Stability analysis for stochastic Cohen-Grossberg neural network with distributed delays and reaction diffusion terms

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Summary
This paper mainly deals with the almost surely exponential stability and exponential $p$-th moment stability for a class of stochastic Cohen–Grossberg neural networks with distributed delays and reaction–diffusion term. By constructing suitable Lyapunov functional, employing the nonnegative semi-martingale convergence theorem and applying matrix theory and stochastic analysis technique, two delay-independent and easily verifiable sufficient conditions are obtained to ensure the existence, uniqueness, almost surely exponential stability and exponential $p$-th moment stability of the equilibrium point for the addressed stochastic Cohen-Grossberg neural network with distributed delays and reaction-diffusion terms.

Keywords:
Cohen–Grossberg neural network; almost surely exponential stability; exponential $p$-th moment stability.

1. Introduction
The Cohen–Grossberg neural network, first proposed and studied by Cohen and Grossberg in 1983 [1], has attracted considerable attention due to its potential applications in classification, parallel computing, associative memory, signal and image processing, especially in solving some difficult optimization problems. In such applications, it is of prime importance to ensure that the designed neural networks are stable [2]. In implementation of neural networks, however, time delays are unavoidably encountered due to the finite switching speed of neurons and amplifiers. It has been found that, the existence of time delays may lead to instability and oscillation in a neural network. Therefore, stability analysis of Cohen–Grossberg neural network with time delays has received much attention [3-10].

Excepting delay effects, strictly speaking, diffusion effects cannot be avoided in the neural networks when electrons are moving in asymmetric electromagnetic fields. So we must consider that the activations vary in space as well as in time. In [11-14], the authors have considered the stability of neural networks with diffusion terms, which are expressed by partial differential equations.

In addition to the delay effects, stochastic effects constitute another source of disturbances or uncertainties in real systems [10]. A lot of dynamical systems have variable structures subject to stochastic abrupt changes, which may result from abrupt phenomena such as stochastic failures and repairs of the components, changes in the interconnections of subsystems or sudden environment switching [15]. Therefore, stochastic perturbations should be taken into account when modeling neural networks. In recent years, the dynamic analysis of stochastic systems (including neural networks) with delays has been an attractive topic for many researchers, and a large number of stability criteria of these systems have been reported [10, 15-20]. Particularly, in [15-16], the authors have considered the exponential $p$-stability of stochastic differential equations with constant delays and obtained several stability conditions for checking the exponential $p$-stability. In [17-20], the problem on stability of stochastic neural networks with constant delays or time-varying delay or bounded distributed delays has been considered and many interesting results have been established by employing a Lyapunov functional approach. To the best of our knowledge, so far, few authors have considered the problem of stability analysis for Cohen–Grossberg neural networks with both distributed delays and reaction-diffusion terms in the simultaneous presence of stochastic effects.

In this paper, we investigate the almost surely exponential stability and exponential $p$-th moment stability for stochastic Cohen–Grossberg neural network with continuously distributed delays and reaction-diffusion terms.

2. Model description and preliminaries
In this paper, we consider the following stochastic Cohen–Grossberg neural network with continuously distributed delays and reaction-diffusion terms.
Let $L^2(X)$ be the space of real Lebesgue measurable functions on $X$. It is a Banach space for the $L_2$-norm
\[\|u\|_2 = \left(\int_X |u(x)|^2 \, dx\right)^{1/2},\]
where $|u|$ denotes the Euclidian norm of a vector $u \in \mathbb{R}^n$ for any integer $n$. The norm $\|u\|$ is defined by
\[\|u\| = \left(\sum_{i=1}^n |u_i|^p\right)^{1/p}, \quad p \geq 1.\]
Note that, $\bar{\xi} = \left(\bar{\xi}_1(s,x), \ldots, \bar{\xi}_n(s,x)\right)^T : s \leq 0$ is $C([-\infty,0] \times \mathbb{R}^m; \mathbb{R}^n)$ -valued function and $F_0$ measurable $\mathbb{R}^n$ -valued random variable, where, for example, $F_0 = F_0$ on $[-\infty,0]$, and $C([-\infty,0] \times \mathbb{R}^m; \mathbb{R}^n)$ is the space of all continuous $\mathbb{R}^n$ -valued functions defined on $[-\infty,0] \times \mathbb{R}^n$.

