A New Scalable Estimation for the Logarithmic Moment Generating Function of Network Traffic Flows

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Summary
Logarithmic moment generating functions play important role in network performance evaluation, especially in the application of large deviation type statistical inequalities for bounding and approximating quality of service measures (like transmission link saturation, buffer overflow rate, or traffic loss ratio due to these resource saturations). In this paper we present a new scalable method for upper bounding the logarithmic moment generating functions, primarily based on conditional zero order and first order statistics of the underlying traffic flows. The power of this new method compared to previously known ones from the literature is also demonstrated through numerical examples.
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Key words:
Logarithmic moment generating functions, effective bandwidth, polygonal approximation

1. Introduction

Some applications require an estimate on the frequency of the occurrence of rare events such as packet loss in telecommunications and accidents covered by insurance in finance. The theory of large deviations is the most powerful mathematical technique for estimating the probability of such rare events. Though less apparent, network design and resource management processes in telecommunications or transport networks also rely on this theory when the bandwidth (throughput) requirement of network traffic has to be determined.

Large deviations based techniques all involve the computation of the logarithmic moment generating function of the some random quantity (e.g. traffic load). Even other techniques used to estimate the frequency of rare events, such as those based on the Chernoff and Hoeffding bounds employ this function. For an $\{X(t), t > 0\}$ stochastic process ($X(t)$ is a random variable describing a stochastic process, e.g. packet arrivals, at the time instant t), the logarithmic moment generating function is defined as

$$LM(s,t) \overset{\text{def}}{=} \log E\left[e^{sX(t)}\right], \; s > 0, t > 0 \; . \qquad (1)$$

Here $E[.]$ is the (probabilistic) expectation: $E\left[f(X(t))\right] \overset{\text{def}}{=} \int_{-\infty}^{\infty} f(x)\, dP(X(t) < x)$ for an arbitrary function $f(.)$. The general form of the logarithmic moment generating function has a broad range of special forms and applications as seen in the.

In practical implementations the logarithmic moment-generating function (1) has to be replaced by a statistical estimator that computes the value of the function from the available samples of the random variable $X(t)$: $X_1(t), X_2(t), \ldots, X_N(t)$.

When it comes to implementation, techniques based on the logarithmic moment-generating function raise a range of difficulties. One of these difficulties lies in selecting an appropriate estimator of the logarithmic moment generating function. The key factor determining the appropriateness of a particular estimator is its distance from the abstract (exact) logarithmic moment generating function. An optimal estimator must meet the two requirements settled below:

Req. 1: The estimator should consistently overestimate the theoretical function in a conservative manner. This is important as an underestimated packet loss probability or accident rate results in compromised service levels in telecom networks and inaccurate insurance risk factors in finance. Thus the overestimation of the original function is a mandatory requirement.

Req. 2: The optimal estimator should provide a tight overestimate of the abstract logarithmic moment generating function. Thus the unnecessarily conservative estimation of the exact function has to be avoided. An excessive overestimation leads to overly cautious predictions for the bandwidth requirement, packet loss or accident rate, leading to underutilised network resources or to an overpriced insurance policy.

2. State-of-the-art

As of today, there are three published methods used for the estimation of the logarithmic moment generating function (1).

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2.1 Classical Empirical Mean

The first common solution to the statistical estimation of the logarithmic moment generating function is the replacement of the expectation with the empirical mean of the exponential powers of the available samples:

$$LM_N(s,t) \overset{\text{def}}{=} \log \frac{1}{N}\sum_{n=1}^{N} e^{sX_n(t)} . \quad (2)$$

This approach was proposed in a number of papers in the literature, e.g. in [1] and [2].

The classical empirical mean type estimator $LM_N(s,t)$ from (2) fails to comply to Req. 1 , as in practice it underestimates the theoretical value of $LM(s,t)$ (0).

The reason for this is that underestimation holds even in the statistical sense: the estimator is biased, that is, the expected value of the estimator is below the correct value:

$$E[LM_N(s,t)] \le LM(s,t) . \quad (3)$$

Although the estimator converges to the theoretical value in the number of the samples ($\lim_{N\to\infty} LM_N(s,t) = LM(s,t)$ with probability 1), for a fixed number of samples it is an underestimator. In practice, the underestimation is usually extensive because the empirical mean needs an extremely high number of samples in order to approach the expectation of the exponential function of the random variable. This is because the exponential function increases the sensitivity of the empirical mean estimator to the values and to the number of samples.

