# Optimization of MIMO Systems in a Correlated Channel

**Jraifi Abdelouahed1† and El Hassan Saidi2††**,

University of Mohammed V, Group CPR, Lab PHE, Rabat, Morocco

**Summary**
Motivated by the study of the optimization of the quality of service for Multiple Input Multiple Output (MIMO) technique in a wireless communication system. The capacity of MIMO channel in independent Rayleigh channels grows linearly as the number of antennas. However, some limitations on the MIMO capacity is due to the correlation between individual sub-channels of the matrix channel. In this paper, we investigate the MIMO channel capacity in correlated channels using a new correlation matrix model. After some mathematical recalls we derive the general upper bound on the MIMO channel capacity. We give the analytical results, which measure the effect of correlation on the MIMO capacity for a proposal model. Then, we use the correlation matrix approach to compute the capacity of MIMO with respect to signal-to-noise (snr) and then predict the number of antennas. By fixing the snr variable to a specific value, we extract information on the optimal numbers of MIMO antennas. Finally, we have given the variation of the MIMO capacity in the limiting case of $N \mapsto \infty$ for our proposal model and the exponential correlation matrix model.

***Key words:***
*MIMO system, Wireless communication, channel capacity, correlation matrix.*

## 1. Introduction

Digital communication using Multiple Input Multiple Output (MIMO) is one of the important techniques used to exploit the spatial diversity in a rich scattering environment in order to improve the spectral efficiency. Due to the great spectral efficiency gain, MIMO systems have known a great revival interest nowadays and have been defined by IEEE 802.16 [1], for fixed broad band wireless access and 3G partnership project (3GPP) for mobile applications. Using MIMO, it has been shown in [2] that spectral efficiency can be improved significantly in a wireless communications in fading environment.
To study MIMO system, we use Rayleigh model as it is the most widely method for indoor and urban channels [3]. Our system is then modeled by $N_R * N_T$ random matrix H. The received vector r is related to the transmitted vector e as:

$$r = He + n,$$

Where n is the noise vector with covariance matrix $\sigma^2 I_{N_R}$.
The component gains $h_{ij}$ of the $H$ channel are independent, identically distributed (iid) and governed by circular complex Gaussian random variables with zero mean and unit variance.
From a diagram of a MIMO wireless transmission system (see Fig. 1), a compressed digital source in the form of a binary data stream is fed to a simplified transmitting block encompassing the functions of error control coding and mapping to complex modulation symbols (BPSQ, QPSK, M-QAM, ...). Each separate symbol streams is mapped onto one of the multiple $N_T$ antennas. After filtering and amplification, the signals are launched into the wireless channel. At the receiver, the signals are captured by $N_R$ antennas and inverse functions are performed to recover the message.

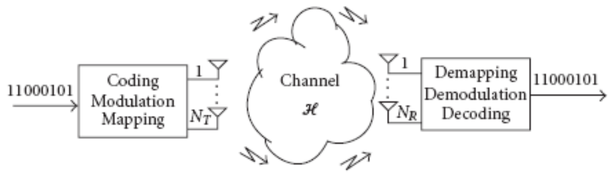


Fig. 1. MIMO Architecture

Information theory can be used to demonstrate the gain of MIMO; if we consider a system with $N_T$ transmitters and $N_R$ receivers, the capacity of the system is given by the relation [4-6],

$$C = \log_2 \det\left[ I_{N_R} + \frac{\rho}{N_T} HH^+ \right] \qquad (1)$$

With unit bit/sec/Hz. Here $(^+)$ stands for the adjoin conjugation, $\rho$ is the signal to noise ratio (snr) and $H$ is the $N_R * N_T$ channel matrix. Note that this equation is based on $N_T$ equal power uncorrelated sources. Foshini

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and Telatar [1] demonstrated that capacity C grows linearly in min( $N_R, N_T$ ).