Furthermore, model (1) comprises the following Cohen–Grossberg neural network model without stochastic effects
\[du_i(t,x) = \sum_{k=1}^n \frac{\partial}{\partial x_k} \left(D_i \frac{\partial u_i(t,x)}{\partial x_k}\right) dt - \alpha_i(u_i(t,x))\beta_i(u_i(t,x)) - \sum_{j=1}^n a_{ij} f_j(u_j(t,x)) - \sum_{j=1}^n b_{ij} \int_{-\infty}^t K_j(t-s) g_j(u_j(s,x)) ds + J_i dt, \quad x \in X,\]
for $i=1,2,\ldots,n$ and $t \geq 0$. In the above system, $n \geq 2$ is the number of neurons in the network, $x_i$ is space variable, $u_i(t,x)$ is the state variable of the $i$-th neuron at time $t$ and in space $x$, $f_j(u_j(t,x))$ and $g_j(u_j(t,x))$ denotes the output of the $j$-th unit at time $t$ on the $i$-th unit and in space $x$, smooth function $D_i = D_i(t,x) \geq 0$ is diffusion operator, $X$ is a compact set with smooth boundary $\partial X$ and measure $X > 0$ in $\mathbb{R}^m$. $\alpha_i(u_i(t,x))$ represents an amplification function; $\beta_i(u_i(t,x))$ is an appropriately behaved function at time $t$; $\xi_i(t,x)$ is the initial boundary value. $a_{ij}, b_{ij}$ and $J_i$ are constants: $a_{ij}$ indicates the strength of the neuron interconnections within the network at time $t$; $b_{ij}$ weights the strength of the $j$-th unit on the $i$-th unit at time $t-s$; $K_j$ is the delay kernel function; $J_i$ denotes the constant input from outside of the network. Moreover, $\omega(t) = (\omega_1(t), \ldots, \omega_n(t))^T$ is $n$ dimensional Brownian motion defined on a complete probability space $(\Omega, F, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ generated by $\{\omega(s) : 0 \leq s \leq t\}$, where we associate $\Omega$ with the canonical space generated by all $\{w_i(t)\}$, and denote by $F$ the associated $\sigma$-algebra. Generated by $\{w(t)\}$ with the probability measure $P$.
Throughout this paper, for system (1), we have the following assumptions:

(A1) \( f_j, g_j \) and \( \sigma_j \) are Lipschitz continuous with Lipschitz constant \( \mu_j > 0, \varphi_j > 0 \) and \( L_j > 0 \), respectively, for \( i,j = 1,2,...,n \).

(A2) The delay kernel \( K_{ij} : [0, +\infty) \to [0, +\infty) \) is a real-valued non-negative continuous function and satisfies

\[
\int_0^{+\infty} e^{\delta s} K_{ij}(s) ds = r_{ij}(\lambda),
\]

where \( r_{ij}(\lambda) \) is continuous function on \( [0, \delta) \), \( \delta > 0 \) and \( r_{ij}(0) = 1 \), \( i,j = 1,2,...,n \).

(A3) \( \sigma_{ij}(u^*_j) = 0 \), where \( u^*_i = (u^*_i, u^*_2,..., u^*_n)^T \) is the equilibrium point of model (2).

(A4) Each function \( \alpha_i(u) \) is bounded, positive and continuous, i.e. there exist a constant \( \bar{\alpha}_i \) such that

\[
0 < \alpha_i(u) \leq \bar{\alpha}_i < +\infty,
\]

for \( u \in R, i = 1,2,...,n \).

