# Wavenumber Integration for Generating an Acoustic Field in the Measurement of Stratified Sea Bottom Properties Using Propagator Matrix Approach

**Prasanna Kumar. S.C**
RVCollege of Engineering,
Instrumentation Technology
Bangalore, India.

**Dr. Ramakanth Kumar. P**
RVCollege of Engineering,
Information Science & Engineering
Bangalore, India.

**Suresh Kumar Thakur**
Deputy Director
Naval Research Board
New Delhi, India.

**Summary**

Determination of total solution for the depth dependence of the field: the depth dependence Green's function for Self Source Acoustic field. In Particular the acoustic field is obtained in terms of the wave number integration of a horizontally stratified ocean. The Inverse Hankel Transform is obtained for shallow water over a Layered inhomogeneous elastic bottom using 'Propagator matrix method' with Fast Field Approximation Technique. Input here the part of summary.

***Key words:*** *Fast Field Approximation, Geoacoustic properties, Green's Function, Helmholtz equation, Hankel transforms, Propagator Matrix, and Underwater acoustic.*

## 1. Introduction

Properly predicting acoustic propagation, especially in shallow water waveguides and/or at low frequencies is very essential in estimating the geoacoustic seabed properties. The problem of determining sea bed geoacoustic properties from measured ocean acoustic fields has received considerable attention in recent years. Traditional methods for measuring seafloor properties are costly and time-consuming. Therefore, in the last few years there has been a growing interest in providing solutions to the inverse problem consisting of determining seafloor properties from the measurement of the acoustic field in the water column. This approach provide an advantage of there is no need for deploying any equipment in the bottom for measurement, and we can cover a much larger area in a single inversion methods as compare to traditional local methods[1].

Geoacoustic inversion represents a strongly nonlinear inverse problem with no direct solution. Matched field inversion (MFI) makes use of the pressure field received on an array of sensors. The measured acoustic field contains the information about the ocean environment, which can be extracted using MFI.

The ocean is an acoustic waveguide limited above by the sea surface and below by the sea floor. The speed of sound in the waveguide is normally related to static pressure, salinity,

is an increasing function of pressure, salinity and temperature, the latter being a function of depth.

In general, all of oceanographic structures have an effect on sound propagation, both as a source of attenuation and of acoustic fluctuations. Considering the upper and lower boundaries of the ocean waveguide, the sea surface is a simple horizontal boundary and nearly a perfect reflector. The sea floor on the other hand, is a lossy boundary with strongly varying topography across ocean basins. Both boundaries have small-scale roughness associated with them which causes scattering and attenuation of sound. The structure of the ocean bottom depends on the local geology, but in general it consists of a thin stratification of sediments overlying the oceanic crust in the deep ocean and relatively thick stratification over continental crust. The nature of stratification is dependent on many factors, including geological age and local geological activity.

The importance of treating the ocean bottom accurately in the numerical models depends on the factors such as source receiver separation, source frequency and ocean depth.

A point source in the ocean is considered which depends on range 'r', depth 'z' and azimuth angle 'Φ'. The acoustic signal produced by the source is realized in the form of pressure signal. Receivers are a group of sensors which receives the pressure signal from the source Ex. Hydrophone array. There is some attenuation when the

Manuscript received November 5, 2008
Manuscript revised November 20, 2008

signal is passed from source to receiver. The received signal is then passed through ADC which converts analog signal to digital samples. The sample thus obtained contains measurement noise, which is called real data.

Generating an acoustic field (Pressure field) using this real data is an important aspect in the measurement of stratified sea bottom properties. The best method to achieve this is wavenumber integration technique.

Basically the wavenumber integration technique is a numerical implementation of the integral transform for horizontally stratified media. The field solution is in the form of a spectral (wavenumber) integral of solutions to the depth-separated wave equation. The wavenumber integration approach evaluates the integrals directly by numerical quadrature. In underwater acoustics, wavenumber integration approach uses FFTs for evaluation of the spectral integrals, hence this method is also called as FFPs (fast field programs).

In the wavenumber integration technique the series of integral transforms are applied to the Helmholtz equation which reduces the original four dimensional partial differential equations (3 space dimensions and 1 time dimension) in to a series of ordinary differential equations in the depth co-ordinate. These equations were then solved analytically within each layer in terms of unknown amplitudes which were determined by matching boundary conditions at the interfaces.

For more efficient analysis, the Fast Field Program were developed which applies an elegant recursive technique to determine the depth-dependent solution for many horizontal wavenumbers simultaneously and is therefore extremely efficient.

