RLC Circuit Response and Analysis (Using State Space Method)

Mohazzab[1] JAVED, Hussain[1] AFTAB, Muhammad[1] QASIM, Mohsin[1] SATTAR

[1]Engineering Department, PAF-KIET, Karachi, Sindh, Pakistan

Abstract-- This paper presents RLC circuit response and analysis, which is modeled using state space method. It provides a method with the exact accuracy to effectively calculate the state space models of RLC distributed interconnect (nodes) and transmission line in closed forms in time domain and transfer functions by recursive algorithms in frequency domain, where their RLC components can be evenly distributed or variously valued. The main features include simplicity and accuracy of the said closed forms of the state space models {A,B,C,D} without involving matrix inverse and matrix multiplication operations, effectiveness and accuracy of the said recursive algorithms of the transfer functions. The response of the RLC is examined from different input functions by using Matlab. A time-varying state-space control model was presented and used to predict the stability and voltages of the RLC series circuit results are shown to validate the method. We find that the effects of changing the resistances and capacitors on the systems are negligent, whereas changing the inductor causes the output to change. The salient features of this algorithm are the inclusion of the parameter variation in the *RLC.*

1. INTRODUCTION

In practice, engineering problems are difficult to solve. Most often, numerical methods are used as analytical solutions to such problems may be non-existent. Numerical methods in themselves are usually iterative in nature requiring several intermediate steps in order to arrive at a solution. An RLC circuit (also known as a resonant circuit or a tuned circuit) is an electrical circuit consisting of a resistor (R), an inductor (L), and a capacitor (C), connected in series or in parallel. An RLC circuit is called a *second-order* circuit as any voltage or current in the circuit can be described by a second-order differential equation for circuit analysis.One very useful characterization of a linear RLC circuit is given by its Transfer Function, which is (more or less) the frequency domain equivalent of the time domain input-output relation. These methods do not use any knowledge of the interior structure of the plant, and as we have seen allows only limited control of the closed-loop behavior when feedback control is used.

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2. STATE-SPACE METHOD

2.1 Definition

The so-called state-space description provide the dynamics as a setoff coupled first-order differential equations in asset of internal variables known as *state* variables, together with a set of algebraic equations that combine the state variables into physical output variables. The concept of the *state* of a non-linear dynamic system [7], refers to a minimum set of variables, known as *state variables* that fully describe the system and its response to any given set of inputs.
This definition asserts that the dynamic behavior of a state-determined system is completely characterized by the response of the set of n variables $xi(t)$, where the number n is defined to be the *order* of the system. The system shown in Fig. 1 has two inputs $u1(t)$ and $u2(t)$, and four output variables$y1(t), \ldots, y4(t)$. If the system is state-determined, knowledge of its state variables($x1(t0), x2(t0), \ldots, xn(t0)$) at some initial time $t0$, and the inputs $u1(t)$ and $u2(t)$ for $t \geq t0$ insufficient to determine all future behavior of the system. The state variables are an *internal* description of the system which completely characterize the system state at any time t, and from which any output variables $yi(t)$ may be computed. Large classes of engineering, biological, social and economic systems may be represented by state-determined system models.

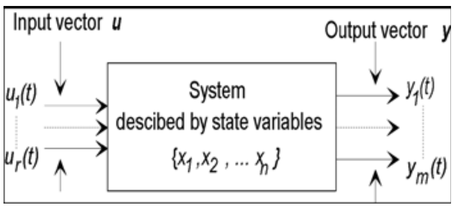

Fig. 1: System inputs and outputs

For such systems the number of state variables, n, is equal to the number of *independent* energy storage elements in the system. The values of the state variables at anytime t specify the energy of each energy storage element within the system and therefore the total system energy and the time derivatives of the state variables determine the rate of change of the system energy. Furthermore, the values of the system state variables at any time t provide sufficient information to determine the values of all other variables in the system at that time. There is no unique set of state variables that describe any given system; many different sets of variables may be selected to yield a complete system description. However, for a given system the order n is unique, and is independent of the particular set of state variables chosen. State variable descriptions of systems may be formulated in terms of physical and measurable variables, or in terms of variables that are not directly measurable. It is possible to mathematically transform one set of state variables to another; the important point is that any set of state variables must provide a complete description of the system.

