Stability analysis for stochastic Cohen-Grossberg neural network with distributed delays and reaction diffusion terms

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Summary
This paper mainly deals with the almost surely exponential stability and exponential *p*-th moment stability for a class of stochastic Cohen–Grossberg neural networks with distributed delays and reaction–diffusion term. By constructing suitable Lyapunov functional, employing the nonnegative semi-martingale convergence theorem and applying matrix theory and stochastic analysis technique, two delay-independent and easily verifiable sufficient conditions are obtained to ensure the existence, uniqueness, almost surely exponential stability and exponential p-th moment stability of the equilibrium point for the addressed stochastic Cohen-Grossberg neural network with distributed delays and reaction-diffusion terms.

Keywords:
Cohen–Grossberg neural network; almost surely exponential stability; exponential *p*-th moment stability.

1. Introduction

The Cohen–Grossberg neural network, first proposed and studied by Cohen and Grossberg in 1983 [1], has attracted considerable attention due to its potential applications in classification, parallel computing, associative memory, signal and image processing, especially in solving some difficult optimization problems. In such applications, it is of prime importance to ensure that the designed neural networks are stable [2]. In implementation of neural networks, however, time delays are unavoidably encountered due to the finite switching speed of neurons and amplifiers. It has been found that, the existence of time delays may lead to instability and oscillation in a neural network. Therefore, stability analysis of Cohen–Grossberg neural network with time delays has received much attention [3-10].

Excepting delay effects, strictly speaking, diffusion effects cannot be avoided in the neural networks when electrons are moving in asymmetric electromagnetic fields. So we must consider that the activations vary in space as well as in time. In [11-14], the authors have considered the stability of neural networks with diffusion terms, which are expressed by partial differential equations.

In addition to the delay effects, stochastic effects constitute another source of disturbances or uncertainties in real systems [10]. A lot of dynamical systems have variable structures subject to stochastic abrupt changes, which may result from abrupt phenomena such as stochastic failures and repairs of the components, changes in the interconnections of subsystems or sudden environment switching [15]. Therefore, stochastic perturbations should be taken into account when modeling neural networks. In recent years, the dynamic analysis of stochastic systems (including neural networks) with delays has been an attractive topic for many researchers, and a large number of stability criteria of these systems have been reported [10, 15-20]. Particularly, in [15-16], the authors have considered the exponential *p*-stability of stochastic differential equations with constant delays and obtained several stability conditions for checking the exponential *p*-stability. In [17-20], the problem on stability of stochastic neural networks with constant delays or time-varying delay or bounded distributed delays has been considered and many interesting results have been established by employing a Lyapunov functional approach. To the best of our knowledge, so far, few authors have considered the problem of stability analysis for Cohen–Grossberg neural networks with both distributed delays and reaction-diffusion terms in the simultaneous presence of stochastic effects.

In this paper, we investigate the almost surely exponential stability and exponential *p*-th moment stability for stochastic Cohen–Grossberg neural network with continuously distributed delays and reaction-diffusion terms.

2. Model description and preliminaries

In this paper, we consider the following stochastic Cohen–Grossberg neural network with continuously distributed delays and reaction-diffusion terms

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$$
\begin{cases}
du_i(t,x) = \sum_{k=1}^{l} \frac{\partial}{\partial x_k}\left(D_i \frac{\partial u_i(t,x)}{\partial x_k} \right) dt \\
\qquad -\alpha_i(u_i(t,x))\big[\beta_i(u_i(t,x)) \\
\qquad -\sum_{j=1}^{n} a_{ij} f_j(u_j(t,x)) \\
\qquad -\sum_{j=1}^{n} b_{ij} \int_{-\infty}^{t} K_{ij}(t-s) g_j(u_j(s,x)) ds \\
\qquad +J_i\big] dt + \sum_{j=1}^{n} \sigma_{ij}(u_j(t,x)) d\omega_j(t) , \\
\qquad\qquad x \in X, \\
\frac{\partial u_i}{\partial m} := (\frac{\partial u_i}{\partial x_1}, \ldots, \frac{\partial u_i}{\partial x_m})^T = 0, \quad t \geq 0, \;\; x \in \partial X , \\
u_i(t_0 + s, x) = \xi_i(s,x), \qquad s \leq 0, \;\; x \in X,
\end{cases} \tag{1}
$$

for $i = 1,2,\ldots,n$ and $t \geq 0$. In the above system, $n \geq 2$ is the number of neurons in the network, x_i is space variable, $u_i(t,x)$ is the state variable of the i-th neuron at time t and in space x, $f_j(u_j(t,x))$ and $g_j(u_j(t,x))$ denotes the output of the j-th unit at time t on the i-th unit and in space x, smooth function $D_{ik} = D_{ik}(t,x) \geq 0$ is diffusion operator, X is a compact set with smooth boundary ∂X and measure mes $X > 0$ in R^m. $\alpha_i(u_i(t,x))$ represents an amplification function; $\beta_i(u_i(t,x))$ is an appropriately behaved function at time t; $\xi_i(t,x)$ is the initial boundary value. a_{ij}, b_{ij} and J_i are constants: a_{ij} indicates the strength of the neuron interconnections within the network at time t; b_{ij} weights the strength of the j-th unit on the i-th unit at time $t-s$; K_{ij} is the delay kernel function; J_i denotes the constant input from outside of the network. Moreover, $\omega(t) = (\omega_1(t),\ldots,\omega_n(t))^T$ is n dimensional Brownian motion defined on a complete probability space (Ω, F, P) with a natural filtration $\{F_t\}_{t\geq 0}$ generated by $\{\omega(s) : 0 \leq s \leq t\}$, where we associate Ω with the canonical space generated by all $\{w_i(t)\}$, and denote by F the associated σ-algebra. Generated by $\{w(t)\}$ with the probability measure P.