(A5) There exists a positive diagonal matrix \( \beta = \text{diag}(\beta_1, \beta_2,..., \beta_n) \) such that

\[
\frac{\beta_i(u) - \beta_i(v)}{u - v} \geq \beta_i
\]

for all \( u, v \in R \ (u \neq v), i = 1,2,...,n \).

**Definition 1** Model (3) is said to be almost surely exponentially stable if there exists a positive constant \( \lambda \) such that for each pair of \( t_0 \) and \( \xi \) there is a positive finite random variable \( K \) such that

\[
\|u(t; t_0, \xi) - u^*\|^p \leq Ke^{-\lambda(t-t_0)}, \quad P - a.s.
\]

for all \( t \geq t_0 \). In this case

\[
\lim_{t \to \infty} \frac{1}{t} \log(\|u(t; t_0, \xi) - u^*\|^p) \leq -\lambda,
\]

(4)

The left hand-side of (4) is called the almost sure Lyapunov exponent of the solution.

**Definition 2** Model (3) is said to be \( p \)-th moment exponentially stable if there exist a pair of positive constants \( \lambda \) and \( K \) such that

\[
E\|u(t; \xi) - u^*\|^p \leq KE\|\xi - u^*\|pe^{-\lambda t}, \quad t \geq 0
\]

for any \( \xi \). In this case

\[
\lim \sup_{t \to \infty} \frac{1}{t} \log(E\|u(t, \xi) - u^*\|^p) \leq -\lambda.
\]

(5)

The left hand-side of (5) is called the \( p \)-th moment Lyapunov exponent of the solution. When \( p = 2 \), it is usually called the exponential stability in mean square.

**Definition 3** [9] A map \( H : R^n \to R^n \) is a homeomorphism of \( R^n \) onto itself, if \( H \in C^0 \), \( H \) is one-to-one, \( H \) is onto and the inverse map \( H^{-1} \in C^0 \).

To prove our results, the following lemmas are necessary.

**Lemma 1** [17] Let \( A(t) \) and \( U(t) \) be two continuous adapted increasing processes on \( t \geq 0 \) with \( A(0) = U(0) = 0 \), Let \( M(t) \) be a real-valued continuous local martingale with \( M(0) = 0 \), a.s. Let \( \zeta \) be a nonnegative \( F_0 \)-measurable random variable with \( E\zeta < \infty \). Define

\[
X(t) = \zeta + A(t) - U(t) + M(t) \quad \text{for} \quad t \geq 0.
\]

If \( X(t) \) is nonnegative, then

\[
\left\{ \lim_{t \to \infty} A(t) < \infty \right\} \subset \left\{ \lim_{t \to \infty} X(t) < \infty \right\} \cap \left\{ \lim_{t \to \infty} U(t) < \infty \right\} \quad \text{a.s.,}
\]
When $B \subset D$ a.s. denotes $P(B \cap D^c) = 0$. In particular, if $\lim_{t \to \infty} A(t) < \infty$ a.s., then for almost all

$\omega \in \Omega$, $\lim_{t \to \infty} X(t, \omega) < \infty$ and $\lim_{t \to \infty} U(t, \omega) < \infty$, i.e., both $X(t)$ and $U(t)$ converge to finite random variables.

**Lemma 2** For $a \geq 0, b_k \geq 0, (k = 1, \ldots, m)$, the following inequality holds

$$a \prod_{k=1}^{m} b_k^{q_k} \leq \frac{1}{r} \sum_{k=1}^{m} q_k b_k^{r} + \frac{1}{r} a^{r},$$

where $q_k > 0, (k = 1, \ldots, m)$, $\sum_{k=1}^{m} q_k = r - 1$ and $r \geq 1$.

**Lemma 3** If $H(x) \in C^0$ satisfies the following conditions

(i) $H(x)$ is injective on $R^n$,

(ii) $\|H(x)\| \to +\infty$ as $\|x\| \to +\infty$, then $H(x)$ is homeomorphism of $R^n$ onto itself.