2.2 The State Equations

A standard form for the state equations is used throughout system dynamics. In the standard form the mathematical description of the system is expressed as a set of n coupled first-order ordinary differential equations, known as the *state equations*, in which the time derivative of each state variable is expressed in terms of the state variables $x_1(t), \ldots, x_n(t)$ and the system inputs $u_1(t), \ldots, u_r(t)$. In the general case the form of the n state equations is:

$$\dot{x}_1 = f_1(\mathbf{x}, \mathbf{u}, t)$$
$$\dot{x}_2 = f_2(\mathbf{x}, \mathbf{u}, t)$$
$$: =:$$
$$\dot{x}_n = f_n(\mathbf{x}, \mathbf{u}, t) \qquad \text{Eq. (1)}$$

where $\dot{x}_i = dx_i/dt$ and each of the functions

It is common to express the state equations in a vector form, the set of n state variables is written as a *state vector* $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$, and the set of r inputs is written as an input vector $\mathbf{u}(t) = [u_1(t), u_2(t), \ldots, u_r(t)]^T$. Each state variable is a time varying component of the column vector $\mathbf{x}(t)$.This form of the state equations explicitly represents the basic elements contained in the definition of a state determined system. Given a set of initial conditions (the values of the x_i at some time t_0) and the inputs for $t \geq t_0$, the state equations explicitly specify the derivatives of all state variables. The value of each state variable at some time Δt later may then be found by direct integration. The system state at any instant may be interpreted as a point in an n-dimensional *state space*, and the dynamic state response $\mathbf{x}(t)$ can be interpreted as a path or trajectory traced out in the state space. In vector notation the set of n equations in Eqs. (1) may be written:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \qquad \text{Eq. (2)}$$

where $\mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ is a *vector* function with n components $f_i(\mathbf{x}, \mathbf{u}, t)$.

For an LTI system of order n, and with r inputs, Eq. (1) becomes a set of n coupled first-order linear differential equations with constant coefficients:

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}u_1 + \ldots + b_{1r}u_r$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}u_1 + \ldots + b_{2r}u_r$$

...

...

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_{n1}u_1 + \ldots + b_{nr}u_r \qquad \text{Eq. (3)}$$

where the coefficients a_{ij} and b_{ij} are constants that describe the system. This set of n equations defines the derivatives of the state variables to be a weighted sum of the state variables and the system inputs.

Equations may be written compactly in a matrix form:

$$\dot{x} = Ax + Bu \qquad \text{Eq. (4)}$$

where the state vector $\mathbf{x}$ is a column vector of length n, the input vector $\mathbf{u}$ is a column vector of length r, $\mathbf{A}$ is an $n \times n$ square matrix of the constant coefficients a_{ij}, and $\mathbf{B}$ is an $n \times r$ matrix of the coefficients b_{ij} that weight the inputs.

2. 3 The Output Equations

A system *output* is defined to be any system variable of interest. A description of a physical system in terms of a set of state variables does not necessarily include all of the variables of direct engineering interest. An important property of the linear state equation description is that all system variables may be represented by a linear combination of the state variables x_i and the system inputs u_i. An arbitrary output variable in a system of order n with r inputs may be written:

$$y(t) = c_1x_1 + c_2x_2 + .. + c_nx_n + d_1u_1 +. . + d_ru_r \qquad \text{Eq (5)}$$

where the c_i and d_i are constants. such equations may be written as:

$$y_1 = c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n + d_{11}u_1 + \ldots + d_{1r}u_r$$
$$y_2 = c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n + d_{21}u_1 + \ldots + d_{2r}u_r$$

...

$$y_m = c_{m1}x_1 + c_{m2}x_2 + \ldots + c_{mn}x_n + d_{m1}u_1 + \ldots + d_{mr}u_r \qquad \text{Eq. (6)}$$

The output equations are commonly written in the compact form:

$$y = Cx + Du \quad \text{Eq. (7)}$$

where **y** is a column vector of the output variables *yi*(*t*), **C** is an *mxn* matrix of the constant coefficients *cij* that weight the state variables, and *D* is an *m x r* matrix of the constant coefficients *dij* that weight the system inputs. For many physical systems the matrix **D** is the null matrix, and the output equation reduces to a simple weighted combination of the state variables:

$$y = Cx. \quad \text{Eq. (8)}$$

2. 4 State Equation Based Modeling Procedure

The complete system model for a linear time-invariant system consists of (i) a set of *n* state equations, defined in terms of the matrices **A** and **B**, and (ii) a set of output equations that relate any output variables of interest to the state variables and inputs, and expressed in terms of the **C** and **D** matrices. The task of modeling the system is to derive the elements of the matrices, and to write the system model in the form:

$$\dot{x} = Ax + Bu \quad \text{state equation} \quad \text{Eq. (9)}$$
$$y = Cx + Du \quad \text{output equation} \quad \text{Eq. (10)}$$

The matrices **A** and **B** are properties of the system and are determined by the system structure and elements. The output equation matrices **C** and **D** are determined by the particular choice of output variables.

