Accelerating Convergence of the Frank-Wolfe Algorithm for Solving the Traffic Assignment Problem

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Summary
The Frank-Wolfe method (FW) is one of the most widely used algorithm for solving routing problems in the telecom and traffic areas, its popularity is attributed to its simplicity and modest memory requirements. However, FW converges very slowly, that makes it less favourable to use without modifications. To improve its performance, different modifications of the Frank-Wolfe method have been suggested. In this paper we describe a new modified version (FWFλ), the algorithm consists to combine Fukushima direction (FWF), with a widened line search technique (FWλ). We also present preliminary computational studies in a C++Builder5, in these we apply (FW), (FWλ), (FWF), and (FWFλ) methods to some Traffic assignment problems. The computational results indicate that the proposed algorithm yield satisfactory results within reasonable computational time comparing to the other methods

Key words:

1. Introduction
The Frank-Wolfe method [4] was first introduced in quadratic programming at once it proved very effective for the resolution of large scale flood problems, the Frank-Wolfe method, is famous for its advantages: it is easy to implement and it performs well far from the optimal solution. However, it has a property that makes it less favourable to use without modifications, namely that it shows very slow asymptotic convergence due to that the feasible solutions tend to zig-zag towards the optimal solution. To improve its performance, different modifications of the Frank-Wolfe method have been suggested, starting with the first "L. J. Leblanc" works [8] until the recent works of "Ziyou Gao & Al" [18].

On our behalf, in this paper we propose a new improvement of the FW method (FWFλ) for solving the traffic assignment problem, this modification consist to combine Fukushima direction [6] (FWF), with a widened line search technique [11] (FWλ).

The main proponent of this attempt is to avoid the zig-zagging in the path described by the solution points of the pure FW method.

For the remainder, Section 2 reviews the FW method and the modified versions: the FWF and FWλ. Section 3 is devoted to the new modified version, we present the possibility of combining the two alternatives presented above, to be able to imbricate them in the FW algorithm, we choose the best possible combination which can join together the maximum effectiveness to FW algorithm and preserve its convergence.

Section 4 establishes the convergences of the modifications cited. Finally, implementation issues and numerical results are discussed in section 5.

2. The Frank-Wolfe Method and modified versions

In this section, we consider the following non-linear programming problem (P):

\[
(P) \quad \min_{x \in X} f(x)
\]

Where \( f : X \rightarrow \mathbb{R} \) is a twice continuously differentiable convex function, \( X \subset \mathbb{R}^n \) is a compact convex polyhedral set.

By continuity of \( f \), (P) has a solution, unique if \( f \) is strictly convex.

2.1 The Frank-Wolfe method

We can summarize the FW algorithm as follows:
At iteration \( k \), FW approximates \( f \) by linearizing at the current iteration \( x^k \), giving an affine minorant \( f_k \) to \( f \):

\[
f_k(y) = f(x^k) + \nabla f(x^k)(y - x^k)
\] (1)
The way taken by the FW algorithm points the widened step effect.

First step: the determination of the search direction, which is done by a linear program:

\[
LP(k) = \min_{y \in X} f_k(y)
\]

Let \( y^k \) the optimal solution of \( LP(k) \). The direction of Frank-Wolfe is then defined by:

\[
d^k = y^k - x^k
\]

In the traffic application, \( LP(k) \) decomposes into a set of shortest path problems [14].

Second step: In this phase, the objective function is minimized along the line segment passing by the point \( x^k \) and the direction \( d^k \). The point obtained from the one-dimensional minimization is \( x^{k+1} \). Analytically this means that:

\[
f(x^{k+1}) = \min_{0 \leq a \leq 1} f(x^k + ad^k)
\]

Convergence test: It is defined from the variation of the objective function between two successive iterations. Let:

\[
LBD_k = f_k(y^k) \quad \text{a lower bound at iteration } k.
\]

\[
UBD_k = f(x^{k+1}) \quad \text{a upper bound at iteration } k.
\]

\[
LBD \quad \text{is the best available lower bound.}
\]

The idea with FW is to iteratively shrink the gap between the so far best found lower bound and the upper bound from the last iteration, until they are sufficiently close to each other. The relative gap at iteration \( k \) can then be defined as [1]:

\[
\frac{UBD_k - LBD}{LBD}
\]

So, the algorithm will stop if the relative gap is less then \( \varepsilon \), \( \varepsilon > 0 \).

2.1.1 Convergence of FW method

Theorem 1 The Frank-Wolfe method iterative process for the resolution of the problem \( (P) \), can be completely described by the following algorithmic map:

\[
A : IR^* \rightarrow IR^*
\]

\[
x^i \rightarrow A(x^i) = x^{i+1} \text{ for } k \in IN^* \text{ with } A = MD
\]

The FW algorithmic map \( A \) is composed of two maps:

The first one is the direction research map:

\[
D : IR^* \rightarrow IR^{2*}
\]

\[
x^i \rightarrow D(x^i) = \{x^k, d^k \} / d^k = y^k - x^k
\]

\( y^k \) the solution of the sub problem \( LP(k) \).