### 3. Main results

In this section, we will give several sufficient conditions on the existence, uniqueness, almost surely exponential stability and exponential p-th moment stability of the equilibrium point for the stochastic Cohen-Grossberg neural network (1).

**Theorem 1** If model (3) satisfies the assumptions (A1)-(A5), and

(A6) There exist constants $\rho_{k,j} \in R, q_k > 0, i,j = 1, \ldots, n, k = 1, \ldots, m+1$; such that

$$\frac{\alpha}{2} \left[ r \beta_i - \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{m} q_k \mu_j q_k - \sum_{j=1}^{n} a_{ij} \mu_j^{\rho_{j+1,i,j}} \right]$$

$$- \sum_{j=1}^{n} p_{ij}^{2} \sum_{k=1}^{m} q_k \varphi_j q_k - \sum_{j=1}^{n} p_{ij}^{2} \varphi_j^{\rho_{j+1,i,j}}$$

$$- \frac{r(r-1)}{2} \sum_{j=1}^{n} r_j^2 > 0,$$

where

$$\sum_{k=1}^{m+1} \rho_{k,j} = 1, \sum_{k=1}^{m} q_k = r - 1, r \geq 1, i,j = 1,2,\ldots,n.$$

then model (2) has a unique equilibrium point, and model (3) is almost surely exponentially stable.

**Proof.** We shall prove this theorem in two steps.

**Step 1:** We will prove the existence and uniqueness of the equilibrium point of model (1) under the given assumptions. Let $H(x) = (H_1(x), H_2(x), \ldots, H_n(x))^T$, where

$$H_i(x) = -\beta_i(x) + \sum_{j=1}^{n} a_{ij} f_j(x_j) + \sum_{j=1}^{n} b_{ij} g_j(x_j) - J_i$$

for $i = 1,2,\ldots,n$. In the following we shall prove that $H(x)$ is a homeomorphism of $R^n$ onto itself.

First, we prove that $H(x)$ is an injective map on $R^n$. In fact, if there exist $x = (x_1,x_2,\ldots,x_n)^T$ and $y = (y_1,y_2,\ldots,y_n)^T \in R^n$ and $x \neq y$ such that $H(x) = H(y)$, then

$$\beta_i(x_i) - \beta_i(y_i) = \sum_{j=1}^{n} a_{ij} (f_j(x_j) - f_j(y_j))$$

$$+ \sum_{j=1}^{n} b_{ij} (g_j(x_j) - g_j(y_j))$$

for $i = 1,2,\ldots,n$. Multiply both sides of (6) by $r|\beta_i - \beta_i|$, it follows from assumptions (A1), (A5) and Lemma 2 that

$$r \beta_i |\beta_i - \beta_i|$$

$$\leq r \sum_{j=1}^{n} |a_{ij}| |x_j - y_j|^{r-1} \mu_j |x_j - y_j|$$

$$+ r \sum_{j=1}^{n} |b_{ij}| |x_j - y_j|^{r-1} \varphi_j |x_j - y_j|$$

$$\leq \sum_{j=1}^{n} |a_{ij}| \sum_{k=1}^{m} q_k \mu_j q_k |x_j - y_j|^r + \mu_j^{\rho_{j+1,i,j}} |x_j - y_j|^r$$

$$+ \sum_{j=1}^{n} |b_{ij}| \sum_{k=1}^{m} q_k \varphi_j q_k |x_j - y_j|^r + \varphi_j^{\rho_{j+1,i,j}} |x_j - y_j|^r,$$

That is,

$$r \beta_i < \sum_{j=1}^{n} |a_{ij}| \sum_{k=1}^{m} q_k \mu_j q_k + \sum_{j=1}^{n} |a_{ij}| \mu_j^{\rho_{j+1,i,j}}$$

$$+ \sum_{j=1}^{n} |b_{ij}| \sum_{k=1}^{m} q_k \varphi_j q_k + \sum_{j=1}^{n} |b_{ij}| \varphi_j^{\rho_{j+1,i,j}}.$$  