2.2 Linear Approximation

Alternatively, as a second solution, a special simplified statistical estimation is sometimes used [3], [4] for the logarithmic moment generating function. This estimator, hereafter referred to as the linear approximation, treats the exponential function inside the expectation as a linear curve and thus only needs the empirical mean of the samples (instead of the empirical mean of the exponential powers of the samples):

$$\overline{LM}_N(s,t) \overset{\text{def}}{=} \log\left(1 + \frac{e^{sp(t)}-1}{p(t)}\frac{M(t)}{N}\right) , \quad (4)$$

where $p(t)$ is an upper bound for the random variable $X(t)$: $0 \le X(t) \le p(t)$ and $M(t) \overset{\text{def}}{=} \sum_{n=1}^{N} X_n(t)$ is the sum of the samples $X_n(t)$.

The linear approximation $\overline{LM}_N(s,t)$ from (4) on the other hand is a coarse overestimator of the theoretical value $LM(s,t)$, which is in contradiction to Req. 2 (0).

The reason for this is that it stems from a very conservative theoretical overestimation of the abstract logarithmic moment generating function (1):

$$LM(s,t) \le \log\left(1 + \frac{e^{sp(t)}-1}{p(t)} E[X(t)]\right) \overset{\text{def}}{=} \overline{LM}(s,t) . \quad (5)$$

This estimation technique is based on the conservative overestimation of the exponential function by a linear function on a finite interval: $e^{sx} \le \frac{e^{sp(t)}-1}{p(t)}x + 1$, if $0 \le x \le p(t)$. The linear approximation $\overline{LM}_N(s,t)$ (4) is closer to the correct value than the upper bound (5). This is due to the nature of the statistical estimation of the expectation from a finite set of samples. Formally, the expected value of the linear approximation $\overline{LM}_N(s,t)$ is below the theoretical conservative bound $\overline{LM}(s,t)$:

$$E\left[\overline{LM}_N(s,t)\right] \le \overline{LM}(s,t) . \quad (6)$$

Yet in practice (4) typically remains an upper bound of (1). This is because the overestimation in (5) is overly conservative.

2.3 Tangent Linear Approximation

A third type of estimator for the computation of the logarithmic moment generating function [3] is closely related to the linear approximation type estimator in (4) and can be written as a function of m for any fixed $0 \le m \le p(t)$:

group of estimators of the

$$\overline{\overline{LM}}_m(s,t) \overset{\text{def}}{=} \frac{e^{sp(t)}-1}{p(t)+(e^{sp(t)}-1)m}\left(m(t)-m\right) + \log\left(1+\frac{e^{sp(t)}-1}{p(t)}m\right) , \quad (7)$$

where $m(t) \overset{\text{def}}{=} \frac{M(t)}{N}$. It is attained by a tangent type overestimation of the linear approximation as a function of $m(t)$ at any arbitrary point $0 \le m \le p(t)$. In fact, this type represents a whole group of estimators, as m is a free variable in the formula.

The third type of estimator $\overline{\overline{LM}}_m(s,t)$ from (7) also overestimates the abstract function $LM(s,t)$ (1) excessively (0), defying the requisite on the tightness of the estimator set forth in Req. 2.

The reason for this is that, by its mathematical construction, $\overline{\overline{LM}}_m(s,t)$ is a conservative upper bound of the linear approximation $\overline{LM}_N(s,t)$:

$$\overline{LM}_N(s,t) \le \overline{\overline{LM}}_m(s,t) \ , \qquad (8)$$

and thus overestimating the linear approximation, which is already behaving as an upper bound. Formally, the function $\overline{\overline{LM}}_m(s,t)$ is obtained as a tangent of $\overline{LM}_N(s,t)$ as a function of $m(t)=\frac{M(t)}{N}$ taken at an arbitrary point *m*.

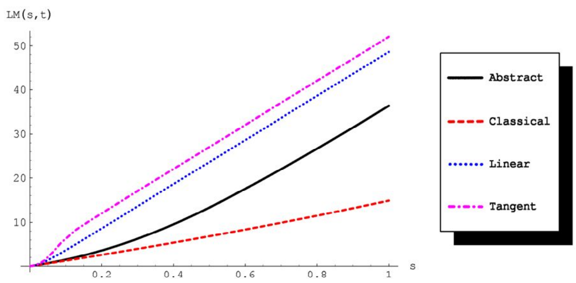

Fig. 1. Comparison of logarithmic moment generating function estimators to the abstract function

2.4. Summary of the Known Solutions

The classical empirical mean estimator (2) violates Req. 1, since it consistently underestimates the abstract function (1) because of the finite number of samples.