Let $L^2(X)$ be the space of real Lebesgue measurable functions on X. It is a Banach space for the L_2-norm

$$\|u\|_2 = \left(\int_X |u(x)|^2 dx \right)^{\frac{1}{2}},$$

where $|u|$ denotes the Euclid norm of a vector $u \in R^n$ for any integer n. The norm $\|u\|$ is defined by

$$\|u\| = \left(\sum_{i=1}^{n} \|u_i\|_2^p \right)^{\frac{1}{p}} , \;\; p \geq 1.$$

Note that, $\xi = \{(\xi_1(s,x),\ldots,\xi_n(s,x))^{\mathrm{T}} : s \leq 0\}$ is $C([-\infty,0] \times R^m; R^n)$-valued function and F_0-measurable R^n-valued random variable, where, for example, $F_0 = F_s$ on $[-\infty,0]$, and $C([-\infty,0] \times R^m; R^n)$ is the space of all continuous R^n-valued functions defined on $[-\infty,0] \times R^m$.

Furthermore, model (1) comprises the following Cohen–Grossberg neural network model without stochastic effects

$$
\begin{aligned}
du_i(t,x) &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k}\left(D_i \frac{\partial u_i(t,x)}{\partial x_k} \right) dt \\
&\quad -\alpha_i(u_i(t,x))\big[\beta_i(u_i(t,x)) \\
&\quad -\sum_{j=1}^{n} a_{ij} f_j(u_j(t,x)) \\
&\quad -\sum_{j=1}^{n} b_{ij} \int_{-\infty}^{t} K_{ij}(t-s) g_j(u_j(s,x)) ds \\
&\quad + J_i\big] dt \qquad x \in X.
\end{aligned} \tag{2}
$$

We will prove in Section 3 that model (2) has a unique equilibrium point $u^* = (u_1^*,\ldots,u_n^*)^{\mathrm{T}}$ by the property of homeomorphism and the inequality technique. Then model (1) admits an equilibrium point $u^* = (u_1^*,\ldots,u_n^*)^{\mathrm{T}}$. At this time, model (1) is equivalent to

$$
\begin{aligned}
& d\left(u_i(t,x)-u_i^*\right) \\
& =\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_{ik}\,\frac{\partial\left(u_i(t,x)-u_i^*\right)}{\partial x_k}\right)dt \\
& \quad +\alpha_i(u_i(t,x))\Big\{-\left[\beta_i\left(u_i(t,x)\right)-\beta_i(u_i^*)\right] \\
& \quad +\sum_{j=1}^{n}a_{ij}\left[f_j(u_j(t,x))-f_j(u_i^*)\right] \qquad (3) \\
& \quad +\sum_{j=1}^{n}b_{ij}\int_{-\infty}^{t}K_{ij}(t-s)\left[g_j(u_j(s,x))-g_j(u_i^*)\right]ds\Big\}dt, \\
& \quad +\sum_{j=1}^{n}\sigma_{ij}(u_j(t,x))d\omega_j(t), \qquad x\in X.
\end{aligned}
$$

Throughout this paper, for system (1), we have the following assumptions:

(**A1**) f_j, g_j and σ_{ij} are Lipschitz continuous with Lipschitz constant $\mu_j>0, \varphi_j>0$ and $L_{ij}>0,$ respectively, for $i,j=1,2,...,n.$

(**A2**) The delay kernel $K_{ij}:[0,+\infty)\to[0,+\infty)$ is a real-valued non-negative continuous function and satisfies

$$\int_0^{+\infty}e^{\lambda s}K_{ij}(s)ds=r_{ij}(\lambda),$$

where $r_{ij}(\lambda)$ is continuous function on $[0,\delta),\ \delta>0$ and $r_{ij}(0)=1,\ i,j=1,2,...,n.$

(**A3**) $\sigma_{ij}(u_j^*)=0$, where $u^*=(u_1^*,u_2^*,...,u_n^*)^{\mathrm{T}}$ is the equilibrium point of model (2).

(**A4**) Each function $\alpha_i(u)$ is bounded, positive and continuous, i.e. there exist a constant $\overline{\alpha}_i$ such that

$$0<\alpha_i(u)\le\overline{\alpha}_i<+\infty,$$

for $u\in R,\ i=1,2,...,n.$

(**A5**) There exists a positive diagonal matrix $\beta=\mathrm{diag}(\beta_1,\beta_2,...,\beta_n)$ such that

$$\frac{\beta_i(u)-\beta_i(v)}{u-v}\ge\beta_i$$

for all $u,v\in R\ (u\ne v),\ i=1,2,...,n.$

Definition 1 Model (3) is said to be almost surely exponentially stable if there exists a positive constant λ such that for each pair of t_0 and ξ there is a positive finite random variable K such that

$$\left\|u(t;t_0,\xi)-u^*\right\|^p\le Ke^{-\lambda(t-t_0)},\qquad P-a.s.$$

for all $t\ge t_0$. In this case

$$\lim_{t\to\infty}\sup\frac{1}{t}\log(\left\|u(t;t_0,\xi)-u^*\right\|^p)\le-\lambda, \qquad (4)$$

The left hand-side of (4) is called the almost sure Lyapunov exponent of the solution.

Definition 2 Model (3) is said to be *p*-th moment exponentially stable if there exist a pair of positive constants λ and K such that

$$E\left\|u(t;\xi)-u^*\right\|^p\le KE\left\|\xi-u^*\right\|pe^{-\lambda t},\ t\ge0$$

for any ξ. In this case

$$\lim_{t\to\infty}\sup\frac{1}{t}\log(E\left\|u(t,\xi)-u^*\right\|^p)\le-\lambda, \qquad (5)$$

The left hand-side of (5) is called the *p*-th moment Lyapunov exponent of the solution. When $p=2$, it is usually called the exponential stability in mean square.