(7)
From (A6) and \( r(r-1) \sum_{j=1}^n L_{ij}^2 \geq 0 \), we can get that
\[
r\beta_i - \sum_{j=1}^n a_{ij} \sum_{k=1}^m q_{kj} \mu_j^{q_{kj}} - \sum_{j=1}^n a_{ji} \mu_j^{\rho_{i_j,j}} \\
- \sum_{j=1}^n b_j q_j \varphi_j^{\omega_j} - \sum_{j=1}^n b_j \varphi_j^{\rho_j} > 0, 
\]
which is a contradiction. So \( H(x) \) is an injective on \( \mathbb{R}^n \).

Second, we prove that \( \|H(x)\| \to +\infty \) as \( \|x\| \to +\infty \).

From Eq.(8), we can choose a small number \( \delta > 0 \), such that
\[
r \beta_i - \sum_{j=1}^n a_{ij} \sum_{k=1}^m q_{kj} \mu_j^{q_{kj}} - \sum_{j=1}^n a_{ji} \mu_j^{\rho_{i_j,j}} \\
- \sum_{j=1}^n b_j q_j \varphi_j^{\omega_j} - \sum_{j=1}^n b_j \varphi_j^{\rho_j} \geq \delta > 0 
\]
for \( i = 1,2,\ldots,n \).

Let \( \tilde{H}(x) = (\tilde{H}_1(x), \tilde{H}_2(x), \ldots, \tilde{H}_n(x))^T \),
where
\[
\tilde{H}_i(x) = -(\beta_i(x) - \beta_i(0)) + \sum_{j=1}^n a_{ij} (f_j(x_i) - f_j(0)) \\
+ \sum_{j=1}^n b_j (g_j(x_i) - g_j(0))
\]
for \( i = 1,2,\ldots,n \). From assumptions (A1), (A5) and Lemma 2, we can get
\[
\sum_{i=1}^n \|x_i\|^{-1} \text{sgn}(x_i) \tilde{H}_i(x_i) \\
\leq \sum_{i=1}^n \left[ -r \beta_i + \sum_{j=1}^n a_{ij} \sum_{k=1}^m q_{kj} \mu_j^{q_{kj}} + \sum_{j=1}^n a_{ji} \mu_j^{\rho_{i_j,j}} \\
+ \sum_{j=1}^n b_j q_j \varphi_j^{\omega_j} + \sum_{j=1}^n b_j \varphi_j^{\rho_j} \right] \|x_i\| \leq -\delta \|x\|^r.
\]
Thus
\[
\delta \|x\|^r \leq r \sum_{i=1}^n \|x_i\|^{-1} \tilde{H}_i(x_i).
\]
By using the Holder inequality, we get
\[
\delta \|x\|^r \leq r \|x\|^{-r} \|\tilde{H}_i(x_i)\|.
\]
That is
\[
\delta \|x\| \leq r \|\tilde{H}_i(x_i)\|.
\]
Obviously, \( \|\tilde{H}(x)\| \to +\infty \) as \( \|x\| \to +\infty \). Thus
\[
\lim_{\|x\| \to +\infty} \|\tilde{H}(x)\| = \lim_{\|x\| \to +\infty} \|\tilde{H}(x)\| = +\infty
\]
By Lemma 3, we know that \( H(x) \) is a homeomorphism on \( \mathbb{R}^n \). Thus equation
\[
- \beta_i(x_i) + \sum_{j=1}^n a_{ij} f_j(x_i) + \sum_{j=1}^n b_j g_j(x_i) - J_i = 0
\]
has a unique solution \( (u_{i_1}^*, u_{i_2}^*, \ldots, u_{i_n}^*)^T \), which is a unique equilibrium point of model (2) due to assumptions (A1) and (A2).