The linear approximation (4) on the other hand overestimates the theoretical value (1) as it is derived from the highly conservative linear upper bound of the exponential function. Because of this property it is non-conformant to Req. 2.

Likewise, the tangent linear approximation (7) fails to meet Req. 2, as it provides an even more conservative upper bound of the theoretical value (1) than the linear approximation (4).

The deviation of the three published estimators, (2) (4) and (7) from the abstract logarithmic moment generating function (1) is depicted in 0. All estimators are computed for the same set of input data, originating from the sampling of traffic made up by 10 on-off type flows. An individual sample represents the amount of traffic generated by the aggregate traffic in a time interval of 1 s. The constituent flows on average stay in the on state for a period of 0.067 s and remain silent for 0.2 s. The data transmission rate in the on state is 5 Mbit/s per flow.

3. A New Polygonal Approximation of *LM(s,t)*

A good estimator of the logarithmic moment generating function has to be a conservative upper bound, in order to stay on the safe side, but it also has to be as close to the theoretical value as possible, in order to avoid overly cautious predictions.

The core of the method is the classification of the input data into non-overlapping, but adjacent intervals set up based on the possible values of the samples. The exponential function inside the logarithmic moment generating function can be approximated by a polygon with linear sections in the variable $X(t)$ over the individual intervals.

If the random variable describing the input process is bounded: $X(t) \le p(t)$, the division of the samples into groups leads to the invented estimator, termed as the polygonal logmoment generator with interval breakpoints

$$0 \overset{\text{def}}{=} p_0(t) < p_1(t) < \ldots < p_K(t) < p_{K+1}(t) \overset{\text{def}}{=} p(t) : \qquad (9)$$

$$\overline{LM}_{(p_1(t),\ldots,p_{K+1}(t))}(s,t) \overset{\text{def}}{=} \log\left(\sum_{k=0}^{K} \frac{e^{sp_{k+1}(t)} - e^{sp_k(t)}}{p_{k+1}(t)-p_k(t)} \frac{M_k(t)}{N} + \frac{p_{k+1}(t)e^{sp_k(t)} - p_k(t)e^{sp_{k+1}(t)}}{p_{k+1}(t)-p_k(t)} \frac{N_k(t)}{N}\right). \qquad (10)$$

Here, $M_k(t) \overset{\text{def}}{=} \sum_{n=1}^{N} X_n(t) I\{p_k(t) < X_n(t) \le p_{k+1}(t)\}$ is the sum of the samples whose values are in the interval $(p_k(t), p_{k+1}(t)]$, $N_k(t) \overset{\text{def}}{=} \sum_{n=1}^{N} I\{p_k(t) < X_n(t) \le p_{k+1}(t)\}$ is the number of samples whose values are in the $(p_k(t), p_{k+1}(t)]$ interval and

$$I\{expression\} \overset{\text{def}}{=} \begin{cases} 1 & \text{if } expression \text{ is true} \\ 0 & \text{if } expression \text{ is false}. \end{cases}$$

This approximation can be interpreted as a generalisation to the linear approximation (4) because for $K=0$ the two estimators are equivalent: $\overline{LM}_{(p_1(t))}(s,t) = \overline{LM}_N(s,t)$.

The estimator overcomes the drawbacks of the other three types of approximations, the underestimation of (2) and the overestimation of (4) and (7). With an appropriate choice of the breakpoints (9) the polygonal logmoment generator (10) exhibits the desired behaviour defined by Req. 1 and Req. 2 (conservative but tight estimator).

The mathematical construction of the novel estimator builds on the fact that the logarithmic moment generating function can be written in the following form by assembling the original expectation from K conditional expectations computed over the intervals created by the breakpoints:

$$LM(s,t) = \log \sum_{k=0}^{K} E\left[e^{sX(t)} \mid p_k(t) < X(t) \le p_{k+1}(t)\right] P\left(p_k(t) < X(t) \le p_{k+1}(t)\right) \tag{11}$$