MIMO systems advantages are numerous; two of them are their ability to turn multi-path propagation, traditionally qualified as a problem of wireless communications, into a benefit for the user. MIMO may be also used to increase operator's revenues. However, the promising advantages of MIMO systems over traditional single antenna systems depend on different parameters that contribute on the limitation on the MIMO channel capacity. Some constraints on the MIMO system are imposed by the correlation between individual sub-channels of the matrix channel [6-7]. The correlation phenomena that appears in MIMO channel is one of the parameters that strongly affect the performance of these systems. When the correlation coefficient increases, the capacity decreases. The effect of correlation on the MIMO channel capacity has been investigated in details in [5]. But, this method does not study the effect the capacity versus the signal to noise ratio (snr). Notice also that the uniform correlation, in which the correlation coefficients are the same, was also considered in the uniform model of [7]. Though modeling correlations in a quite reasonable manner, this uniform model is however not real because it considers the correlation between sub-channels in the same way independently of the antenna separation distance. To get a more realistic system, we propose a new model where one takes into account the rapid attenuation of the correlations between distant channels. We notice that this limitation has been addressed in [7] where attenuation has been modeled by using an inverse power law correlation function $r^{|i-j|}$ known in the literature as the exponential matrix model.

In this paper, we develop a matrix model proposal for studying the MIMO system in cases where spatial diversity does not allow the implementation of several antennas such as in 3G generation at the level of user equipment (UE). Using a theoretical approach, we reconsider the computation of the channel capacity to predict the optimal number of antennas in MIMO correlated sub-channel systems.

The presentation of this paper is as follows. In section 1, we give some mathematical tools. In section 3, we study MIMO capacity in correlated sub-channel systems. In section 4, we introduce our correlated matrix model proposal which takes into account strong correlations between neighboring sub-channels and neglect distant ones. Finally, we compare our model proposal with that of [7] and give a conclusion.

## 2. Mathematical recalls

In this section, we recall mathematical properties about Jensen's inequality and the concavity of functions. These results are useful in the finding of bounds of certain performance measures, including MIMO systems. The determination of bound permits to bring a complicated expression to something simpler and then allow getting more insight in ways to improve the MIMO system performances.

One of the important inequalities to be used in this paper is Jensen's inequality. Writing down this inequality, we start by giving the definitions of convex and concave functions.

**Definition 1**: A real function $f(x)$ is said to be convex over an interval $[a,b]$ if for every $x_1$ , $x_2 \in [a,b]$ and $\lambda_1 > 0, \lambda_2 \leq 1$, with $\lambda_1 + \lambda_2 = 1$, we have:

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

In an other way by taking $\lambda_1 = \lambda$ and $\lambda_2 = 1 - \lambda$ , we also have,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

A function is strictly convex if equality holds only if $\lambda = 0$ or $\lambda = 1$ .

Notice that $\lambda x_1 + (1-\lambda)\, x_2$ is simply a line segment connecting $x_1$ and $x_2$ (in the x direction) and $\lambda f(x_1) + (1-\lambda) f(x_2)$ is a line segment connecting $f(x_1)$ and $f(x_2)$. Thus, the function $f(x)$ is convex if it lies below the straight line segment connecting any $f(y)$ and $f(z)$ for any points $y$ and $z$ in the interval $[a,b]$.

**Definition 2**: A real function $g(x)$ is said concave if -$g(x)$ is convex, that is

$$g(\lambda x_1 + (1-\lambda)x_2) \geq \lambda g(x_1) + (1-\lambda) g(x_2)$$

Where $x_1$, $x_2$ and $\lambda$ are as before. Strict concavity of $g$ is equivalent to strict convexity of ( $-g$ )

**Examples**: the real functions $x^2$, $e^x$, $|x|$, $xLogx$ are convex while $Logx$ and $\sqrt{x}$ are concave.

**Theorem**: Jensen Inequality:

If the function $f$ is convex and $X$ is a random real variable:

$$f[E(X)] \leq E[f(X)]$$

Where E(X) denotes the expectation of the random real variable. Put differently,

$$f\left(\sum_k p_k x_k\right) \le \sum_k x p_k f(x_k)$$

Where $\sum p_k = 1$ and where we have set $p_k = p(x_k)$.

If $f$ is strictly convex, then the equality $f[E(X)] = E[f(X)]$ follows from the identity $E(Y) = Y$ with $Y$ standing for $X$ or $f(X)$.