Definition 3 [9] A map $H:R^n\to R^n$ is a homeomorphism of R^n onto itself, if $H\in C^0$, H is one-to-one, H is onto and the inverse map $H^{-1}\in C^0$.

To prove our results, the following lemmas are necessary.

Lemma 1[17] Let $A(t)$ and $U(t)$ be two continuous adapted increasing processes on $t\ge0$ with $A(0)=U(0)=0$, Let $M(t)$ be a real-valued continuous local martingale with $M(0)=0,a.s.$ Let ς be a nonnegative F_0 -measurable random variable with $E\varsigma<\infty.$ Define

$$X(t)=\varsigma+A(t)-U(t)+M(t)\quad\text{for}\quad t\ge0.$$

If $X(t)$ is nonnegative, then

$$
\begin{aligned}
&\left\{\lim_{t\to\infty}A(t)<\infty\right\} \\
&\subset\left\{\lim_{t\to\infty}X(t)<\infty\right\}\cap\left\{\lim_{t\to\infty}U(t)<\infty\right\}\ a.s.,
\end{aligned}
$$

When $B \subset D\ a.s.$ denotes $P(B \cap D^c) = 0.$ In particular, if $\lim_{t\to\infty} A(t) < \infty\ a.s.$, then for almost all $\omega \in \Omega,\ \lim_{t\to\infty} X(t,\omega) < \infty$ and $\lim_{t\to\infty} U(t,\omega) < \infty$, i.e., both $X(t)$ and $U(t)$ converge to finite random variables.

Lemma 2 For $a \ge 0, b_k \ge 0, (k = 1,\ldots,m),$ the following inequality holds

$$a\prod_{k=1}^{m} b_k^{q_k} \le \frac{1}{r}\sum_{k=1}^{m} q_k b_k^r + \frac{1}{r}a^r,$$

where $q_k > 0,\ (k = 1,\ldots,m)$, $\sum_{k=1}^{m} q_k = r-1$ and $r \ge 1$.

Lemma 3 [9] If $H(x) \in C^0$ satisfies the following conditions

(i) $H(x)$ is injective on R^n,

(ii) $\|H(x)\| \to +\infty\ as\ \|x\| \to +\infty$, then $H(x)$ is homeomorphism of R^n onto itself.

3. Main results

In this section, we will give several sufficient conditions on the existence, uniqueness, almost surely exponential stability and exponential p-th moment stability of the equilibrium point for the stochastic Cohen-Grossberg neural network (1).

Theorem 1 If model (3) satisfies the assumptions (A1)-(A5), and

(A6) There exist constants $\rho_{k,j} \in R, q_k > 0,$
$i,j = 1,\ldots,n,\quad k = 1,\ldots,m+1;$ such that

$$\overline{\alpha}_i\Bigg[r\beta_i - \sum_{j=1}^{n}|a_{ij}|\sum_{k=1}^{m} q_k \mu_j^{\frac{r\rho_{k,j}}{q_k}} - \sum_{j=1}^{n}|a_{ji}|\mu_i^{r\rho_{m+1,j}} - \sum_{j=1}^{n}|b_{ij}|\sum_{k=1}^{m} q_k \varphi_j^{\frac{r\rho_{k,j}}{q_k}} - \sum_{j=1}^{n}|b_{ji}|\varphi_i^{r\rho_{m+1,j}}\Bigg] - \frac{r(r-1)}{2}\sum_{j=1}^{n} L_{ij}^2 > 0,$$

where

$$\sum_{k=1}^{m+1}\rho_{k,j} = 1,\ \sum_{k=1}^{m} q_k = r-1,\ r \ge 1,\ i,j = 1,2,\ldots,n.$$

then model (2) has a unique equilibrium point, and model (3) is almost surely exponentially stable.

Proof. We shall prove this theorem in two steps.

Step 1: We will prove the existence and uniqueness of the equilibrium point of model (1) under the given assumptions.

Let $H(x) = (H_1(x), H_2(x),\ldots,H_n(x))^{\mathrm{T}}$, where

$$H_i(x) = -\beta_i(x_i) + \sum_{j=1}^{n} a_{ij} f_j(x_j) + \sum_{j=1}^{n} b_{ij} g_j(x_j) - J_i$$

for $i = 1,2,\ldots,n.$ In the following we shall prove that $H(x)$ is a homeomorphism of R^n onto itself.

First, we prove that $H(x)$ is an injective map on R^n. In fact, if there exist $x = (x_1, x_2,\ldots,x_n)^{\mathrm{T}}$ and $y = (y_1, y_2,\ldots,y_n)^{\mathrm{T}} \in R^n$ and $x \ne y$ such that $H(x) = H(y),$ then

$$\beta_i(x_i) - \beta_i(y_i) = \sum_{j=1}^{n} a_{ij}(f_j(x_j) - f_j(y_j)) + \sum_{j=1}^{n} b_{ij}(g_j(x_j) - g_j(y_j)) \qquad (6)$$

for $i = 1,2,\ldots,n.$ Multiply both sides of (6) by $r|x_i - y_i|^{r-1}$, it follows from assumptions (A1), (A5) and Lemma 2 that

$$\begin{aligned} &r\beta_i |x_i - y_i|^r \\ &\le r\sum_{j=1}^{n}|a_{ij}||x_i - y_i|^{r-1}\mu_j |x_j - y_j| \\ &\quad + r\sum_{j=1}^{n}|b_{ij}||x_i - y_i|^{r-1}\varphi_j |x_j - y_j| \\ &\le \sum_{j=1}^{n}|a_{ij}|\left[\sum_{k=1}^{m} q_k \mu_j^{\frac{r\rho_{k,j}}{q_k}}|x_i - y_i|^r + \mu_j^{r\rho_{m+1,j}}|x_j - y_j|^r\right] \\ &\quad + \sum_{j=1}^{n}|b_{ij}|\left[\sum_{k=1}^{m} q_k \varphi_j^{\frac{r\rho_{k,j}}{q_k}}|x_i - y_i|^r + \varphi_j^{r\rho_{m+1,j}}|x_j - y_j|^r\right], \end{aligned}$$