The other one is the line search map:

\[
M : IR^{2*} \rightarrow IR^*
\]

\[
(x^k, d^k) \rightarrow M(x^k, d^k) = \{x^{k+1} / f(x^{k+1}) = \min_{0 \leq a \leq 1} f(x^k + ad^k)\}
\]

Lemma 1 The FW algorithmic map \( A \) is closed.

Lemma 2 Let suppose that the feasible area \( X \), has \( p \) nods remotely finished: \( y^1, ..., y^p \).

Let \( x^i \) a starting feasible solution of the FW algorithm applied to the problem \( (P) \), then \( \{x^k\}^\infty_{k=i} \subset C = Co(x^i, y^1, ..., y^p) \), \( C \) is compact and \( C \subset X \).

I.e. the sequence of the points generated by the FW algorithm is contained in the convex closing envelop of \( X \).

To establish the FW convergence, we will apply the Convergence Theorem [17]:

Theorem 2 (Convergence Theorem)

Let \( A \) be an algorithm on \( X \), \( Z \) the evaluation function, \( \Omega \subset X \) the solution set, let the sequence \( \{x^k\}^\infty_{k=i} \) generated by the FW algorithm for the resolution of the problem \( (P) \), and suppose:

i) All points \( x^k \) are contained in a compact \( S \subset X \).

ii) There is a continuous function \( Z \) on \( X \) such that:

\[
\text{if } x \notin \Omega \text{ then: } Z(y) < Z(x) \text{ for all points } y \in A(x).
\]

\[
\text{if } x \in \Omega \text{ then: } Z(y) \leq Z(x) \text{ for all points } y \in A(x).
\]

iii) The mapping \( A(x) \) is closed at points outside \( \Omega \).

Then the limit of any convergent subsequence of \( \{x^k\}^\infty_{k=i} \) is a solution

2.2 The widened step technique

This technique consists in using, during a certain iteration count a descent step larger than the one obtained from one-dimensional optimization, while keeping the basic movement directions. It results a way in wider zigzag thus from it, where the algorithm points would advance more quickly towards the optimal solution (see Fig. 1):
2.2.1 Presentation of the FW\(\lambda\) method [11]

We will introduce this alternative in the form of mixed algorithm:

Let:

\[ A_k : IR^n \rightarrow IR^n \] the algorithmic map of the FW method.

\[ B : IR^n \rightarrow IR^n \] the modified version map.

With for \(k \in K \cap K^0\), \(|K^0|\) finished \(B = A_k\).

And for all \(k \in K^0\), the following additional stages are required in the pure FW algorithm:

1- Set \(\gamma_k = \lambda_k \alpha_k\) ; \(\lambda_k\) is the step descent multiplier to the iteration \(k\), with \(\lambda_k > 1\).

2- Set \(\beta_k = \min (\gamma_k, 1)\).

(This condition let us in the feasible area).

3- Let \(x^{k+1} = x^k + \beta_k d^k\).

If \(f(x^{k+1}) < f(x^k)\) set \(x^{k+1} = x^{k+1}\).

else set \(x^{k+1} = x^k + \alpha_k d^k\).

(If by modifying the descent step, we don’t improve the objective function, we take \(\lambda_k = 1\)).

2.3 Descente direction modification

In this part, we present another modified version of the FW algorithm, it concerned the Fukushima method [6], we label it by FWF.

2.3.1 Presentation of the FWF method

We suppose that the optimal solution belongs to a face “\(S\)” of the feasible area (we show that the convergence is less slow if the optimum is an interior point of the feasible area [7]).

\(S\) is a convex polyhedron; therefore \(S\) can be represented by the convex envelope of a finished number of extreme points from the feasible area \(y^1, y^2, ..., y^n\).

Let \(x^*\) the optimal solution of the problem;

\(x^* \in S, \) thus there exists \(\delta_i, i = 1, ..., n\) such as:

\[ x^* = \sum \delta_i y^i \] with \(\sum \delta_i = 1, \delta_i \geq 0 \) for \(i = 1, ..., n\)

These enable us to write \(x^*\) in the form:

\[ x^* = x^k + \alpha_k d^k / \alpha_k = 1 \]

And

\[ d^k = (\sum \delta_i y^i) - x^k \] \((*)\)

As it is practically impossible to determine with exactitude the \(y^i\) and the \(\delta_i\), however, we note that the FW algorithm generates \(y^i\) as solutions of sub-problems LP(i), which let us think that a convex combination of a certain number of the \(y^i\) previously generated by the algorithm could be used to approximate \((*)\), and thus provide, a more refined
direction than that of FW, which uses the \(y^i\) the most recently generated.