## 2. Wavenumber Integration

This technique is a numerical implementation of the integral transform for horizontally stratified media. The field solution is in the form of a spectral (wavenumber) integral of solutions to the depth-separated wave equation. The wavenumber integration approach evaluates the integrals directly by numerical quadrature. In underwater acoustics, wavenumber integration approach uses FFTs for evaluation of the spectral integrals, hence this method is also called as FFPs (fast field programs).

### 2.1 Integral Transform Solution

For a source distribution along a vertical axis in horizontally stratified environment, cylindrical co-ordinate system (r, φ, z) is introduced with z axis passing through the sources making the field independent of angle φ.

The acoustic field with time dependence exp(-iωt) in layer m containing the source can be expressed in terms of scalar displacement potentials $\Psi_m$ (r, z) which satisfy the Helmholtz equation[2],

$$[\nabla^2 + k^2_m(z)]\, \Psi_m (r, z) = f_s(z, w)\, \delta(r)\ /2\pi r \qquad \text{----} \quad (1)$$

Equation (1) represents the Helmholtz equation for Displacement Potentials

Where $k_m(z)$ is the medium wavenumber

$$k_m(z) = \frac{\omega}{c(z)} \qquad \text{------------------------} \quad (2)$$

### 2.2 Helmholtz equation:

The Helmholtz equation represents the time-independent form of the original equation, results from applying the technique of separation of variables to reduce the complexity of the analysis.

The wave equation is:

$$\frac{1}{c^2}\frac{d^2}{dt^2} - \nabla^2)\Psi(\overrightarrow{x}, t) = \rho(\overrightarrow{x}, t) \qquad \text{-------------} \quad (3)$$

where

- $c$ is the velocity of propagation,
- $\rho(\overrightarrow{x}, t)$ is a prescribed source
- and $\nabla^2 = \Delta$

Helmholtz equation

$$[\nabla^2 + k^2(r)]\psi(r, \omega) = 0 \qquad \text{---------} \quad (4)$$

With $k(r) = \dfrac{\omega}{c(r)}$

For layers without sources the field must satisfy the homogeneous Helmholtz equation with $f_s(z, \omega) = 0$.

### 2.3 Hankel transform

$$f(r, z) = \int_0^\infty f(k_r, z) J_0(k_r r) k_r dk_r$$

$$f(k_r, z) = \int_0^\infty f(r, z) J_0(k_r r) r dr \qquad \text{-----} (5)$$

Applying the forward Hankel transform to equation (1) the depth separated wave equation i.e an ordinary differential equation in depth is obtained.[7]

$$[\frac{d^2}{dz^2} - [k_r^2 - k_m^2(z)]]\psi_m(k_r, z) = \frac{f_s(z)}{2\pi} \qquad \text{-------} (6)$$

Equation (6) represents the depth separated wave equation.

The total solution for the depth dependence of the field i.e depth dependent Green's function[1] is

$$\psi_m(k_r, z) = \hat{\psi}_m(k_r, z) + A_m^+(k_r)\psi_m^+(k_r, z) + A_m^-(k_r)\psi_m^-(k_r, z) \qquad \text{-------} (7)$$

Where $A_m^+(k_r)$ and $A_m^-(k_r)$ are arbitrary coefficients to be determined from the boundary conditions at the interfaces between the layers. The particular solution to the equation (2) is chosen to be the field produced by the sources in the absence of boundaries. When the unknown coefficients are found, the total field at the angular frequency ω is found at the range r by evaluating the inverse Hankel transform.

The field at each interface now has two distinct integral representations, one from the layer above and one from the layer below. Depending on the type of interface, a certain set of boundary conditions must be satisfied. [2]

- At a fluid-fluid interface, both the vertical displacement $\omega$ and the normal stress $\sigma_{zz}$ must be continuous. If one of the media is a vacuum the normal stress vanishes.
- At a fluid-solid interface, both displacement and normal stress must be continuous while the tangential stress $\sigma_{rz}$ vanishes. If the fluid layer is replaced by a vacuum, both the stress must vanish.
- At a 'welded' interface between two solid media, all the four parameters must be continuous.

Since the boundary conditions have to be satisfied at all ranges 'r', it is clear that they must be satisfied by the kernels in the integral representations as well. By imposing the appropriate interface conditions together with the radiation conditions for $z \to \pm\infty$, we obtain a linear system of equations in the unknown coefficients $A^+, A^-, B^+, B^-$. In principle this system has to be solved for all values of the horizontal wavenumber $k_r$, and the total field can then be determined by evaluating the inverse transforms. Both the solution of the linear system of equations and the evaluation of the inverse transforms must done numerically, requiring truncation and discretization of the horizontal wavenumber axis.