If $f$ is concave, then

$$Ef(X) \le f(EX)$$

Notice moreover that the Jensen inequality is important in the sense that it allows us to pull a function g outside of the summation and permits simplifications in the analytic computations generally difficult to handle exactly.

## 3. MIMO capacity

For narrow bandwidth, the MIMO channel is modeled by a $N_R * N_T$ random matrix $H$ , and the received vector $y$ (dimension $N_R$) is given by the equation below:

$$y = Hx + n \qquad (2)$$

Where $n$ is the noise vector (dimension $N_R$) and $x$ the transmitted vector (dimension $N_T$ ). When the bandwidth is large, Orthogonal Frequency Division Multiplexing (OFDM) [8] can be used to divide the large bandwidth into narrow ones [2]. Eq (2) remains valid for each sub-band.

The most widely used model for indoor or urban channels is the Rayleigh model [9]: the $h_{ij}$ components of $H$ are independent identically distributed (i.i.d) and are circular complex Gaussian random variables with zero mean and variance $\sigma^2$ . When the transmitted signal vector $x$ is composed of i.i.d equal power components each with a Gaussian distribution and the receiver knows the channel, the capacity of MIMO system for $N_R = N_T = N$ is [2]:

$$C = \log_2 \det\left(I + \frac{\rho}{N} HH^+\right) \qquad (3)$$

Where $I$ is the $N * N$ identity matrix; $\rho$ describe the average signal to noise ratio (snr); $H$ is the normalized channel matrix and ( $^+$ ) means adjoin conjugate. Here we consider the normalization condition as follow:

$$\frac{1}{N}\sum_{i,j=1}^{N} \left|h_{ij}\right|^2 = 1 \qquad (4)$$

Where $h_{ij}$ denote the gains between the $jth$ transmit and the $ith$ receive of MIMO system. When the parallel sub-channel are uncorrelated ( $H = I$ ), $\frac{\rho}{N}$ denotes the signal to noise ratio per receive branch. A rough way to investigate the MIMO channel capacity in the MIMO channel capacity in correlated channels is to assume that all the received powers are equal. In this case, $\sum_j \left|h_{ij}\right|^2 = 1$ and Eq (1) is reduced to [7]:

$$C = \log_2 \det\left(I + \frac{\rho}{N} C\right) \qquad (5)$$

Where C is the normalized channel correlation matrix,
C
is the component of C defined as:

$$C_{ij} = \sum_k h_{ik} h^*{}_{jk} \qquad (6)$$

Where (¤) denotes complex conjugate. The equation (5) is justified because of the assumption of equal received powers.

Also, we can deduce from Eq (6) that = ¤ ..

In the case of random (stochastic) channel, also the capacity
is random. The capacity ergodic can be defined [3]. Referring
to the mathematical recall in section 1, we can obtain the upper bound on the mean (ergodic) capacity as:

$$\overline{C} = \langle C \rangle \le C_u = \log_2 \det\left[I + \frac{\rho}{N}\langle C \rangle\right] \qquad (7)$$

denotes the expectation over the channel matrix. For simplicity, we assume further that the channel is deterministic, and the same results hold true for $C_u$ . If we denote by A the matrix under the determinant in Eq (7), we obtain:

$$\det(A) = \left(I + \frac{\rho}{N}\right)^N B_N \qquad (8)$$

With

$$B_N = \det(M) \qquad (9)$$

=

| | | | | |
|---|---|---|---|---|
| 1 | $bC_{12}^*$ | $bC_{12}^*$ | … | $bC_{1n}$ |
| $bC_{12}^*$ | 1 | $bC_{12}^*$ | … | $bC_{2n}$ |
| $bC_{13}^*$ | $bC_{23}^*$ | 1 | … | $bC_{3n}$ |
| .<br>.<br>. | .<br>.<br>. | .<br>.<br>. | 1 | .<br>.<br>. |
| $bC_{1n}^*$ | … | … | … | 1 |