That is,

$$r\beta_i < \sum_{j=1}^{n}|a_{ij}|\sum_{k=1}^{m} q_k \mu_j^{\frac{r\rho_{k,j}}{q_k}} + \sum_{j=1}^{n}|a_{ji}|\mu_i^{r\rho_{m+1,j}} + \sum_{j=1}^{n}|b_{ij}|\sum_{k=1}^{m} q_k \varphi_j^{\frac{r\rho_{k,j}}{q_k}} + \sum_{j=1}^{n}|b_{ji}|\varphi_i^{r\rho_{m+1,j}}. \qquad (7)$$

From (A6) and $\frac{r(r-1)}{2}\sum_{j=1}^{n} L_{ij}^2 \geq 0$, $\overline{\alpha}_i > 0$, we can gat that

$$r\beta_i - \sum_{j=1}^{n}|a_{ij}|\sum_{k=1}^{m} q_k \mu_j^{\frac{r\rho_{k,j}}{q_k}} - \sum_{j=1}^{n}|a_{ji}|\mu_i^{r\rho_{m+1,j}}$$

$$-\sum_{j=1}^{n}|b_{ij}|\sum_{k=1}^{m} q_k \varphi_j^{\frac{r\rho_{k,j}}{q_k}} - \sum_{j=1}^{n}|b_{ji}|\varphi_i^{r\rho_{m+1,j}} > 0, \qquad (8)$$

which is a contradiction. So $H(x)$ is an injective on R^n.

Second, we prove that $\|H(x)\| \to +\infty \ as \ \|x\| \to +\infty$.

From Eq.(8), we can choose a small number $\delta > 0$, such that

$$r\beta_i - \sum_{j=1}^{n}|a_{ij}|\sum_{k=1}^{m} q_k \mu_j^{\frac{r\rho_{k,j}}{q_k}} - \sum_{j=1}^{n}|a_{ji}|\mu_i^{r\rho_{m+1,j}}$$

$$-\sum_{j=1}^{n}|b_{ij}|\sum_{k=1}^{m} q_k \varphi_j^{\frac{r\rho_{k,j}}{q_k}} - \sum_{j=1}^{n}|b_{ji}|\varphi_i^{r\rho_{m+1,j}} \geq \delta > 0 \qquad (9)$$

for $i = 1,2,\ldots,n$.

Let $\tilde{H}(x) = (\tilde{H}_1(x), \tilde{H}_2(x), \ldots, \tilde{H}_n(x))^{\mathrm{T}}$,

where

$$\tilde{H}_i(x) = -(\beta_i(x_i) - \beta_i(0)) + \sum_{j=1}^{n} a_{ij}\left(f_j(x_j) - f_j(0)\right)$$
$$+\sum_{j=1}^{n} b_{ij}\left(g_j(x_j) - g_j(0)\right)$$

for $i = 1,2,\ldots,n$. From assumptions (A1), (A5) and Lemma 2, we can get

$$\sum_{i=1}^{n} r|x_i|^{r-1}\operatorname{sgn}(x_i)\tilde{H}_i(x_i)$$

$$\leq \sum_{i=1}^{n}\Bigg[-r\beta_i + \sum_{j=1}^{n}|a_{ij}|\sum_{k=1}^{m} q_k \mu_j^{\frac{r\rho_{k,j}}{q_k}} + \sum_{j=1}^{n}|a_{ji}|\mu_j^{r\rho_{m+1,j}}$$
$$+\sum_{j=1}^{n}|b_{ij}|\sum_{k=1}^{m} q_k \varphi_j^{\frac{r\rho_{k,j}}{q_k}} + \sum_{j=1}^{n}|b_{ji}|\varphi_j^{r\rho_{m+1,j}}\Bigg]|x_i|^r \leq -\delta\|x\|^r.$$

Thus

$$\delta\|x\|^r \leq r\sum_{i=1}^{n}|x_i|^{r-1}\left|\tilde{H}_i(x_i)\right|.$$

By using the Holder inequality, we get

$$\delta\|x\|^r \leq r\|x\|^{r-1}\left\|\tilde{H}_i(x_i)\right\|,$$

that is

$$\delta\|x\| \leq r\left\|\tilde{H}_i(x_i)\right\|.$$

Obviously, $\left\|\tilde{H}(x)\right\| \to +\infty$ as $\|x\| \to +\infty$. Thus.

$$\lim_{\|x\|\to+\infty}\|H(x)\| = \lim_{\|x\|\to+\infty}\left\|\tilde{H}(x)\right\| = +\infty$$

By Lemma 3, we know that $H(x)$ is a homeomorphism on R^n. Thus equation

$$-\beta_i(x_i) + \sum_{j=1}^{n} a_{ij} f_j(x_j) + \sum_{j=1}^{n} b_{ij} g_j(x_j) - J_i = 0,$$

has a unique solution $(u_1^*, u_2^* \ldots, u_n^*)^{\mathrm{T}}$,which is a unique equilibrium point of model (2) due to assumptions (A1) and (A2).

Step 2: We prove that Eq. (3) is almost surely exponentially stable.