Now, using the linear bound of the exponential function on the finite intervals $\left(p_k(t), p_{k+1}(t)\right]$ for the corresponding conditional expectations:

$$e^{sx} \le \frac{e^{sp_{k+1}(t)} - e^{sp_k(t)}}{p_{k+1}(t) - p_k(t)}\left(x - p_k(t)\right) + e^{sp_k(t)} = \frac{e^{sp_{k+1}(t)} - e^{sp_k(t)}}{p_{k+1}(t) - p_k(t)} x + \frac{p_{k+1}(t)e^{sp_k(t)} - p_k(t)e^{sp_{k+1}(t)}}{p_{k+1}(t) - p_k(t)}, \text{ if } p_k(t) \le x \le p_{k+1}(t),$$

the next theoretical upper bound can be deduced for the logarithmic moment generating function:

$$LM(s,t) \le \log\left(\sum_{k=0}^{K} \frac{e^{sp_{k+1}(t)} - e^{sp_k(t)}}{p_{k+1}(t) - p_k(t)} E\left[X(t) \mid p_k(t) < X(t) \le p_{k+1}(t)\right] P\left(p_k(t) < X(t) \le p_{k+1}(t)\right) + \frac{p_{k+1}(t)e^{sp_k(t)} - p_k(t)e^{sp_{k+1}(t)}}{p_{k+1}(t) - p_k(t)} P\left(p_k(t) < X(t) \le p_{k+1}(t)\right)\right) \stackrel{\text{def}}{=} LM_{(p_1(t),\ldots,p_{K+1}(t))}(s,t). \tag{12}$$

For any number of breakpoints ($K > 0$) this theoretical upper bound $LM_{(p_1(t),\ldots,p_{K+1}(t))}(s,t)$ is by definition tighter than the linear type upper bound $\overline{LM}(s,t)$ from (5):

$$LM_{(p_1(t),\ldots,p_{K+1}(t))}(s,t) \le \overline{LM}(s,t) \ . \tag{13}$$

The replacement of the conditional expectations in (12) to the corresponding empirical conditional means results in the polygonal logmoment generator (10), whose expected value is below the theoretical upper bound similarly to (3) and (6):

$$E\left[\overline{LM}_{(p_1(t),\ldots,p_{K+1}(t))}(s,t)\right] \le LM_{(p_1(t),\ldots,p_{K+1}(t))}(s,t) \ . \tag{14}$$

The presence of the breakpoints empowers the invented estimator $\overline{LM}_{(p_1(t),\ldots,p_{K+1}(t))}(s,t)$ (10) with the right amount of flexibility to deliver an arbitrarily tight upper bound for the logarithmic moment generating function $LM(s,t)$ from (1).

The polygonal logmoment generator (10) is a conservative but tight estimator of the logarithmic moment generating function (1), which makes it fully conformant to both Req. 1 and Req. 2. These properties are essential when deploying the solution in real-life applications. In all conceivable applications (network resource requirement estimation in telecommunications and transport systems, insurance risk calculation in finance, etc.) the resource estimation from the logarithmic moment generating function is sensitive to both underestimation and overestimation. The underestimation brings about the violation of the most important requirement, the sustenance of contracted service levels in telecommunications and profitable operations in the insurance. Too cautious overestimation, on the other hand, results in underutilised network resources and overpriced insurance rates resulting in lost profit for telecom operators and insurance companies alike.

The invented estimator (10) attains a very high precision from simple to obtain input data: it needs the empirical conditional probabilities and empirical conditional averages of the samples classified by the breakpoints (9). These two simple quantities are easily estimated even from a very low number of samples. Such precision in the approximation could only be reached with other methods at a price of complex input requirements such as the knowledge of quantities like the second and all higher moments of the samples. Furthermore these alternative methods raise intractable numerical difficulties in the computations they require.

The invented polygonal logmoment generator (10) delivers an overwhelming improvement in the tightness of the estimator (0) when compared to the method of the linear approximation (4), yet the computations involved are only slightly more complex (empirical conditional probabilities and averages instead of only the empirical average).

The invented polygonal logmoment generator (10) inherits the safe conservative nature from the linear approximation (Req. 1) as they are derived from the same type of upper bound, yet the invented estimator is by its mathematical construction tighter. Unlike its ancestor, the invented method allows the control of the tightness of the estimation through the use of the breakpoints (9).

The assumption made on the bounded nature of the input data does not restrict the applicability of the method, since in practice the estimator use a finite number of samples, therefore there exists a maximal sample value, which can be used as the theoretical upper bound. This means that the estimator can be used for arbitrary, not necessarily bounded, random variables.

