n-1}{k} p^k (1-p)^{n-1-k}} \quad (1)$$

Proof:

Let $G(n,p)$ be a *k*-regular Erdős and Rényi random graph, with probability of an edge between any two nodes is *p*. We know that

$$\chi(G,p) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}} \quad (2)$$

$$P(d_G(v_i)=k) = k\binom{n-1}{k}p^k(1-p)^{n-1-k} \quad (3)$$

where each $v_i \in V(G)$ .Since graph is k-regular, $d_G(v_i)$ is same and its values is $P(d_G(v_i)=k)$. Substituting this in equation (1) and also there are $\frac{n}{2}$ *u-v* pairs in a graph, the equation (1) becomes

$$\chi(G,p) = \sum_{uv\in E(G)} \frac{1}{\sqrt{k\binom{n-1}{k}p^k(1-p)^{n-1-k}\,k\binom{n-1}{k}p^k(1-p)^{n-1-k}}} \quad (4)$$

$$\text{Let } P = \frac{1}{\sqrt{k\binom{n-1}{k}p^k(1-p)^{n-1-k}\,k\binom{n-1}{k}p^k(1-p)^{n-1-k}}} \quad (5)$$

$$\text{Then} \quad \chi(G,p) = \frac{1}{P} + \frac{1}{P} + \ldots\ldots\ldots\ldots \text{ upto } \frac{nk}{2} \text{ times} \quad (6)$$

Then $\chi(G,p) = \left(\frac{nk}{2}\right)\frac{1}{P}$, Substituting P value, $\chi(G)$ is written as

$$\chi(G,p) = \left(\frac{nk}{2}\right) \frac{1}{k\binom{n-1}{k}p^k(1-p)^{n-1-k}} \quad (7)$$

$$\therefore \chi(G,p) = \left(\frac{n}{2}\right) \frac{1}{\binom{n-1}{k}p^k(1-p)^{n-1-k}} \quad (8)$$

Hence, the proof is completed.

2. The connectivity index of *complete* Erdős and Rényi random graph is

$$\left(\frac{n}{2}\right) \quad (9)$$

Proof:

A graph in which every node is associated to all the other nodes of a graph is called *complete graph*. Hence the probability that a node is associated with every other node is equal to $p=1$. Substituting this in equation (9) with $k=n-1$

$$P(d_G(v_i)=n-1) = n-1 \quad (10)$$

$\therefore$ From equations (4) and (10), we get

$$\therefore \quad \chi(G,p) = \sum_{uv\in E(G)} \frac{1}{\sqrt{(n-1)(n-1)}} \quad (11)$$

$$\therefore \quad \chi(G,p) = \frac{1}{(n-1)} + \frac{1}{(n-1)} + \cdots \text{ upto } \frac{n(n-1)}{2} \text{ terms} \quad (12)$$

$$\therefore \quad \chi(G,p) = \frac{\frac{n(n-1)}{2}}{(n-1)} \quad (13)$$

$$\boxed{\therefore \quad \chi(G,p) = \frac{n}{2}} \quad (14)$$

Hence, the proof is completed.

5. Analysis of Connectivity Index: A Simulation Approach

A math-lab simulation is done to analyze the variation of connectivity index with respect to variation in number of links for a network with 100,200,300 and 1000 nodes. The simulation process is carried out with following steps.

1. Initially a minimally connected network with N nodes is simulated. (Network having N-1 links and each node has degree two except node 1 and node N).
2. The connectivity Index of the network is computed for the current network.
3. If the number of links of the network does not reach maximum, that is $\left(\frac{N(N-1)}{2}\right)$, a link is added between two nodes randomly which are not adjacent to each other and step 2 is repeated.
4. After reaching maximum links simulation in the network, the graph with Number of edges versus connectivity index is plotted.

Figure-1 describes the flow chart of the simulation process.

Figure-2 is the result of the above described simulation with process for a MANET with the number of nodes equal to 100 and the maximum number of links simulated is 5000.

Figure-3 is the result of the above described simulation process for a MANET with number of nodes 1000 and the maximum number of links simulated is 499500. Note that behavior of two curves follows the same path and also as proved mathematically in section 4 (lemma-2), the CI is maximum when the network is fully connected, which is equal to $\frac{N}{2}$, that is for N=1000, CI= 500. This is same as what we proved in *lemma 2*. The Links versus CI graph for the simulation results for a network with 100,200,300… 1000 nodes exhibits the similar curve paths.