Let $u(t,x) = (u_1(t,x), u_2(t,x), \ldots, u_n(t,x))^{\mathrm{T}}$ be any solution of the model (1). It follows from (A6) that

$$\overline{\alpha}_i\Bigg[r\beta_i - \sum_{k=1}^{m}|a_{ij}|q_k \mu_j^{\frac{r\rho_{k,j}}{q_k}} - \sum_{k=1}^{m}|a_{ji}|\mu_i^{r\rho_{m+1,j}}$$
$$-\sum_{k=1}^{m}|b_{ij}|q_k \varphi_j^{\frac{r\rho_{k,j}}{q_k}} - \sum_{k=1}^{m}|b_{ji}|\varphi_i^{r\rho_{m+1,j}}\Bigg]$$
$$-\frac{r(r-1)}{2}\sum_{j=1}^{n} L_{ij}^2 > 0,$$

and there exists a sufficiently small constant $c > 0,$ and $\frac{c}{\overline{\alpha}_i} > 0$ is also a sufficiently small constant, such that

$$\overline{\alpha}_i\Bigg[(r\beta_i - \frac{c}{\overline{\alpha}_i}) - \sum_{k=1}^{m}|a_{ij}|q_k \mu_j^{\frac{r\rho_{k,j}}{q_k}} - \sum_{k=1}^{m}|a_{ji}|\mu_i^{r\rho_{m+1,j}}$$

$$-\sum_{k=1}^{m}|b_{ij}|q_k \varphi_j^{\frac{r\rho_{k,j}}{q_k}} \ . \qquad (10)$$

Let $z_i = u_i - u_i^*$, and applying Itô's formula to z_i^2 and integrating both side with respect to x, we have

$$d\|z_i\|_2^2$$

$$= \int_X 2z_i \Bigg\{ \sum_{k=1}^{l} \frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial z_i}{\partial x_k}\Big) - \alpha_i(u_i(t,x))\big[\beta_i(u_i(t,x)) - \beta_i(u_i^*)\big]$$

$$+ \alpha_i(u_i(t,x))\Bigg[\sum_{j=1}^{n} a_{ij}\big[f_j(u_j(t,x)) - f_j(u_j^*)\big]$$

$$+ \sum_{j=1}^{n} b_{ij}\int_{-\infty}^{t} K_{ij}(t-s)\big[g_j(u_j(t,x)) - g_j(u_j^*)\big]ds\Bigg]\Bigg\} dtdx$$

$$+ \int_X 2z_i \sum_{j=1}^{n} \sigma_{ij}(u_j(t,x))d\omega_j(t)dx$$

$$+ \int_X \sum_{j=1}^{n} \sigma_{ij}^2(u_j(t,x))dtdx\,.$$

Taking $V(z(t),t) = e^{ct}\sum_{i=1}^{n}\|z_i(t)\|_2^r$, applying itô's formula to $V(z(t),t)$, and integrating both side with respect to x , we have

$$V(z(t),t)$$

$$= \sum_{i=1}^{n}\|z_i(0)\|_2^r + \int_0^t ce^{cs}\sum_{i=1}^{n}\|z_i(s)\|_2^r ds$$

$$+ \int_0^t e^{cs} r\sum_{i=1}^{n}\|z_i(s)\|_2^{r-2}\int_X z_i \Bigg\{\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial z_i}{\partial x_k}\Big)$$

$$- \alpha_i(u_i(t,x))\big[\beta_i(u_i(t,x)) - \beta_i(u_i^*)\big]$$

$$+ \alpha_i(u_i(t,x))\Bigg[\sum_{j=1}^{n} a_{ij}\big[f_j(u_j(t,x)) - f_j(u_j^*)\big]$$

$$+ \sum_{j=1}^{n} b_{ij}\int_{-\infty}^{t} K_{ij}(t-s)\big[g_j(u_j(t,x)) - g_j(u_j^*)\big]ds\Bigg]\Bigg\} dtdx$$

$$+ \int_0^t e^{cs} r\sum_{i=1}^{n}\|z_i(s)\|_2^{r-2}\int_X z_i(s)\sum_{j=1}^{n}\sigma_{ij}(u_j(s,x))\,d\omega_j(s)dx$$

$$+ \int_0^t e^{cs}\frac{r}{2}\sum_{i=1}^{n}\|z_i(s)\|_2^{r-2}\int_X \sum_{j=1}^{n}\sigma_{ij}^2(u_j(s,x))\,dsdx$$

$$+ \int_0^t e^{cs}\,\frac{r(r-2)}{2}\sum_{i=1}^{n}\|z_i(s)\|_2^{r-4}$$

$$\times \sum_{j=1}^{n}\Big(\int_X z_i(s)\ \sigma_{ij}(u_j(s,x))dx\Big)^2\ ds$$

Notice that, it follows from the boundary condition that [9]

$$\sum_{k=1}^{l}\ \int_X z_i\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial z_i}{\partial x_k}\right)dx = -\sum_{k=1}^{l}\int_X D_{ik}\left(\frac{\partial z_i}{\partial x_k}\right)^2 dx \qquad (11)$$