Step 2: We prove that Eq. (3) is almost surely exponentially stable.

Let \( u(t,x) = (u_1(t,x), u_2(t,x), \ldots, u_n(t,x))^T \) be any solution of the model (1). It follows from (A6) that
\[
\alpha_i \left[ \beta_i - \sum_{j=1}^m a_{ij} q_{kj} \mu_j^{q_{kj}} - \sum_{j=1}^m a_{ji} \mu_j^{\rho_{i_j,j}} \\
- \sum_{j=1}^m b_j q_j \varphi_j^{\omega_j} - \sum_{j=1}^m b_j \varphi_j^{\rho_j} \right] \\
- \frac{r(r-1)}{2} \sum_{j=1}^n L_{ij}^2 > 0,
\]
and there exists a sufficiently small constant \( c > 0 \), and
\[
\frac{c}{\alpha_i} > 0 \text{ is also a sufficiently small constant, such that}
\]
\[
\alpha_i \left[ \beta_i - \frac{c}{\alpha_i} \sum_{j=1}^m a_{ij} q_{kj} \mu_j^{\rho_{i_j,j}} - \sum_{j=1}^m a_{ji} \mu_j^{\rho_{i_j,j}} \\
- \sum_{j=1}^m b_j q_j \varphi_j^{\omega_j} - \sum_{j=1}^m b_j \varphi_j^{\rho_j} \right] \\
- \frac{r(r-1)}{2} \sum_{j=1}^n L_{ij}^2 > 0.
\]

Let \( z_i = u_i - u_{i_i}^* \), and applying Itô's formula to \( z_i^2 \) and integrating both side with respect to \( x \), we have
\begin{align*}
\|z(t)\|_2^2 &= \int_\mathbb{R} z_j(t) \left[ \sum_{j=1}^n \frac{\partial}{\partial x_j} D_{a,j} \frac{\partial }{\partial x_j} \right] \left[ -\alpha_j(u(t,x)) \right] \|\beta_j(u(t,x)) - \beta_j(u^*)\|_2^2 \\
&\quad + \alpha_j(u(t,x)) \left[ \sum_{j=1}^n a_j \left[ f_j(u_j(t,x)) - f_j(u^*_j) \right] \right] \\
&\quad + \sum_{j=1}^n b_j \int_\mathbb{R} K_j(t-s) \left[ g_j(u_j(t,x)) - g_j(u^*_j) \right] ds \\
&\quad + \int_\mathbb{R} 2z_j \left[ \sum_{j=1}^n \sigma_j(u_j(t,x)) \right] d\omega_j(t) dx \\
&\quad + \int_\mathbb{R} \sum_{j=1}^n \sigma_j^2(u_j(t,x)) dtdx.
\end{align*}

Taking \( V(z(t),t) = e^{\sigma t} \sum_{i=1}^n \|z_i(t)\|_2^2 \) and integrating both sides with respect to \( t \), we have

\begin{align*}
V(z(t),t) &= \sum_{i=1}^n \|z_i(0)\|_2^2 + \int_0^1 e^{\sigma t} \sum_{i=1}^n \|z_i(t)\|_2^2 dt \\
&\quad + \int_0^1 e^{\sigma t} \left[ \sum_{i=1}^n \left[ \frac{\partial}{\partial x_i} \left( D_{a,i} \frac{\partial}{\partial x_i} \right) \right] \right] \left[ -\alpha_i(u_i(t,x)) \right] \|\beta_i(u_i(t,x)) - \beta_i(u^*_i)\|_2^2 \\
&\quad + \sum_{i=1}^n b_i \int_\mathbb{R} K_i(t-s) \left[ g_i(u_i(t,x)) - g_i(u^*_i) \right] ds \\
&\quad + \int_0^1 e^{\sigma t} \sum_{i=1}^n \left[ 2z_i \left( \sum_{i=1}^n \sigma_i \right) \right] d\omega_i(t) dx \\
&\quad + \int_0^1 e^{\sigma t} \left[ \sum_{i=1}^n \sigma_i^2 \right] dtdx \\
&\quad + \int_0^1 e^{\sigma t} \left[ \sum_{i=1}^n \int_\mathbb{R} z_i(s) \sigma_i(u_i(s,x)) d\omega_i(s) dx \right] ds \\
&\quad + \int_0^1 e^{\sigma t} \left[ \sum_{i=1}^n \int_\mathbb{R} z_i(s) \right] dtdx
\end{align*}

