On Order-convexity Of Syntopogenous Structures (X,S,$\leq$)

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Summary
In this paper, fuzzy syntopogenous spaces is introduced. First, the *-increasing and *-decreasing spaces have been studied. Second,order-convexity on syntopogenous structures (X,S,$\leq$) has been defined. Third, the equivalent description of order-convexity has been given. Finally, some important properties of order-convexity have been obtained.
Key words:
Fuzzy topology, Order, Algebra, Convexity

1. Preliminaries

In this paper, L=<L, $\wedge$, $\vee$, '> always denotes a completely distributive lattice with order–reversing involution " ' ". Let 0 be the least element and 1 be the greatest one in L. Suppose X is a nonempty (usual) set, an L-fuzzy set in X is a mapping A: X$\to$L, and L^x will denote the family of all L-fuzzy sets in X. It is clear that L^x=<L^x, $\leq$,$\wedge$,$\vee$,'> is a fuzzy lattice ,which has the least element $\underline{0}$ and the greatest one $\underline{1}$, where $\underline{0}$ (x)=0, $\underline{1}$(x)=1,for any x$\in$ X.

Definition1.1. A binary relation $\ll$ on L^x is called an L-fuzzy semi-topogenous order if it satisfies the following axiom: (1) $\underline{0} \ll \underline{0}$ and $\underline{1} \ll \underline{1}$; (2) A $\ll$ B implies A$\leq$ B; (3) $A_1 \leq A \ll B \leq B_1$ implies $A_1 \ll B_1$. The complement of an L-fuzzy semi-topogenous order $\ll$ is the L-fuzzy semi-topogenous order $\ll^c$ defined by A $\ll^c$ B iff B'$\ll$ A'. An L-fuzzy semi-topogenous order $\ll$ is called: (Ⅰ) symmetrical if $\ll = \ll^c$; (Ⅱ) topogenous if $A_1 \ll B_1$ and $A_2 \ll B_2$ implies $A_1 \wedge A_2 \ll B_1 \wedge B_2$ and $A_1 \vee A_2 \ll B_1 \vee B_2$; (Ⅲ) perfect if Aj $\ll$ Bj , j$\in$J implies $\vee$Aj $\ll$ $\vee$Bj ; (Ⅳ)co-perfect if Aj $\ll$ Bj , j$\in$J implies $\wedge$Aj $\ll$ $\wedge$Bj ; (Ⅴ)biperfect if perfect and co-perfect.

Suppose that $\ll_1, \ll_2$ are L-fuzzy semi-topogenous order on X , we call $\ll_1$ is finer than $\ll_2$ (i.e.$\ll_2$ is coarser than $\ll_1$) if for any A, B$\in L^x$, A $\ll_2$ B implies A $\ll_1$B, denoted by $\ll_2 \leq \ll_1$. For a given L-fuzzy semi-topogenous order $\ll$, we define $\ll^p, \ll^1$ as follows: A $\ll^p$ B iff there exist Ai , i$\in$I, suth that A=$\vee$Ai , for any i$\in$I, Ai $\ll$ B; A $\ll^1$ B iff there exist Bj , j$\in$J suth that B=$\wedge$Bj and A $\ll$ Bj , for each j$\in$J.

Definition1.2. An L-fuzzy syntopogenous structure on X is a nonempty family S of L-fuzzy semi-topogenous order on X having the following two properties:(LFS1) S is directed in the sense that given any two members of S there exist a member of S finer than both;(LFS2) For each $\ll$ in S there exist $\ll_1$ in S such that A $\ll$ B implies the existence of an L-fuzzy set C with A $\ll_1$C $\ll_1$B.