And $b = \frac{\rho}{N}\left(1 + \frac{\rho}{N}\right)^{-1}$

By substituting Eq (9) in Eq (7), the MIMO capacity can be cast as the sum of two terms like

$$C = N \log_2\left[1 + \frac{\rho}{N}\right] + \log_2(B_N) \qquad (10)$$

From Eq (10), it is easy to see that The first term denotes the MIMO capacity of N parallel independent sub-channels [2] $C_{ij} = 0$ and the second one captures the effect of the sub-channel correlation. The first term is positive while the contribution of the correlation is negative due to the property $\log x \prec 0$ for $0 \prec x \prec 1$. At high snr ($b \approx 1$), $B_N$ depends only on the correlation matrix. It is then an interesting issue to determine the MIMO capacity in a correlated channel. In this regards, the evaluation of the matrix $B_N$ turns out to be a crucial.

## 4. Correlation Matrix Model

In this section, we propose a new correlation matrix model which takes into account the distance between different branches at transmitter/receiver. For our model, the components of **C** are given by:

$$C_{ij} = \begin{cases} e^{-C(j-i)} \rightarrow i \le j \\ C^*_{ji} \rightarrow i \succ j \end{cases} \qquad (11)$$

Where $C$ is the correlation coefficient of neighboring receive branches. The above model may be not the exact one for some real world scenarios but this is simple model which allows us to investigate the impact of correlation on the MIMO capacity. We note, however, that the proposal model
is physically reasonable because of closely dependency of correlation and distance between receive antennas (correlation decreases with increasing distance between receive antennas). From mathematical point of view, the proposal model decreases rapidly with the distance between receive antennas. This model may be a realistic model for some real-world scenarios such as in 3G because of electromagnetic interaction of antenna elements on small platform (on User Equipment: UE) and the implementation of diversity at user mobile (UE) in 3G cannot support the number of antennas as in the base station (Node B).

Now, substituting Eq (11) back into $B_N$ and using some transformations of the determinant (we multiply the second row by $e^{-C}$ and subtracts it from the second row an so on ), then, we obtain the following $B_N$ :

BN=det (

| | | | | |
|---|---|---|---|---|
| $1-b\left\|e^{-C}\right\|^2$ | $(b-1)e^{-C}$ | 0 | … | 0 |
| $b(e^{-C})^*\left(1-b\left\|e^{-C}\right\|^2\right)$ | $1-b\left\|e^{-C}\right\|^2$ | $(b-1)e^{-C}$ | … | 0 |
| $b(e^{-C})^{2*}\left(1-b\left\|e^{-C}\right\|^2\right)$ | $b(e^{-C})^*\left(1-b\left\|e^{-C}\right\|^2\right)$ | $1-b\left\|e^{-C}\right\|^2$ | … | 0 |
| … | … | … | … | … |
| $b\left((e^{-C})^*\right)^{n-1}$ | $b\left((e^{-C})^*\right)^{n-2}$ | $b\left((e^{-C})^*\right)^{n-3}$ | … | 1 |

) (12)

From $B_N$ , it's difficult to derive a closed form analytical expression. But a simple form expression can be used in a practically case of high snr ($\frac{\rho}{N} >> 1$), when $1 - b << 1$. In this case,

$$B_N \approx \left(1 - b\left|e^{-C}\right|^2\right)^{N-1} \qquad (13)$$

And

$$C \approx N \log_2\left[1 + \frac{\rho}{N}\left(1 - \left|e^{-C}\right|^2\right)\right] + \log_2\left(\frac{1 + \frac{\rho}{N}}{1 + \frac{\rho}{N}\left(1 - \left|e^{-C}\right|^2\right)}\right)$$

(14)

For $N >> 1$, we obtain:

$$C \approx N \log_2 \left[ 1 + \frac{\rho}{N} \left( 1 - \left| e^{-C} \right|^2 \right) \right] \quad (15)$$

In the limit case of $N \to \infty$, we obtain from Eq (15)

$$C \approx \frac{\rho}{\log_2} \left( 1 - \left| e^{-C} \right|^2 \right) \quad (16)$$

And

$$C_\infty = \frac{\rho}{\ln 2} \quad (17)$$

## 5. Illustration of Results

To show how to predict the number of antenna in a MIMO system, we give the following figure (fig2) illustrating the variation of the capacity MIMO with respect to snr (signal to noise ratio). We recall that we have set the value of the correlation coefficient to 0.2.