Moreover, in order to guarantee a maximum effectiveness of the modified algorithm, the two directions are compared in term of directional derivative, and those of the smallest value are selected to be the current descent direction.

The FWF can be summarized in the following steps:

\textit{Step0.} Let \(x^1\) a starting feasible solution, and let \(\ell\) a positive integer. Set \(k = 1\).

\textit{Step1.} Resolve the sub-problem LP\((k)\), let \(y^k\) the optimal solution.

\textit{Step2.} If \(\forall f(x^k)(y^k - x^k) = 0\) stop, else go to \textit{Step3}

\textit{Step3. Let \(\mu_i, i = k-q, ..., k\) chosen such as:}

\[ \sum_{i=k-q}^{k} \mu_i = 1, \mu_i \geq 0 \forall i = k-q, ..., k \]

Set:

\[ w^k = (\sum \mu_i y^i) - x^k \]

Such as \(q = \min \{k, \ell\} - 1\),

And

\[ w^k = y^k - x^k \]

\textit{Step4. Calculate the directional derivatives:}

\[ \gamma^k_1 = \nabla f(x^k) y^k / \| y^k \| \] \((6)\)

\[ \gamma^k_2 = \nabla f(x^k) w^k / \| w^k \| \] \((7)\)

Put:

\[ d^k = \begin{cases} v^k \text{ if } \gamma^k_1 < \gamma^k_2 \\ w^k \text{ else} \end{cases} \]

\((8)\)

\textit{Step5. Resolve the one-dimensional problem:}

\[ \min_{\delta_k \in \mathbb{R}} f(x^k + \alpha d^k) \]

Let \(\alpha_k\) its optimal solution.

Set \(x^{k+1} = x^k + \alpha_k d^k, k = k+1\), return to \textit{Step1}.

2.4 The combined method, FW\(\lambda\)

Now, we study the possibility of combining the two methods presented above, to be able to imbricate them in the FW algorithm.

We will choose the best possible combination which can join together the maximum effectiveness to FW algorithm; for a matter of convergence, this alternative must have the mixed algorithm form, which consequently leaves us two possibilities to consider:
• First possibility: Consider (FWF) as the basic algorithm, and the descent step technique as the modification to be incorporated.

• Second possibility: Taking (FWF) as the basic algorithm, and Fukushima’s modification like compensation measures.

As the (FWF) method acts on the descent directions which are responsible of the zigzag phenomenon; seen as this alternative equivalent to the FW method, in most unfavourable cases, which represents a certain guarantee of improvement, contrary to (FWF) which rests on a heuristic technique. Thus, the first possibility is the best adapted to design a mixed algorithm.

2.4.1 Presentation of the FWF\(\lambda\) method

In this algorithm, the widened step technique will be apply on the « \(\ell\) » first iterations, where the (FW) direction is preserved, considering the number « \(\ell\) » of \(y^j\) used in Fukushima modification is not reached yet. Then, we use the (FWF) algorithm:

\textbf{Step0.} (Initialisation), Let \(x^j\) a starting feasible solution, set \(k=1\).

\textbf{Step1.} (FWF\(\lambda\) direction determination), Calculate the search direction of Fukushima “ \(d^k\) “.

\textbf{Step2.} (Line search) Find the step \(\alpha_k\), which minimizes the objective: \(\min \left\{ f(x^k + \alpha d^k) \right\}\) by application of the (FWF) technique, if \((k \leq \ell)\) by application of the (FWF) method, else.

\textbf{Step3.} if a stopping condition is satisfied stop, else \(k = k+1\); return to \textbf{Step1}.

3. Convergence of the modified versions

Consider that: \(B: IR^\times \rightarrow IR^\times\) the algorithmic map of the modified versions. The convergence of the modified versions rises from the convergence theorem of the mixed algorithms.

3.1 The alternative (FWF)

It is thus enough to check the following conditions of convergence:

1. For any \(k\in K, f(x^{k+1}) \leq f(x^k)\)
2. For any \(k\in K, x^k \in X\), where \(X\) is compact.
3. Let \(x^*\in \Omega\), and let \(y\) a point of the algorithm; if \(f(y) \leq f(x^*)\), then \(y \in \Omega\).

3.2 The alternative (FWF)

Suppose that the \(\mu_i, i = k-q, \ldots, k\) are the continuous functions of \(y^{k-q}, \ldots, y^k\) and \(x^j\).

I.e. \(\mu_i = \mu_i(y^{k-q}, \ldots, y^k, x^j)\), for any \(k\) and \(j\neq 0\).

Then, the FWF\(\lambda\) algorithm is finishes in an optimal solution or generates an infinite continuation, of which any accumulation point is an optimal solution.

Thus, the Frank-Wolfe modifications proposed are globally convergent.