If S is an L-fuzzy syntopogenous structure on X, then the pair (X,S) is called an L-fuzzy syntopogenous space. An L-fuzzy syntopogenous structure S consisting of a single semi-topogenous order is called a topogenous structure and the pair (X,S) is called an L-fuzzy topogenous space. S is called perfect (resp. biperfect) if each member of S is perfect (resp. biperfect). An L-fuzzy syntopogenous structure S_1 is called finer than another one S_2 , if for each $\ll$ in S_2 there exist a member of S_1 finer than $\ll$. In this case we also say that S_2 is coarser than S_1 , denoted by $S_2 \leq S_1$. If S_1 is finer than S_2 and S_2 is finer than S_1, then S_1 ,S_2 are called equivalent, denoted by $S_1 \sim S_2$. To every L-fuzzy syntopogenous structure corresponds an L-fuzzy topology τ (S) given by the interior operator μ^0= sup $\{\rho : \rho \ll \mu$ for some $\ll \in S\}$. If $\{\ll_\alpha : \alpha \in \wedge\}$ is a family of L-fuzzy semi-topogenous order on X then $\ll = \vee_{\alpha \in \wedge} \ll_\alpha$ is the L-fuzzy semi-topogenous order defined by $\mu \ll \rho$ iff $\mu \ll_\alpha \rho$ for some $\alpha \in \wedge$. If S is a L-fuzzy syntopogenous structure, then it is easy to see that $\ll_s = \vee \{\ll : \ll \in S\}$ is an L-fuzzy topogenous order and $\{\ll_s\}$ is an L-fuzzy topogenous structure. Moreover, $\mu \in \tau$ (S) iff $\mu \ll^p_s \mu$. To every fuzzy topology τ on X corresponds a perfect L-fuzzy topogenous structure $S_\tau = \{\ll\}$, where $\mu \ll \rho$ iff there exists $\sigma \in \tau$ with $\mu \leq \sigma \leq \rho$. Moreover , $\tau = \tau$ (S). Conversely, to every perfect L-fuzzy topogenous structure S= $\{\ll\}$ corresponds the L-fuzzy topology $\tau = \tau$ (S), where $\mu \in \tau$ iff $\mu \ll \mu$. To two different L-fuzzy topologies correspond different perfect L-fuzzy topogenous structure[1][2].

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2. *-increasing and *-decreasing spaces

A preorder on X is a binary relation "$\leq$" on X which is reflexive and transitive. preorder on X which is also anti-symmetric is called a partial order or simply an order. By a preordered (resp. an ordered) set we mean a set with a preorder (resp. a partial order) on it

Definition2.1. (Katsaras [4]) Let $(X,\leq)$ be a preorder set, $A\in L^x$ is called: (i) * -increasing if $x\leq y$ implies $A(x)\leq A(y)$; (ii) *-decreasing, if $x\leq y$ implies $A(y)\leq A(x)$; (iii) order-convex, if $y\leq x\leq z$ implies $A(y)\wedge A(z)\leq A(x)$.

Definition2.2. Let $(X,\leq)$ be a preorder set, define mappings p,a,c: $L^x\rightarrow L^x$ as follows: for any $A\in L^x, x\in X$, $p(A)(x)=\vee\{A(y): y\leq x$; $a(A)(x)=\wedge\{A(y) : x\leq y$; $c(A)=p(A)\wedge a(A)$.

Theorem2.1. Let (X,S) be an L-fuzzy syntopogenous space, define a binary relation $\leq_s$ on X as follows: for any $x, y\in X$, $x\leq_s y$ iff for $A\in L^x, \langle\langle\in S, \lambda\in L, \lambda\neq 0$ and $x_\lambda \langle\langle A$ implies $y_\lambda\leq A$, then "$\leq_s$" is a preorder on X, it is called the preorder generated by S on X.

proof.(1) (Reflexivity) We can get immediately from $x_\lambda \langle\langle A$ implies $x_\lambda\leq A$.

(2) (Transitivity) Suppose $x\leq_s y$, $y\leq_s z$ and for $A\in L^x$, $\langle\langle\in S$, $\lambda\in L, \lambda\neq 0$, $x_\lambda \langle\langle A$. By (LFS_2) there exist $\langle\langle_1\in S$, $B\in L^x$, such that $x_\lambda \langle\langle_1 B \langle\langle_1 A$. Since $x\leq_s y$, thus $y_\lambda\leq B \langle\langle_1 A$, so $y_\lambda \langle\langle_1 A$. Also because $y\leq_s z$, hence $z_\lambda\leq A$, i.e. $x\leq_s z$.

Definition2.3. Let $(X,\leq)$ be a preorder set, S be an L-fuzzy syntopogenous structure on X, then $(X,S,\leq)$ is called *-increasing (*-decreasing) if for $x,y\in X$, $x\leq y$ implies $x\leq_s y$ ($y\leq_s x$).