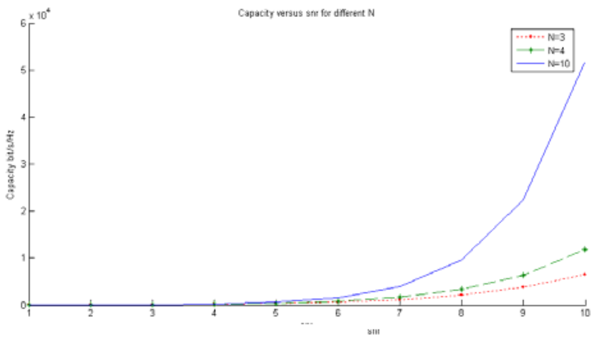


Fig. 2. MIMO capacity in terms of snr for different N

From fig2, we learn the following:

(i) For given snr and for a desired value of MIMO capacity, we can determine the number of antennas to install. Choosing a MIMO performance with MIMO capacity as:

$$C_N(snr) \geq 4 * 10^4$$

At snr=6dB, we find that the required number of antennas is at least $N = 10$. As we can see, this number is too high because of the required high performance of MIMO capacity. Relaxing this requirement by choosing for instance $C_N(snr) \geq 10^4$ we get $N = 4$. The number of antennas strongly depends then on the precision of $C_N(snr)$.

(ii) Knowing that the choice of the MIMO capacity depends on the type of service we want to send on the channel (voice, data, and image), we can, by help of Fig. 2, determine the optimal value of the received antennas.

(iii) To optimize the number of antenna in a MIMO system, the same analytical approach may also be used for other correlation model such as uniform, exponential, ...,

Bellow, we give the figure3 describing the variation of capacity $C$ with respect to signal-to-noise ration (snr) respectively for our proposal model and the exponential model of [7], this figure shows the MIMO capacity of a deterministic channels and by using approximate formulas in the limiting case of $N \to \infty$ (Eq. 16). The MIMO channel capacity $C_\infty$ evaluated using the exponential correlation matrix model [7] is also shown for comparison. We recall also that the used coefficient of correlation is set to 0.5 for the two models. As we can see from this figure that our proposal model predicts lower capacity because the correlation matrix model of our model between distant receive branches decrease rapidly.

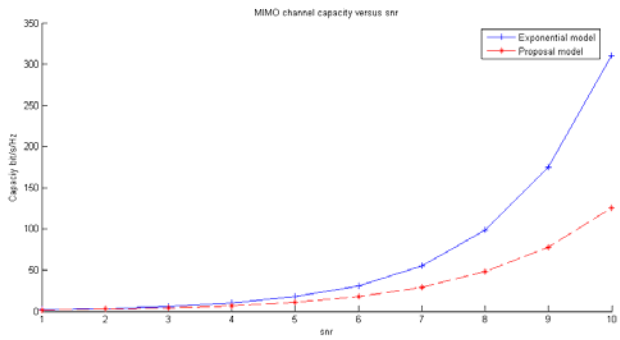


Fig. 3. MIMO capacity for proposal model and exponential matrix model

## 6. Conclusion

In this paper, we have presented a review of MIMO system and the impact of wireless channel correlation. We show that MIMO channel capacity depends substantially on the correlation between individual sub-channels of the matrix channel. By using the Jensen inequality, we estimate the upper bound of MIMO channel capacity. Then, we have proposed a new correlation matrix model and we derive the analytical formula for the MIMO channel capacity. To predict the optimal number of antennas in a correlated MIMO channel, we use the correlation matrix approach to compute the capacity of MIMO with respect to signal-to-noise (snr) and then predict the number of antennas. By fixing the snr variable to a specific value, we extract information on the optimal numbers of MIMO antennas. Finally, we show that the exponential correlation matrix model predicts higher channel capacity than our proposal model because the correlation coefficient in our model decreases rapidly.

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