3. APPLYING STATE SPACE METHOD ON RLC CIRCUIT

3.1 Series RLC Circuit

Consider the series RLC circuit given below:

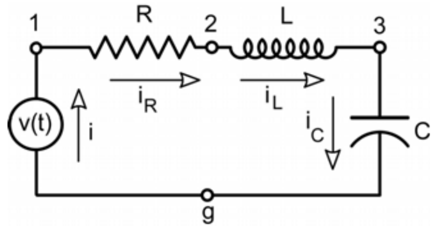

Fig. 2: Series RLC circuit

Table 1: Power Variables

	Across variable	Through variable
Voltage source	known	i
Resistor	V_{12}	i_R
Inductor	V_{23}	i_L
Capacitor	V_{3g}	i_C

We reduced this circuit in the "Big Picture" handout to yield a second order differential equation relating the input v_{1g} and the voltage across the capacitor v_{3g}.

$$V_{1g}=LC(d^2V_{3g}/d^2t)+RC(dV_{3g}/dt)+ V_{3g} \quad \text{Eq. (11)}$$

The state-space representation can be thought of as a partial reduction of the equation list to a set of simultaneous differential equations rather than to a single higher order differential equation. Although the state variables of a system are not unique and definition of many non-physical variables is possible, we will work with physical variables, specifically the energy storage variables of a system. There are two independent energy storages in RLC circuit, the capacitor which stores energy in an electric field and the inductor which stores energy in a magnetic field. The state variables are the energy storage variables of these two elements,V_{3g} and i_L.

The energy storage elements of a system are what make the system dynamic. The flow of energy into or out of a storage element occurs at a finite rate and is described by a differential equation relating the derivative of the energy storage variable (a state variable) to the other power variable of the element. We will first formulate the state equations to find state variables, V_{3g} and i_L. We will then formulate the output equations to calculate all of the remaining unknowing power variables using the input v, and the state variables V_{3g} and i_L.

3.2 Important Points

- Always start a state equation reduction with the elemental equation of an energy storage element. You will have as many state equations as there are independent energy storages in the system.

- Rearrange the energy storage elemental equation to place the derivative of the state variable on the left side by itself.
- Proceed to eliminate all power variables except for the input variable and the state variables.

3.3 State Equations

Inductor Elemental Eq: $V_{23}=L di_L/dt$
Rearrange to put the derivative of the state variable i_L on the left side.

$$di_L/dt=V_{23}/L$$

Eliminate v_{23} because it is neither the input nor a state variable.

$$di_L/dt=1/L(V-V_{12}-V_{3g})$$

Eliminate v_{12} because it is neither the input nor a state variable.

$$di_L/dt=1/L(V-Ri_R-V_{3g})$$

Eliminate i_R because it is neither the input nor a state variable.

$$di_L/dt=1/L(V-Ri_L-V_{3g}) \qquad \text{Eq. (12)}$$

This is a state equation; v is the input, v_{3g} and i_L are state variables.

Capacitor Elemental Eq: $i_C=C dv_{3g}/dt$
Eliminate i_C since it is neither the input nor a state variable.

$$dv_{3g}/dt=i_L/C \qquad \text{Eq. (13)}$$

This is a state equation; v_{3g} and i_L are state variables.

Vector-Matrix Form of the State Equations: Now write the state equations in vector matrix form.
The general vector-matrix form of the state equations is:

$$d/dt=Ax+Bu \qquad \text{Eq. (14)}$$

where **x** is the vector of state variables, called the "state vector", and **u** is the vector of inputs. In this example, we have just one input, so u would be a single function, not a vector of functions.