Proposition2.2. (1) Let $S_1, S_2\in S(X)$ and if $S_2\leq S_1$, then for $x, y\in X$, $x\leq_{S1} y$ implies $x\leq_{S2} y$. And if $S_1\sim S_2$, then $\leq_{S1}=\leq_{S2}$. (2) If $(X,S_1,\leq)$ is *-increasing (*-decreasing) and $S_2\leq S_1$, then $(X,S_2,\leq)$ is *-increasing (*-decreasing).

Theorem2.3. If $\leq$ is a preorder on X, $S\in S(X)$, then (1) $(X,S,\leq_S)$ is *-increasing; (2) $(X,S_\leq,\leq)$ is *-increasing; (3) $(X,S,\leq)$ is *-increasing iff $S\leq S_\leq$.

Proof. (1) Obvious. (2) If $x\leq y$, $S_\leq=\{\langle\langle\}$ [6], for $A\in L^x$, $\lambda\in L, \lambda\neq 0$ and $x_\lambda \langle\langle A$ implies $x_\lambda(x)\leq A(y)$,i.e. $y_\lambda\leq A$. thus $x\leq_{S\leq} y$, by Definition2.3 $(X,S_\leq,\leq)$ is *-increasing.

(3)"$\Rightarrow$"If $S\leq S_\leq$,by (2) $(X,S_\leq,\leq)$ is *-increasing, from Proposition2.2(2)then$(X,S,\leq)$is *-increasing.

"$\Leftarrow$"Suppos $\langle\langle_1\in S$, $S_\leq=\{\langle\langle\}$,A $\langle\langle_1 B$ and $x\leq y$, if $A(x)=0$,easily $A(x)\leq A(y)$. If $A(x)\neq 0$, because $x_{A(X)}\leq A \langle\langle_1 B$, then $x_{A(X)} \langle\langle_1 B$, as $(X,S,\leq)$ is *-increasing, so $y_{A(X)}\leq B$, i.e.$A(x)\leq B(y)$, thus $A \langle\langle B$, $S\leq S_\leq$.

Corollary2.4. Let $(X,\leq)$ be a preorder set, $H_i=\{E\in L^x$: E is increasing on $(X,\leq)\}$, define binary relation $\langle\langle_{Hi}$ as follows: A $\langle\langle_{Hi}$ B iff there exists $E\in H_i$ such that $A\leq E\leq B$. Then $S\in S(x)$, $(X,S,\leq)$ is *-increasing iff $S\leq\{\langle\langle_{Hi}\}$.
proof. Easily by Theorems 3.8[6] and 2.3.

Theorem2.5. The supremum of any mumber of *-increasing (*-decreasing) L-fuzzy syntopogenous structures on X is also *-increasing (*-decreasing)

The proof is omitted.

Corollary2.6. S^u (S^1) is the finest one of all *-increasing (*-decreasing) L-fuzzy syntopogenous structures which is coarser than S on S(X). Where $S^u=\vee\{S'\in S(X):(X, S',\leq)$ *-increasing, $S'\leq S\}$; $S^1=\vee\{S'\in S(X):(X, S',\leq)$ *-decreasing, $S'\leq S\}$. And (1) $S_1\leq S$ implies $S^u{}_1\leq S^u$, $S^1{}_1\leq S^1$; (2) $S_1\sim S$ implies $S_1{}^u\sim S^u$, $S^1{}_1\sim S^1$.

Proposition2.7. (1) If f: $(X,\leq)\rightarrow(Y,S,\leq')$ is an increasing mapping, $(Y,S,\leq')$ is *-increasing (*-decreasing), then $(X,f^{-1}(S),\leq)$ is *-increasing (*-decreasing); (2)If f is a decreasing mapping, $(Y,S,\leq')$ is *-increasing (*-decreasing), then $(X, f^{-1}(S),\leq)$ is *-decreasing (*-increasing).

Theorem2.8. If $\{(X_\lambda,S_\lambda,\leq_\lambda): \lambda\in\wedge\}$is a family of *-increasing (*-decreasing) L-fuzzy syntopogenous space, then the prodact $(\Pi_{\lambda\in\wedge}X_\lambda,\Pi_{\lambda\in\wedge}S_\lambda,\leq)$ is *-increasing (*-decreasing), where $\{x_\lambda\}\leq\{y_\lambda\}$ iff for any $\lambda\in\wedge$, $x_\lambda\leq_\lambda y_\lambda$.
Proof. By Corollary 2.6 (1) and Def. 7.1 .([4]).