Hence, using (A1), (A4), (A5) and (11), Holder inequality and Lemma 2, we obtain

$$V(z(t),t)$$

$$\le \sum_{i=1}^{n}\|z_i(0)\|_2^r + \int_0^t ce^{cs}\sum_{i=1}^{n}\|z_i(s)\|_2^r ds$$

$$+ \int_0^t e^{cs}\sum_{i=1}^{n}\Big\{- r\overline{\alpha}_i\beta_i\|z_i(s)\|_2^r$$

$$+ r\overline{\alpha}_i\sum_{j=1}^{n}|a_{ij}|\,\mu_j\,\|z_i(s)\|_2^{r-1}\,\|z_j(s)\|_2$$

$$+ r\overline{\alpha}_i\sum_{j=1}^{n}|b_{ij}|\int_{-\infty}^{s} K_{ij}(s-\sigma)\ \varphi_j\,\|z_i(s)\|_2^{r-1}\,\|z_j(\sigma)\|_2\mathrm{d}\sigma\Big\} ds$$

$$+ \int_0^t\ re^{cs}\sum_{i=1}^{n}\|z_i(s)\|_2^{r-2}\int_X z_i(s)\sum_{j=1}^{n}\sigma_{ij}(u_j(s,x))d\omega_j(s)dx$$

$$+ \int_0^t\ \frac{r}{2}e^{cs}\sum_{i=1}^{n}\|z_i(s)\|_2^{r-2}\sum_{j=1}^{n}L_{ij}^2\|z_i(s)\|_2^2 ds$$

$$+ \int_0^t\frac{r(r-2)}{2}e^{cs}\sum_{i=1}^{n}\sum_{j=1}^{n}\|z_i(s)\|_2^{r-2}\,L_{ij}^2\,\|z_i(s)\|_2^2 ds$$

$$\le \sum_{i=1}^{n}\|z_i(0)\|_2^r + \int_0^t ce^{cs}\sum_{i=1}^{n}\|z_i(s)\|_2^r ds$$

$$+ \int_0^t\overline{\alpha}_i e^{cs}\sum_{i=1}^{n}\Bigg\{- r\beta_i\|z_i(s)\|_2^r$$

$$+ \sum_{j=1}^{n}|a_{ij}|\left[\, r\Big(\mu_j^{\rho_{m+1,i}}\|z_j(s)\|_2\Big)\prod_{k=1}^{m}\left(\mu_j^{\frac{\rho_{k,j}}{q_k}}\|z_i(s)\|_2\right)^{q_k}\,\right]$$

$$+\int_{-\infty}^{s} k_{ij}(s-\sigma)\sum_{j=1}^{n}\left|b_{ij}\right|\left[r\left(\varphi_j^{\rho_{m+1,i}}\left\|z_j(\sigma)\right\|_2\right)\prod_{k=1}^{m}\left(\varphi_j^{\frac{\rho_{k,j}}{q_k}}\left\|z_i(s)\right\|_2\right)^{q_k}\right]d\sigma\Bigg\}$$

$$+\int_0^t re^{cs}\sum_{i=1}^{n}\left\|z_i(s)\right\|_2^{r-2}$$

$$\times\int_X z_i(s)\sum_{j=1}^{n}\sigma_{ij}(u_j(s,x))d\omega_j(s)dx$$

$$+\int_0^t \frac{r-1}{2}e^{cs}\sum_{i=1}^{n}\sum_{j=1}^{n}L_{ij}^2\left[r\left\|z_i(s)\right\|_2\prod_{k=1}^{m}\left(\left\|z_i(s)\right\|_2\right)^{q_k}\right]ds$$

$$\le \sum_{i=1}^{n}\left\|z_i(0)\right\|_2^r+\int_0^t ce^{cs}\sum_{i=1}^{n}\left\|z_i(s)\right\|_2^r ds$$

$$+\int_0^t \overline{\alpha}_i e^{cs}\sum_{i=1}^{n}\Bigg\{-r\beta_i\left\|z_i(s)\right\|_2^r$$

$$+\sum_{j=1}^{n}\left|a_{ij}\right|\left[\sum_{k=1}^{m}q_k\left(\mu_j^{\frac{\rho_{k,j}}{q_k}}\left\|z_i(s)\right\|_2\right)^r+\left(\mu_j^{\rho_{m+1,j}}\left\|z_j(s)\right\|_2\right)^r\right]$$

$$+\int_{-\infty}^{s}K_{ij}(s-\sigma)\sum_{j=1}^{n}\left|b_{ij}\right|\left[\sum_{k=1}^{m}q_k\left(\varphi_j^{\frac{\rho_{k,j}}{q_k}}\left\|z_i(s)\right\|_2\right)^r$$

$$+\left(\varphi_j^{\rho_{m+1,j}}\left\|z_j(\sigma)\right\|_2\right)^r\right]\mathrm{d}\sigma\Bigg\}ds$$

$$+\int_0^t re^{cs}\sum_{i=1}^{n}\left\|z_i(s)\right\|_2^{r-2}\int_X z_i(s)\sum_{j=1}^{n}\sigma_{ij}(u_j(s,x))d\omega_j(s)dx$$

$$+\int_0^t \frac{r-1}{2}e^{cs}\sum_{i=1}^{n}\sum_{j=1}^{n}L_{ij}^2\left(\sum_{k=1}^{m}q_k\left\|z_i(s)\right\|_2^r+\left\|z_i(s)\right\|_2^r\right)ds. \tag{12}$$

Notice that

$$\int_{-\infty}^{s}K_{ij}(s-\sigma)\sum_{j=1}^{n}\left|b_{ij}\right|\left[\sum_{k=1}^{m}q_k\left(\varphi_j^{\frac{\rho_{k,j}}{q_k}}\left\|z_i(s)\right\|_2\right)^r$$

$$+\left(\varphi_j^{\rho_{m+1,j}}\left\|z_j(\sigma)\right\|_2\right)^r\right]\mathrm{d}\sigma$$

$$=\sum_{j=1}^{n}\left|b_{ij}\right|\sum_{k=1}^{m}q_k\varphi_j^{\frac{r\rho_{k,j}}{q_k}}\left\|z_i(s)\right\|_2^r$$

$$+\int_0^{+\infty}K_{ij}(\sigma)\sum_{j=1}^{n}\left|b_{ij}\right|\varphi_j^{r\rho_{m+1,j}}\left\|z_j(s-\sigma)\right\|_2^r\mathrm{d}\sigma,$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ij}\right|\int_0^{+\infty}K_{ij}(\sigma)\int_0^t e^{cs}\varphi_j^{r\rho_{m+1,j}}\left\|z_j(s-\sigma)\right\|_2^r dsd\sigma$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ij}\right|\int_0^{+\infty}K_{ij}(\sigma)e^{c\sigma}\int_0^t e^{cs}\varphi_j^{r\rho_{m+1,j}}\left\|z_j(s)\right\|_2^r dsd\sigma$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ij}\right|\int_0^{+\infty}K_{ij}(\sigma)e^{c\sigma}\int_{-\sigma}^{0}e^{cs}\varphi_j^{r\rho_{m+1,j}}\left\|z_j(s)\right\|_2^r dsd\sigma$$