We can also write as:

$$\frac{d}{dt}\begin{bmatrix} i_L \\ V_{3g} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & \frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_l \\ V_{3g} \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} V$$

3.4 Output Equations

We know the source v and the state variables v_{3g} and i_L as functions of time, then we can calculate every unknown power variable as functions of time.

$$i=i_C=i_R=i_L \qquad \text{Eq. (15)}$$

The output equation for the voltage drop across the resistor requires a substitution.

$$V_{12}=Ri_R \qquad \text{Eq. (16)}$$

Vector-Matrix Form of the output Equations: The general vector-matrix form of the output equations is:

$$Y=Cx+Du \qquad \text{Eq. (17)}$$

where **y** is the vector of output variables, **x** is the vector of state variables and **u** is the vector of inputs.

We can also write as:

$$\begin{bmatrix} i_C \\ i_R \\ i \\ v_{12} \\ v_{23} \end{bmatrix} = \begin{bmatrix} i_L \\ V_{3g} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ R & 0 \\ -R & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} V$$

The state equations are coupled simultaneous first order differential equations. They are first order differential equations since the vector **x** of state variables is differentiated with respect to time. They are coupled simultaneous equations because they represent the response of a physical system in which the state variables and the input determine the future state of the system.

3.5 Parallel RLC Circuit

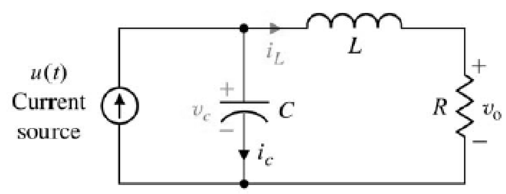

Fig. 3: parallel RLC circuit

By KCL and KVL [6], we have that:

$$L\frac{di_L}{dt} = v_C - i_L R \qquad \text{Eq. (18)}$$

Take

$$i_C = C\frac{di_L}{dt} = -i_L + u(t) \qquad \text{Eq. (19)}$$

$$x_1 = v_c; x_2 = i_L; y = v_0 \qquad \text{Eq. (20)}$$

Then the state equation becomes

$$x_1^{\cdot} = -\frac{1}{C}x_2 + \frac{1}{C}u$$
$$x_2^{\cdot} = -\frac{1}{L}x_1 + \frac{R}{L}x_2$$

The output equation

$$y = i_R(t) = \frac{1}{R}v_C = \frac{1}{R}x_1 \qquad \text{Eq. (21)}$$

In matrix form

$$\begin{bmatrix} x_1^{\cdot} \\ x_2^{\cdot} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ \frac{1}{L} & \frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

3.6 Stability

We can also find the stability of the system by the position of the poles. First putting the values of R,L,C in the matrix A and using the equation which is stated below.

$$\lambda I - A \qquad \text{Eq. (22)}$$

Take determinant and plot the graph. If the poles are at right side then it means that the system is unstable at those values.

4. THE MATLAB

In practice, engineering problems are difficult to solve. Most often, numerical methods are used as analytical solutions to such problems may be non-existent [1]. Numerical methods in themselves are usually iterative in nature requiring several intermediate steps in order to arrive at a solution. This means much time will be needed and more energy expended in an effort to solve a particular problem. In order to lessen these difficulties, there is a need to develop computer algorithms that can solve the problem at hand. Here again, the user is posed with the problem of developing a computer program either in BASIC, FORTRAN, or C++ [2]. These programming languages are high level languages which require the user to have expert knowledge and ability to develop subroutines that can handle matrix inversion, determinant, curve-fitting, matrix multiplication, and graphics [3]. MATLAB is a software package for high performance computation and visualization. The combination of analysis capabilities, flexibilities, reliability and powerful graphics makes MATLAB the premier software package for engineers and scientists [4]. MATLAB provides an iterative environment with more than hundreds of reliable and accurate built-in mathematical functions. These functions provide solutions to a broad range of mathematical problems including: Matrix Algebra, Complex Arithmetic, Linear Systems, Differential Equations, Signal Processing, Optimization and other types of scientific computations [5]. In this paper, the need for MATLAB as a pedagogical tool in engineering research and teaching is highlighted

4.1 Response of RLC Circuit through MATLAB

Coding:

```
function rlc(R,L,C)
A = [0 1/C; -1/L -R/L];
B = [0; 1/L];
C = [1 0];
D = [0];
% Create a Matlab state-space model.
sys = ss(A, B, C, D);
close 'all';
% Compute and plot the step response.
figure;
[y, t] = step(sys);
plot(t, y);
title('Step Response');
% Compute and plot the impulse response.
figure;
[y, t] = impulse(sys);
plot(t, y);
title('Impulse Response');
% Compute and plot the zero-state response to a
sinusoidal input.
figure;
dt = 0.002;
tf = 4;
t = 0 : dt : tf;
```

```
u = sin(t);
[ysine, t] = lsim(sys, u, t);
plot(t, ysine);
title('Sine Response');
```