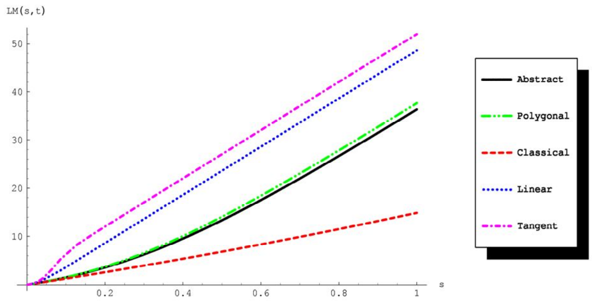

Fig. 2. Comparison of the invented and existing estimators to the abstract logarithmic moment generation function

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Appendix

There are some special well-known logarithmic moment generating functions which are included in the general definition (1):

Rate type Logarithmic Moment Generating Function

If the time dependence is omitted, the stochastic process is reduced to a random variable: $X(t) \overset{\text{def}}{=} X$ and therefore the logarithmic moment generating function of the random variable X is attained:

the rate type logarithmic moment generating function:

$$LM(s) \overset{\text{def}}{=} \log E\left[e^{sX}\right],\ s > 0 \qquad (15)$$

and the rate type effective bandwidth function used for bufferless multiplexing for example as in [3] and [4]:

$$\alpha(s) \overset{\text{def}}{=} \frac{1}{s}\log E\left[e^{sX}\right] = \frac{1}{s}LM(s)\ . \qquad (16)$$

In case of rate type input quantities, $X_1, X_2, \ldots, X_N$ (samples of the random variable X), the rate type logarithmic moment generating function can be computed using the following form of the invented estimator:

$$\overline{LM}_{(p_1,\ldots,p_{K+1})}(s) \overset{\text{def}}{=} \log\left(\sum_{k=0}^{K} \frac{e^{sp_{k+1}(t)} - e^{sp_k(t)}}{p_{k+1}(t) - p_k(t)} \frac{M_k}{N} + \frac{p_{k+1}(t)e^{sp_k(t)} - p_k(t)e^{sp_{k+1}(t)}}{p_{k+1}(t) - p_k(t)} \frac{N_k}{N}\right). \qquad (17)$$

Here $M_k \overset{\text{def}}{=} \sum_{n=1}^{N} X_n\, I\{p_k < X_n \le p_{k+1}\}$ is the sum of samples whose values fall into the interval $(p_k, p_{k+1}]$, $N_k \overset{\text{def}}{=} \sum_{n=1}^{N} I\{p_k < X_n \le p_{k+1}\}$ is the number of samples whose values are in the interval $(p_k, p_{k+1}]$.

Input Process type Logarithmic Moment Generating Function

If the random variable $X(t)$ of the stochastic process describes the amount of arrivals in the time interval $[0,t)$: $X(t) \overset{\text{def}}{=} X[0,t)$, the logarithmic moment generating function can be used to arrive at the well-known effective bandwidth function defined by Kelly [5]:

$$\alpha(s,t) \overset{\text{def}}{=} \frac{1}{st}\log E\left[e^{sX[0,t)}\right] = \frac{1}{st}LM(s,t),\ \text{or} \qquad (18)$$

the scaled cumulant generating function used by Duffield et al. [1]:

$$\lambda(\theta) \overset{\text{def}}{=} \lim_{t\to\infty}\frac{1}{t}\log E\left[e^{\theta X[0,t)}\right]. \qquad (19)$$

The invented estimator for the input process type logarithmic moment generating function from the available samples $X_1[0,t), X_2[0,t), \ldots, X_N[0,t)$ (instances of the random variable $X[0,t) \le p(t) \overset{\text{def}}{=} tp$) takes the form of

$$\overline{LM}_{(tp_1,\ldots,tp_{K+1})}(s,t) \overset{\text{def}}{=} \log\left(\sum_{k=0}^{K} \frac{e^{stp_{k+1}} - e^{stp_k}}{tp_{k+1} - tp_k} \frac{M_k[0,t)}{N} + \frac{p_{k+1}e^{stp_k} - p_k e^{stp_{k+1}}}{p_{k+1} - p_k} \frac{N_k[0,t)}{N}\right). \qquad (20)$$

Here $M_k[0,t) \overset{\text{def}}{=} \sum_{n=1}^{N} X_n[0,t)\, I\{tp_k < X_n[0,t) \le tp_{k+1}\}$ is the sum of samples whose values fall into the interval $(tp_k, tp_{k+1}]$, $N_k[0,t) \overset{\text{def}}{=} \sum_{n=1}^{N} I\{tp_k < X_n[0,t) \le tp_{k+1}\}$ is the number of samples whose values are in the interval $(tp_k, tp_{k+1}]$.