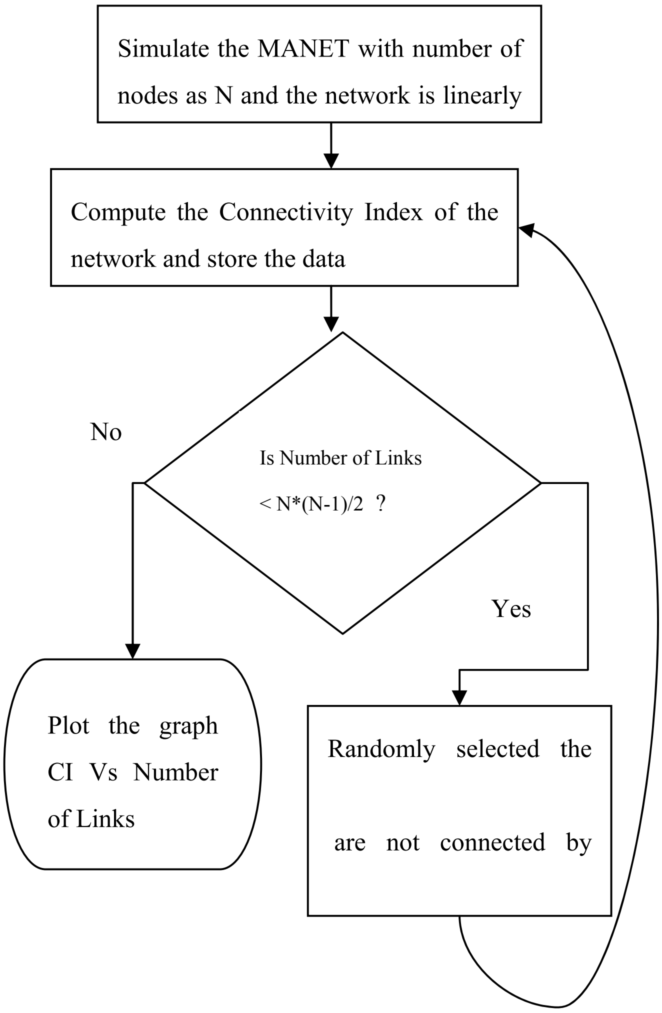

Fig: 1 Flow chart for the simulation process.

6. Conclusions

CI can be used as an important entity in studying the network behavior. It can be effectively used as one of the input parameters for network topology control sparsification and clustering algorithms. The study of signal interference in relation to CI can be of great use to decide whether to add or drop a link, in the direction of reducing the interference. Further work is in progress in these directions.

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7. Appendix

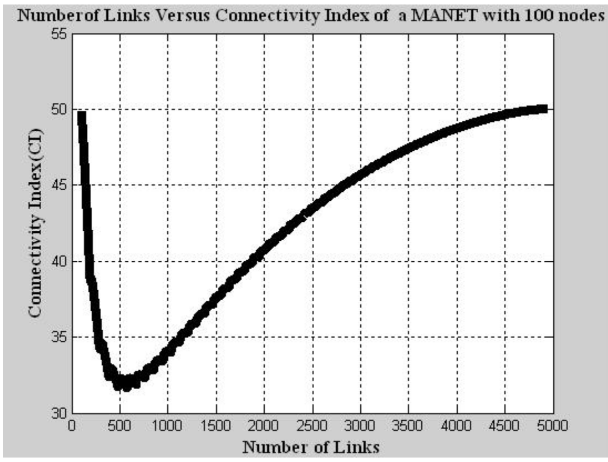

Fig: 2 Number Of Links Vs CI of a network with 100 nodes

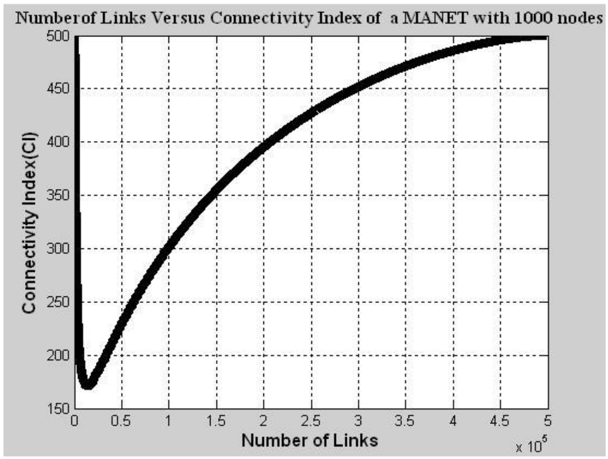

Fig: 3 Number Of Links Vs CI of a network with 1000 nodes.

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