4. Computational Results

The main areas of the FW method applications and its modifications are traffic assignment problem; this last can be formulated as an optimization program with a nonlinear objective function and linear constraints [14]. Consider an urban traffic network represented as a graph \(G(N, A)\) where \(N\) and \(A\) are the sets of nodes and links, respectively. \(O\) is the set of origins and \(D\) is the set of destinations. The user-equilibrium traffic assignment problem can be stated as:

\[
\min z = \sum_{\omega \in \mathcal{D}} t_\omega(\omega) d \omega
\]

\[
\sum_{p \in \omega} f_p \delta_{p,\omega} = q_{\omega d} \quad \forall \omega \in O, d \in D
\]

\[
f_p \geq 0 \quad \forall p \in C_{\omega d}, \forall \omega \in O, d \in D
\]

\[
x_a = \sum_{\omega \in \mathcal{D}} \sum_{p \in \omega} f_p \delta_{a, p} \quad \forall a \in A
\]

Where:

\(x_a\) the total flow on link \(a\)

\(t_\omega(\omega)\) link cost function which is continuously differentiable and convex

\(q_{\omega d}\) the total traffic demand between \(o\) and \(d\)

\(f_p\) the flow on path \(p\)

\(C_{\omega d}\) the set of paths connecting \(o\) and \(d\)

\(\delta_{a, p}\) the path-link incidence matrix.

\(\delta'_{p, \omega}\) the path-pair\((\omega, d)\) incidence matrix.

To demonstrate the efficiency of the new algorithm (FWF\(\lambda\)), in this section we present the results of the numerical examples in which the three algorithms: FW, FW\(\lambda\), FWF\(\lambda\) and the FW algorithm, are applied for solving some traffic assignment problems. In this study, the computer programs are coded in C++Builder5.

• Dijkstra’s algorithm and Golden section method were used respectively for the resolution of the shortest path sub-problems and the one-dimensional problems.
The starting feasible solution is obtained by application of the all-or-nothing method.

The strategies of the descent step modification were realized by repeated tests. The most representative strategies are:

For \( N \leq 25 \), \( \lambda_k = 1.6 \) \( K_0 = \{k \in K / k \leq 5\} \)

For \( N > 25 \), \( \lambda_k = 1.4 \) or 1.5 \( K_0 = \{k \in K / k \leq 5\} \) or \( \{k \in K / k \leq 10\} \)

\( \mu_i = \frac{1}{q_i}, \forall i \).

\( \ell = |K_0| \).

The convergence test is defined as:

\[
\frac{UBD_i - LBD}{LBD} < 10^{-4}
\]

4.1 First test

A first test was applied on a real problem suggested by L. J. Leblanc [8], which considers the Sioux Falls network (South of Dakota; USA), composed of 24 nodes and 76 arcs.

We applied the following strategy:

\( K_0 = \{k \in K / k \leq 10\} \) and \( \lambda_k = 1.5 \) for any \( k \in K_0 \).

On this example (see Table 1), we can note that the FWF\( _{\lambda} \) algorithm is definitely higher than the FW method:

Table 2: Comparison between FW, FW\( _{\lambda} \), FWF and FWF\( _{\lambda} \).

The global comparison is definitely favourable to FWF\( _{\lambda} \). The results obtained indicate that:

- FWF\( _{\lambda} \) gives a very good approximation of the optimum.
- The superiority of FWF\( _{\lambda} \) is very remarkable. It reduces on the average more than 85% the iteration count required by the FW method, and more than 55% these required by FWF method.
- We also record, a good performance of FWF and FW\( _{\lambda} \).
- We also note that for some problems FW\( _{\lambda} \) is higher in relation to FWF.
According to the results obtained, we noted that the advantage of FWF\(\lambda\) becomes more significant in general, when the size of the network increases, and the required precision is finer, and for feasible starting solutions enough far from the optimum.

Fig 2 illustrates the improvement effect carried through the evaluation of the objective function during the first iterations of the first problem:

5. Conclusion

The FW method is widely applied for solving the urban traffic assignment problem, particularly for the user equilibrium assignment problem, which has a special structure that consists of an iterative process of linear programming. However, the FW algorithm converges very slowly when iterations are closing to the optimal solution. In order to expedite the rate of convergence of the Frank-Wolfe algorithm, the paper has specified a new algorithm for solving the traffic assignment problem. A numerical example has been given to demonstrate the effectiveness of the proposed algorithm. The computational results indicate that the performance of the FWF\(\lambda\) is better than the pure FW and the other methods discussed. Finally, we have here, at least, in the case of the FW method a thesis confirmation on which the new algorithm FWF\(\lambda\) was conceived and which is: The Combination of certain techniques of acceleration the convergence rate within an algorithm generates further improvement, even to define the good compatibilities conditions.

References