3. order-convexityity

Definition3.1. Let S be an L-fuzzy syntopogenous structure on $(X,\leq)$, $(X,S,\leq)$ will be said to be order-convexity iff $S\sim(S^u\vee S^1)^a$, for $a\in\{i,p,b\}$ where i is identity.

Proposition3.1 If $(X,S,\leq)$ is order-convexity, then $S\sim S^a$.
Proof. If $(X,S,\leq)$ is order-convexity, then $S^a\sim(S^u\vee S^1)^{aa}=(S^u\vee S^1)^a\sim S$, so $S\sim S^a$.

Theorem3.2. $(X,S,\leq)$ is order-convexity iff $S \sim (S_1 \vee S_2)^a$,where $(X,S_1,\leq)$ ($(X,S_2,\leq)$) is *-decreasing (*-increasing).

Proof. The necessity is obvious. Conversely, if $S \sim (S_1 \vee S_2)^a$, then $S_i \leq (S_1 \vee S_2) \leq (S_1 \vee S_2)^a \leq S$,(i=1,2), $S_1 \leq S^u$, $S_2 \leq S^1$, therefore $S \sim (S_1 \vee S_2)^a \leq (S^u \vee S^1)^a \leq S^a$,but $S^a \sim (S_1 \vee S_2)^{aa} = (S_1 \vee S_2)^a \sim S$, so that $(S^u \vee S^1)^a \sim S$.

Proposition3.3. If $(X,S,\leq)$ is order-convexity, $a' \in \{i,p,b\}$ is an elementary operation such that aa' is also an elementary operation, then $(X,S^{a'},\leq)$ is aa'-order-convex.

Proof. If $S \sim (S^u \vee S^1)^a$, then $S^{a'} \sim (S^u \vee S^1)^{aa'}$. From Theorem 3.2 have $(X, S^{a'},\leq)$ is aa'-order-convex.

Theorem3.4. If $(X,S,\leq)$ is order-convexity, then $(X,S^{ta},\leq)$ is also order-convexity.

Proof. If $(X,S,\leq)$ is order-convexity, then $S^{ta} \sim (S^u \vee S^1)^{ata} = (S^u \vee S^1)^{ta} \sim (S^u \vee S^1)^a \sim (S^{tu} \vee S^{t1})^a$, by Theorem 3.2 $(X,S^{ta},\leq)$ is order-convexity..

Theorem3.5. Let $\{S_i : i \in I \neq \Phi\}$ be a family of order-convexity L-fuzzy syntopogenous structure on the preorder set $(X,\leq)$, then $(\vee_{i \in I} S_i)^a$ is also order-convexity on $(X,\leq)$.

Proof. Put $S= (\vee_{i \in I} S_i)^a$, $S_1 = \vee_{i \in I} S_i^u$ and $S_2 = \vee_{i \in I} S_i^1$, then S_1 is *-increasing, S_2 is *-decreasing on $(X,\leq)$. As $S_i \sim (S_i^u \vee S_i^1)^a$, then $(\vee_{i \in I} S_i)^a \sim (\vee_{i \in I}(S_i^u \vee S_i^1)^a)^a \sim ((\vee_{i \in I} S_i^u) \vee (\vee_{i \in I} S_i^1))^a \sim (S_1 \vee S_2)^a$, $S \sim (S_1 \vee S_2)^a$, by Theorem 3.2 $(\vee_{i \in I} S_i)^a$ is also order-convexity on $(X,\leq)$.

Theorem3.6. Let $(X,\leq)$, $(X',\leq')$ be preordered set, f is a preorder preserving mapping from X to X'. If $(X', S',\leq')$ is order-convexity, then $(X, f^{-1}(S'),\leq)$ is also order-convexity.

Proof. If $(X', S',\leq')$ is order-convexity, then $f^{-1}(S') \sim f^{-1}((S'^u \vee S'^1)^a) = f^{-1}(S'^u \vee S'^1)^a = (f^{-1}(S'^u) \vee f^{-1}(S'^1))^a$, by Proposition2.7, $f^{-1}(S'^u)$ is *-increasing, $f^{-1}(S'^1)$ is *-decreasing on $(X,\leq)$. Also by Theorem 3.2 then $(X,f^{-1}(S'),\leq)$ is also order-convexity.

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