Notice that, it follows from the boundary condition that

\begin{equation}
\sum_{i=1}^n \int_\mathbb{R} z_i(s) \left[ \sum_{i=1}^n \sigma_i(u_i(s,x)) \right] d\omega_i(s) dx = -\sum_{i=1}^n \int_\mathbb{R} D_{a,i} \left( \frac{\partial}{\partial x_i} \right) \left[ \sum_{i=1}^n \sigma_i(u_i(s,x)) \right] dx
\end{equation}

Hence, using (A1), (A4), (A5) and (11), Holder inequality and Lemma 2, we obtain

\begin{align*}
V(z(t),t) &\leq \sum_{i=1}^n \|z_i(0)\|_2^2 + \int_0^1 e^{\sigma t} \sum_{i=1}^n \|z_i(t)\|_2^2 dt \\
&\quad + \int_0^1 e^{\sigma t} \left[ \sum_{i=1}^n b_i \int_\mathbb{R} K_i(t-s) \left[ g_i(u_i(t,x)) - g_i(u^*_i) \right] ds \\
&\quad + \sum_{i=1}^n \int_\mathbb{R} z_i(s) \left[ \sum_{i=1}^n \sigma_i(u_i(s,x)) \right] d\omega_i(s) dx \right] ds \\
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&\quad + \int_0^1 e^{\sigma t} \left[ \sum_{i=1}^n b_i \int_\mathbb{R} K_i(t-s) \left[ g_i(u_i(t,x)) - g_i(u^*_i) \right] ds \\
&\quad + \sum_{i=1}^n \int_\mathbb{R} z_i(s) \left[ \sum_{i=1}^n \sigma_i(u_i(s,x)) \right] d\omega_i(s) dx \right] ds \\
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\begin{align*}
V(z(t),t) &\leq \sum_{i=1}^n \|z_i(0)\|_2^2 + \int_0^1 e^{\sigma t} \sum_{i=1}^n \|z_i(t)\|_2^2 dt \\
&\quad + \int_0^1 e^{\sigma t} \left[ \sum_{i=1}^n b_i \int_\mathbb{R} K_i(t-s) \left[ g_i(u_i(t,x)) - g_i(u^*_i) \right] ds \\
&\quad + \sum_{i=1}^n \int_\mathbb{R} z_i(s) \left[ \sum_{i=1}^n \sigma_i(u_i(s,x)) \right] d\omega_i(s) dx \right] ds \\
&\quad + \int_0^1 e^{\sigma t} \left[ \sum_{i=1}^n \int_\mathbb{R} z_i(s) \right] dtdx
\end{align*}
\begin{align}
&+ \int_{-\infty}^{s} K_{y}(s-\sigma) \sum_{j=1}^{n} q_{j} \left( \mu_{j} \| z_{j}(s) \|_{2} \right) \left( \mu_{j}^{\rho_{\alpha_{1}}} \| z_{j}(s) \|_{2} \right)^{r} \rho_{\alpha_{1}} j \left( \mu_{j} \| z_{j}(s) \|_{2} \right) \left( \mu_{j}^{\rho_{\alpha_{1}}} \| z_{j}(s) \|_{2} \right)^{r} ds \nonumber \\
&+ \int_{-\infty}^{s} e^{-x} \sum_{i=1}^{n} \left( \mu_{i} \| z_{i}(s) \|_{2} \right)^{r} \rho_{\alpha_{1}} j \left( \mu_{j} \| z_{j}(s) \|_{2} \right) \left( \mu_{j}^{\rho_{\alpha_{1}}} \| z_{j}(s) \|_{2} \right)^{r} \rho_{\alpha_{1}} j \left( \mu_{j} \| z_{j}(s) \|_{2} \right) \left( \mu_{j}^{\rho_{\alpha_{1}}} \| z_{j}(s) \|_{2} \right)^{r} ds, \nonumber \\
&+ \int_{0}^{r} \rho_{\alpha_{1}} j \left( \mu_{j} \| z_{j}(s) \|_{2} \right) \left( \mu_{j}^{\rho_{\alpha_{1}}} \| z_{j}(s) \|_{2} \right)^{r} \rho_{\alpha_{1}} j \left( \mu_{j} \| z_{j}(s) \|_{2} \right) \left( \mu_{j}^{\rho_{\alpha_{1}}} \| z_{j}(s) \|_{2} \right)^{r} ds \nonumber \\
&+ \int_{0}^{r} e^{-x} \sum_{i=1}^{n} \left( \mu_{i} \| z_{i}(s) \|_{2} \right)^{r} \rho_{\alpha_{1}} j \left( \mu_{j} \| z_{j}(s) \|_{2} \right) \left( \mu_{j}^{\rho_{\alpha_{1}}} \| z_{j}(s) \|_{2} \right)^{r} \rho_{\alpha_{1}} j \left( \mu_{j} \| z_{j}(s) \|_{2} \right) \left( \mu_{j}^{\rho_{\alpha_{1}}} \| z_{j}(s) \|_{2} \right)^{r} ds. \nonumber \\
\end{align}