The difference between the various numerical implementations concerns the method used for solving the linear system of equations in the unknown amplitudes and the numerical evaluation of the inverse Hankel transform.

The numerical solution of the full wavefield problem divides naturally into two parts. First, the depth-dependent Green's function is found at a discrete number of horizontal wavenumbers for the selected receiver depths. Secondly, the wavenumber integral is evaluated, yielding the transfer function at the selected depths and ranges. To yield the total response in time the above two steps are repeated and at selected frequencies, 'frequency integration' has to be performed. The overall efficiency of the wavenumber integration approach is closely related to the efficiency with which the depth equation is solved.

## 2.4 Propagator matrix approach:

Since the trapped waveguide field generally dominates the ocean acoustic field, the invariant embedding approach is inconvenient. The global matrix approach is unconditionally stable. Hence propagator matrix approach is chosen for simplicity. The numerical solutions of the depth separated wave equation attempt to reduce both computational and memory requirements by developing the propagator matrix scheme for the solution.
The vector of field parameters at interface m, bounding layer m below, is given by the matrix relation,

$$v_m(k_r, z_m) = c_m(k_r, z_m)a_m(k_r) \qquad \text{----} (12)$$

Where the vector $v_m(k_r, z_m)$ contains the displacements and stresses at interface m, and $a_m(k_r)$ is a vector containing the wavefield amplitudes in layer m. similarly at the top interface, m-1, of layer m,

$$v_m(k_r, z_{m-1}) = c_m(k_r, z_{m-1})a_m(k_r) \qquad \text{-------} (13)$$

On solving we get,

$$v_m(k_r, z_{m-1}) = P_m(k_r)v_m(k_r, z_m) \qquad \text{------- -} (14)$$

With $P_m(k_r)$ being the propagator matrix for layer m,

$$P_m(k_r) = c_m(k_r, z_{m-1})[c_m(k_r, z_m)]^{-1} \quad \text{-----} (15)$$

The inverse can be obtained in closed form, leading to closed form expressions also for the coefficients of the propagator matrix. Using the continuity of the field parameters at the interfaces, Eq. (14) can be used recursively to establish a matrix relation between the field parameters at some interface m and the parameters at a lower interface n,

$$v_m(k_r, z_m) = R_n^m(k_r) v_{n+1}(k_r, z_n) \quad \text{---------} (16)$$

With

$$R_n^m(k_r) = \prod_{l=m+1}^{n} [P_l(k_r)] \quad \text{----------} (17)$$

For a source in fluid medium the normal stress or pressure is continuous and the vertical displacement is discontinuous. Now we can use Eq.(16) to 'propagate' the solution from the lowermost interface to an artificial interface number s introduced at the source depth ($z_s$), adding the discontinuity in the field parameters and continue to propagate the solution upto the uppermost interface yielding,

$$v_1(k_r, z_1) = R_s^1(k_r) \left[ R_{N-1}^s(k_r) v_N(k_r, z_{N-1}) + \hat{v}(k_r, z_s) \right] \quad \text{---} (18)$$

The boundary conditions at the uppermost and lowermost interfaces, the radiation conditions in the limiting halfspaces, provide the necessary additional equations to determine the unknowns.

The propagator matrix approach reduces the number of equations to be solved from 2(N-1) in the DGM approach to just 2 in the purely fluid case and from 4(N-1) to 4 in the elastic case. In addition the coefficient matrices in Eq. (18) are determined by successive multiplication of small matrices. Consequently, the propagator matrix approach has insignificant memory requirements and is easily implemented.

In the propagator matrix approach the received field is determined by introducing a dummy interface at the receiver depth and then using the recurrence, Eq. (16) to determine the kernel at the receiver depth, once the field parameters are found at the lowermost interface from the solution of Eq. (18).

Since the standard propagator matrix approach couples the field at the lowermost and uppermost interfaces in the stratification, it is not surprising that this approach encounters numerical instability problems.