Hence, we have

$$V(z(t),t)\le\sum_{i=1}^{n}\left\|z_i(0)\right\|_2^r+\int_0^t ce^{cs}\sum_{i=1}^{n}\left\|z_i(s)\right\|_2^r ds$$

$$+\int_0^t\overline{\alpha}_i e^{cs}\sum_{i=1}^{n}\Big\{-r\beta_i\left\|z_i(s)\right\|_2^r$$

$$+\sum_{j=1}^{n}\left|a_{ij}\right|\left[\sum_{k=1}^{m}q_k\mu_j^{\frac{r\rho_{k,j}}{q_k}}\left\|z_i(s)\right\|_2^r+\mu_j^{r\rho_{m+1,j}}\left\|z_j(s)\right\|_2^r\right]$$

$$+\sum_{j=1}^{n}\left|b_{ij}\right|\left[\sum_{k=1}^{m}q_k\varphi_j^{\frac{r\rho_{k,j}}{q_k}}\left\|z_i(s)\right\|_2^r\right.$$

$$\left.+\varphi_j^{r\rho_{m+1,j}}\int_0^{+\infty}e^{c\sigma}K_{ij}(\sigma)\left\|z_j(s)\right\|_2^r\mathrm{d}\sigma\right]\Big\}ds$$

$$+\overline{\alpha}_i\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ij}\right|\int_0^{+\infty}\int_{-\sigma}^{0}e^{cs}K_{ij}(\sigma)\varphi_j^{r\rho_{m+1,j}}e^{c\sigma}\left\|z_j(s)\right\|_2^r dsd\sigma$$

$$+\int_0^t re^{cs}\sum_{i=1}^{n}\left\|z_i(s)\right\|_2^{r-2}\int_X z_i(s)\sum_{j=1}^{n}\sigma_{ij}(u_j(s,x))d\omega_j(s)dx$$

$$+\int_0^t\frac{r-1}{2}e^{cs}\sum_{i=1}^{n}\sum_{j=1}^{n}L_{ij}^2 r\left\|z_i(s)\right\|_2^r ds$$

$$= \sum_{i=1}^{n} \|z_i(0)\|_2^r - \int_0^t e^{cs} \sum_{i=1}^{n} \Bigg[(\overline{\alpha}_i r\beta_i - c) - \overline{\alpha}_i \sum_{j=1}^{n} |a_{ij}| \sum_{k=1}^{m} q_k \mu_j^{\frac{r\rho_{k,j}}{q_k}}$$

$$- \overline{\alpha}_i \sum_{j=1}^{n} |a_{ji}| \mu_j^{r\rho_{m+1,j}} - \overline{\alpha}_i \sum_{j=1}^{n} |b_{ij}| \sum_{k=1}^{m} q_k \varphi_j^{\frac{r\rho_{k,j}}{q_k}}$$

$$+ \frac{r(r-1)}{2} \sum_{j=1}^{n} L_{ij}^2 \Bigg] \|z_i(s)\|_2^r ds$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| \int_0^{+\infty} \int_{-\sigma}^{0} e^{cs} K_{ij}(\sigma)\, \overline{\alpha}_i \varphi_j^{r\rho_{m+1,j}} e^{c\sigma} \|z_i(s)\|_2^r ds d\sigma$$

$$+ \int_0^t re^{cs} \sum_{i=1}^{n} \|z_i(s)\|_2^{r-2}$$

$$\times \int_X z_i(s) \sum_{j=1}^{n} \sigma_{ij}(u_j(s,x)) d\omega_j(s) dx,$$

from (A6) and Eq. (10), we can get

$$V(z(t),t)$$

$$\leq \sum_{i=1}^{n} \|z_i(0)\|_2^r$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| \int_0^{+\infty} \int_{-\sigma}^{0} e^{cs} K_{ij}(\sigma)\, \overline{\alpha}_i \varphi_j^{r\rho_{m+1,j}} e^{c\sigma} \|z_i(s)\|_2^r ds d\sigma$$

$$+ \int_0^t re^{cs} \sum_{i=1}^{n} \|z_i(s)\|_2^{r-2} \int_X z_i(s) \sum_{j=1}^{n} \sigma_{ij}(u_j(s,x)) d\omega_j(s) dx. \tag{13}$$

It is obvious that the right hand-side of (13) is a nonnegative semimartingale. From Lemma 1, it can be easily seen that its limit is a.s. finite as $t \to \infty$, which shows that

$$\limsup_{t\to\infty} V(u(t),t) < +\infty, \quad P-a.s..$$

Since

$$\limsup_{t\to\infty} \left[e^{ct} \sum_{i=1}^{n} \|z_i(t)\|_2^r \right] < +\infty, \quad P-a.s..$$

Which implies

$$\limsup_{t\to\infty} \frac{1}{t} \log\left[\sum_{i=1}^{n} \|z_i(t)\|_2^r \right] < -c, \quad P-a.s.,$$

that is

$$\limsup_{t\to\infty} \frac{1}{t} \log\left[\|z\|^r \right] < -c, \quad P-a.s..$$

The proof is completed.