Notice that

\[ V(z(t), t) \leq \sum_{i=1}^{n} \| z_{i}(0) \|_{2}^{2} + \int_{0}^{r} e^{-x} \sum_{i=1}^{n} \left( \mu_{i} \| z_{i}(s) \|_{2} \right)^{r} \rho_{\alpha_{1}} j \left( \mu_{j} \| z_{j}(s) \|_{2} \right) \left( \mu_{j}^{\rho_{\alpha_{1}}} \| z_{j}(s) \|_{2} \right)^{r} \rho_{\alpha_{1}} j \left( \mu_{j} \| z_{j}(s) \|_{2} \right) \left( \mu_{j}^{\rho_{\alpha_{1}}} \| z_{j}(s) \|_{2} \right)^{r} ds. \]
such that

\[ \sum_{j=1}^{n} a_{ji} \mu_{j}^{p_{ji}+1} - \sum_{j=1}^{n} b_{ji} q_{j} \phi_{j}^{q_{j}} \]

shows that

Since

Which implies

The proof is completed.

**Theorem 2** Under the assumptions of Theorem 1, model (3) is \( p \)-th moment exponentially stable.

**Proof.** Taking expectations for both hand-sides of (13), it follows from

\[
E \int_{0}^{t} e^{\alpha s} \sum_{i=1}^{n} \left\| z_{i}(s) \right\|_{2}^{2} \, ds 
\]

Clearly, there exists some positive constant \( K \) such that

\[
E \left\| z(t) \right\|_{2}^{2} = KE \left( \sum_{i=1}^{n} \left\| z_{i}(0) \right\|_{2}^{2} \right) e^{-\gamma t}.
\]

The proof is completed.

**4 Conclusions**

In this paper, the almost surely exponential stability and exponential \( p \)-th moment stability have been studied for a class of stochastic Cohen–Grossberg neural networks with distributed delays and reaction–diffusion term. Two delay-independent and easily verifiable sufficient conditions have been obtained to ensure the existence, uniqueness, almost surely exponential stability and exponential \( p \)-th moment stability of the equilibrium point for the addressed stochastic Cohen-Grossberg neural network with distributed delays and reaction–diffusion terms.

**References**


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