Let layer m be an isovelocity fluid of thickness $h_m$. Inserting the equation

$$c_m(k_r, z) = d_m(k_r) e_m(k_r, z),$$

(where $d_m(k_r)$ is a depth independent matrix and $e_m(k_r, z)$ is a diagonal matrix containing the exponentials)

into the Eq. (15) yields the following propagator matrix,

$$P_m(k_r) = d_m(k_r) e_m(k_r, z_{m-1}) \left[ e_m(k_r, z_m) \right]^{-1} \left[ d_m(k_r) \right]^{-1} \quad \text{--} (19)$$

Where the product of the two diagonal matrices including the exponentials are shown to be

$$e_m(k_r, z_{m-1}) \left[ e_m(k_r, z_m) \right]^{-1} = \begin{bmatrix} e^{-ik_{z,m}h_m} & 0 \\ 0 & e^{ik_{z,m}h_m} \end{bmatrix} \quad \text{------------} (20)$$

For wavenumbers, the field in layer m one of the exponentials in the above equation becomes large and the other small. Truncation errors may therefore magnify significantly across the layer, in turn yielding unstable solutions to Eq. (18).

To determine the acoustic field parameters at a particular receiver range r and depth z, we must numerically evaluate the inverse Hankel transform of the solution to the depth separated wave equation at depth z [1],

$$g(r, z) = \int_0^\infty g(k_r, z) J_m(k_r r) k_r dk_r \quad \text{------} (21)$$

Where,

$g(r, z)$ represents the field parameter of interest, e.g., acoustic pressure, or a particular displacement or stress component.

$g(k_r, z)$ is the associated wavenumber kernel.

The order of Bessel function is m=0 except for horizontal displacement and shear stress. The evaluation of this numerical integral is complicated by the following features,

- Infinite upper integration limit
- The oscillatory nature of the Bessel function (especially for long ranges)
- Waveguide problems i.e. poles on or close to the real wavenumber axis.

### 2.5 Fast Field Approximation:

Accurate evaluation of the inverse Hankel Transform, can be obtained by the Fast Field Program (FFP) integration technique, in this technique first the Bessel function is expressed in terms of Hankel functions.

$$J_m(k_r r) = \frac{1}{2}\left[H_m^{(1)}(k_r r) + H_m^{(2)}(k_r r)\right] \quad \text{----} \quad (22)$$

Where

$H_m^{(1)}$ with the present choice of the time frequency transform corresponds to outgoing waves and $H_m^{(2)}$ to incoming waves. $H_m^{(2)}$ is important only for representing the standing wavefield at very short ranges and is therefore neglected. Now replacing $H_m^{(1)}(k_r r)$ by its asymptotic form,

$$\lim_{k_r \to \infty} H_m^{(1)}(k_r r) = \sqrt{\frac{2}{\pi k_r r}}\, e^{i[k_r r - (m+\frac{1}{2})\frac{\pi}{2}]} \quad \text{---} \quad (23)$$

To arrive at the following expression for the inverse Hankel transform.

$$g(r,z) = \sqrt{\frac{1}{2\pi r}} e^{-i(m+\frac{1}{2})\frac{\pi}{2}} \int_0^\infty g(k_r, z)\sqrt{k_r} e^{ik_r r} dk_r \quad \text{-----} \quad (24)$$

The approximation of Eq. (21) by Eq. (24) does not remove any of the complications concerning the integration interval or the oscillatory nature of the integrand. However the exponential function is more suitable for numerical integration than the Bessel function, particularly in terms of computation time.

To numerically evaluate the FFP integral, Eq. (24), we must either use a quadrature scheme for semi-infinite integration intervals or truncate the integration interval at a wavenumber beyond which the contribution to the integral is insignificant.

## 3. Simulated Result

The following wave forms shows the acoustic field created for various ranges.

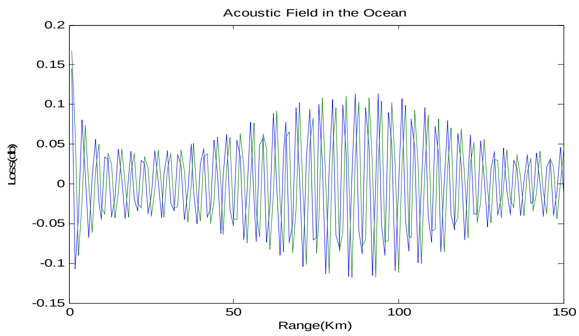


**Fig.1** Acoustic field for frequency of 225Hz, source depth Zs = 7 km, receiver depth Zr = 45 km and Range varying between 0 and 150 km.