4.2 Graphs

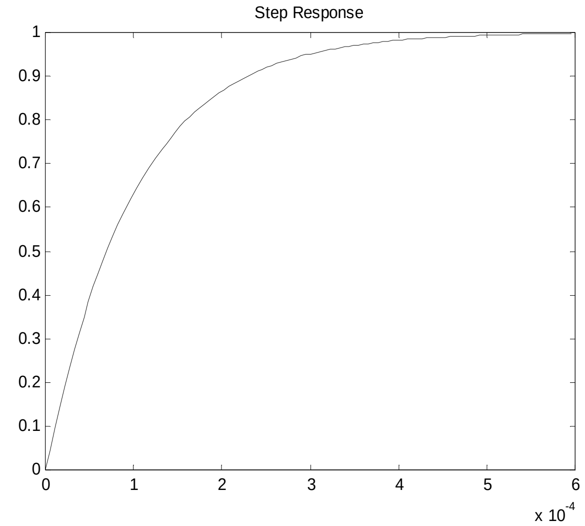

Fig.4: Step Response

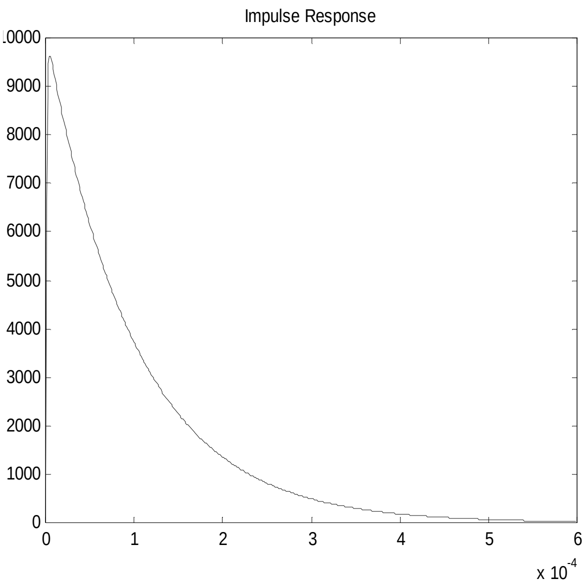

Fig.5: Impulse Response

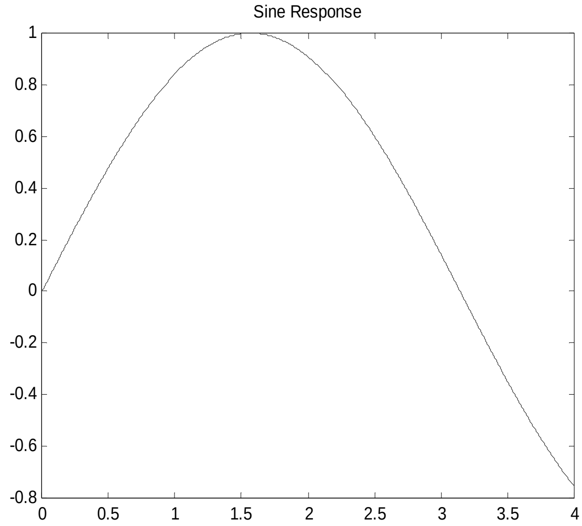

Fig. 6: Sine Response

5. CONCLUSION AND FUTURE WORK

In this paper we have concluded that using state space method we can easily find the response and stability of the RLC circuit and also with the help of MATLAB the analysis of an RLC circuit becomes too simpler.

The future work is to make such algorithm which is effective and more accurate to calculate the response and analysis of an RLC circuit.

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Mohazzab Javed doing B.E in Electronic Engineering from PAF-KIET.During 2003-2005,I done my intermediate from SAINT PATRIC COLLEGE.I am interested in research works.

Hussain Aftab doing B.E in Electronic Engineering from PAF-KIET.During 2003-2005,I had done my intermediate from GOVT.COLLEGE FORMEN. I wish to move forward in the field of INDUSTRIAL CONTROL SYSTEMS.

Muhammad Qasim doing B.E in Electronic Engineering from PAF-KIET.During 2003-2005,I had done my intermediate from FAZAIA DEGREE COLLEGE.My aim is to work on ROBOTICS.