Theorem 2 Under the assumptions of Theorem 1, model (3) is p-th moment exponentially stable.

Proof. Taking expectations for both hand-sides of (13), it follows from

$$E\int_0^t re^{cs} \sum_{i=1}^{n} \sum_{j=1}^{n} \|z_i(s)\|_2^{r-2} \int_X z_i(s) \sigma_{ij}(u_j(s,x)) dx dw_j(s) = 0$$

that

$$e^{ct} \sum_{i=1}^{n} E\|z_i(t)\|_2^r \leq \sum_{i=1}^{n} E\|z_i(0)\|_2^r$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}| \int_{-\sigma}^{0} e^{cs} \int_0^{+\infty} K_{ij}(\sigma) \varphi_j^{r\rho_{m+1,j}} e^{c\sigma} E\|z_j(s)\|_2^r d\sigma ds$$

Clearly, there exists some positive constant K such that

$$E\|z(t)\|^r = E\left(\sum_{i=1}^{n} \|z_i(t)\|_2^r \right) \leq KE\left(\sum_{i=1}^{n} \|z_i(0)\|_2^r \right) e^{-ct}.$$

The proof is completed.

4 Conclusions

In this paper, the almost surely exponential stability and exponential p-th moment stability have been studied for a class of stochastic Cohen–Grossberg neural networks with distributed delays and reaction–diffusion term. Two delay-independent and easily verifiable sufficient conditions have been obtained to ensure the existence, uniqueness, almost surely exponential stability and exponential p-th moment stability of the equilibrium point for the addressed stochastic Cohen-Grossberg neural network with distributed delays and reaction-diffusion terms.

References

[1] M.A. Cohen, S. Grossberg, Absolute stability of global pattern formation and parallel memory storage by competitive neural networks, IEEE Transactions on Systems, Man, and Cybernetics, 13 (5) (1983) 815–826.
[2] L. Wang, X.F. Zou, Harmless delays in Cohen–Grossberg neural networks, Physica D 170 (2) (2002) 162–173.
[3] T.P. Chen, L.B. Rong, Robust global exponential stability of Cohen–Grossberg neural networks with time delays, IEEE Transactions on Neural Networks, 15 (1) (2004) 203–206.
[4] J.D. Cao, J.L. Liang, Boundedness and stability for Cohen–Grossberg neural network with time-varying delays, Journal of Mathematical Analysis and Applications, 296 (2) (2004) 665–685.

[5] S. Arik, Z. Orman, Global stability analysis of Cohen-Grossberg neural networks with thime-vary delays, Phy. Lett. A, 341(2005)410-421.
[6] J.Y. Zhang, Y. Suda, H. Komine, Global exponential stability of Cohen-Grossberg neural networks with variable delays, Phy. Lett. A 338 (2005) 44-55.
[7] K. Yuan, J.D. Cao, An analysis of global asymptotic stability of delayed Cohen–Grossberg neural networks via nonsmooth analysis, IEEE Transactions on Circuits and Systems I 52 (9) (2005) 1854–1861.
[8] K.N. Lu, D.Y. Xu, Z.C. Yang, Global attraction and stability for Cohen-Grossberg neural networks with delays, Neural Networks 19 (10) (2006) 1358-1549.
[9] Q.K. Song, J.D. Cao, Stability analysis of Cohen–Grossberg neural network with both time-varying and continuously distributed delays, Journal of Computational and Applied Mathematics 197 (1) (2006) 188–203.
[10]Q.K. Song, Z.D. Wang, Stability analysis of impulsive stochastic Cohen-Grossberg neural networks with mixed time delays, Physica A, 387 (2008) 3314–3326.
[11] Q.K. Song, J.D. Cao, Global exponential stability and existence of periodic solutions in BAM networks with delays and reaction–diffusion terms, Chaos, Soliton and Fractals , 23 (2005) 421–430.
[12] Q.K. Song, J.D. Cao, Z.J. Zhao, Periodic solutions and its exponential stability of reaction–diffusion recurrent neural networks with continuously distributed delays, Nonlinear Anal.: RWA, 7 (2006) 65–80.
[13] Z.J. Zhao, Q.K. Song, J. Zhang, Exponential periodicity and stability of neural networks with reaction–diffusion terms and both variable and unbounded delays, Comput. Math. Appl. 51 (2006) 475–486.
[14] W. Allegretto, D. Papini, Stability for delayed reaction–diffusion neural networks, Physics Letters A, 360 (2007) 669–680.
[15] Z.G. Yang, D.Y. Xu, L. Xiang, Exponential *p*-stability of impulsive stochastic differential equations with delays, Physics Letters A 359 (2) (2006) 129–137.
[16] H. J. Wu, J. T. Sun, *p*-moment stability of stochastic differential equations with impulsive jump and Markovian switching, Automatica 42(10) (2006) 1753-1759.
[17] H.Y. Zhao, N. Ding, Dynamic analysis of stochastic Cohen–Grossberg neural networks with time delays, Applied Mathematics and Computation 183 (1) (2006) 464–470.
[18] Z.D.Wang, Y.R. Liu, M. Li, X.H. Liu, Stability analysis for stochastic Cohen–Grossberg neural networks with mixed time delays, IEEE Transactions on Neural Networks 17 (2006) 814–820.
[19] Z.D. Wang, Y.R. Liu, K. Fraser, X.H. Liu, Stochastic stability of uncertain Hopfield neural networks with discrete and distributed delays, Physics letters A 354 (4) (2006) 288-297.
[20] J.H. Zhang, P. Shi, J.Q. Qiu, Novel robust stability criteria for uncertain stochastic Hopfield neural networks with time-varying delays, Nonlinear Analysis RWA 8 (4) (2007) 1349–1357.

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