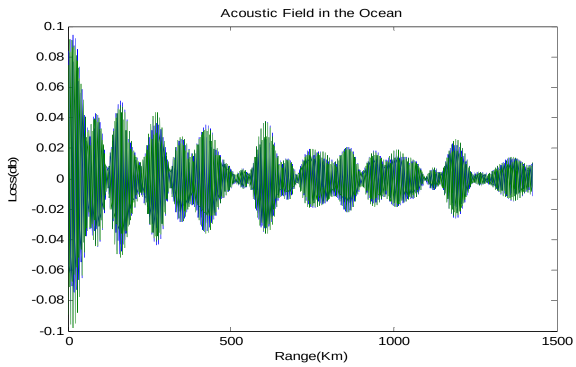


**Fig.2** Acoustic field for Frequency = 225Hz , source depth $z_s$ = 7km, receiver depth $z_r$ = 45km and Range varying between 0 and 1500km

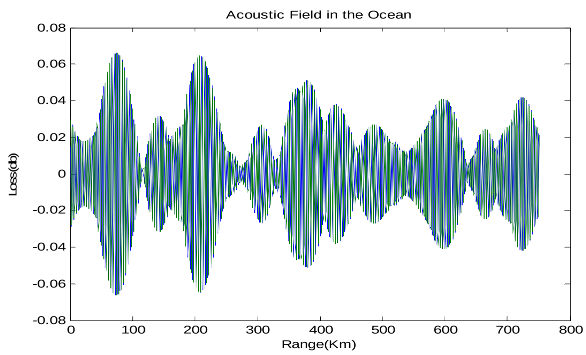


**Fig.3** Acoustic field for Frequency = 200 Hz, source depth $z_s$ = 7km, receiver depth $z_r$ = 45km And Range varying between 0 and 800km.

## 4. Conclusion

This approach presents a fast calculation method for two dimensional wavenumber integration, which can be used to model the acoustic field in ocean generated by ship generated source in shallow water, the propagator Matrix approach is relatively easier to implement. Only few lines of code are needed to complete the computation of the propagator matrices.

Propagation matrix approach requires the addition of a dummy interface at every receiver depth. Consequently, the computation time is proportional to the sum of the number of layers and the number of receivers. Hence multiple sources are not treated efficiently by the propagator matrix approach

## References

[1] Sergo M. Jesus, "Can maximum Likelihood Estimators Improve Genetic Algorithm Search in Geoacoustic Inversion?" *Journal of computational Acoustics of IMACS.*

[2] F. B. Jensen, W. A. Kuperman, M. B. Porter, and H. Schmidt, "Computational Ocean Acoustics", New York: *AIP, 1994*.

[3] C.M. Linton, "The Green's function for the two-dimensional Helmholtz equation in periodic domains". *J Eng Math* 33 (1998), pp. 377–402.

[4] C. T. Tindle and N. R. Chapman, "A phase function for finding normal mode eigenvalues over a layered elastic bottom," *J. Acoust. Soc. Amer.*, vol. 96, no. 3, pp. 1777–1782, Sep. 1994.

[5] W. T. Thomson, "Transmission of elastic waves through a stratified solid," *J. Appl. Phys.*, vol. 21, pp. 89–93, 1950.

[6] N. A. Haskell, "The dispersion of surface waves in multilayered media," *Bull. Seism. Soc. Amer.*, vol. 43, pp. 17–34, 1953.

[7] H. Schmidt and F. B. Jensen, "A full wave solution for propagation in multilayered viscoelastic layered media with application to Gaussian beam reflection at fluid-solid interfaces," *J. Acoust. Soc. Amer.*, vol. 77, pp. 813–825, 1985.

[8] M. West, R. A. Sack, F. Walkden, "The Fast Field Program (FFP). A Second Tutorial: Application to Long Range Sound Propagation in the Atmosphere. Applied Acoustics," Vol. 33, pp, 199-228, 1991.

[9] M.K. Jain, S.R.K. Iyengar ,"Numerical Methods for scientific & Engineering Computation".

**Prasanna Kumar. S.C,** Asst. Professor department of Instrumentation Technology, RV College of Engineering, Mysore Road Bangalore. Completed M.Tech, in "Bio-Medical Instrumentation" from SJ College of Engineering, Mysore, pursuing PhD at Avinashilingam University. Life Member of ISTE.

**Dr. Ramakanth Kumar. P,** Professor of CSE department, RV College of Engineering, Bangalore, Completed M.S in Computer Science from BITS pillani, Doctorate From Mangalore University in Computer Science and Engineering.

**Suresh Kumar Thakur,** has completed MBA in Finance and M Tech (Radar & Communication Engg.) With specialization in Under Water Electronics, Presently Dy Director at Directorate of Naval R&D, DRDO HQs, New Delhi; as well as looking after other technical aspects on Naval R&D as Joint Director.