Correlated MIMO Rayleigh Channels: Eigenmodes and Capacity Analyses

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Abstract
Equipping a wireless system's transmitter and receiver with multiple antennas represents one of the most promising solutions to improve the bandwidth efficiency and system reliability without need to use extra bandwidth or transmitting more power into the channel. Such a system is called Multiple Input Multiple Output (MIMO) and offers high capacity. This gain in capacity is affected by the channel correlation and the number of receive and transmit antennas. In this paper, the MIMO Rayleigh channel matrix will be analyzed. The behaviour of channel matrix eigenmodes and the power levels allocated to them by the water-filling algorithm with SNR and with the channel correlation will be explored. Capacity of the channel eigenmodes and total capacity of the channel will also be studied. It was found that the transmitter allocates all transmitted power to the strongest eigenmode at low SNR, and divides this power equally among the eigenmodes at high SNRs. The strongest eigenmode of a channel matrix increases with correlation, while the other eigenmodes of the channel decrease. When the channel correlation approaches 1, the strongest eigenmode approaches the M_r * M_t, while the other eigenmodes approach zero. At low SNRs, the transmitter allocates the power to the strongest eigenmode only, so that the capacity of the channel increases with the channel correlation. The channel matrix is considered perfectly known at both sides of the wireless link.
Key words:
Channel correlation, Channel capacity, Channel eigenmodes, Low and high SNR.

1. Introduction

There are always increasing demands for high data rates and better quality of services [1] – [10]. These demands can be fulfilled using the conventional systems, i.e. Single Input Single Output (SISO) -which are limited by multipath fading and interference- by increasing either the channel bandwidth, the transmit power, or both. However, this simplistic solution is not attractive for the following reasons. First, the transmit power cannot exceed a certain value for its biological hazards, this is from one side. On the other side, building linear receivers with sensitivity beyond 30-35 dB is technically difficult and costly [11]. Second, frequency spectrum is a scarce and expensive resource especially below the 6 GHz. This makes it very difficult and costly to increase the channel bandwidth [11]. For these reasons, new techniques must be introduced to realize the needs of the modern wireless systems. These techniques must be affordable in terms of cost and biologically unharmful.

Using multiple antennas at both sides of the wireless link, represents one of the most promising solutions to improve the bandwidth efficiency and system reliability without need to use extra bandwidth or transmitting more power into the channel [1], [2], [4]. Such a system is called Multiple Input Multiple Output (MIMO).

In MIMO systems, transmitter is equipped with more than one antenna to transmit the data and receiver is equipped with more than one antenna to receive this data. The capacities achieved by MIMO systems are very high comparing with the conventional systems (SISO, SIMO, and MISO). It has been proven that capacities of these systems increase linearly with the number of antenna pairs if the channel is highly scattering and rich with multipath [1], [2].

These gains in capacity and reliability depend on the number of antennas at both sides, the statistics of the channel, and the channel knowledge at the transmitter [12]. Channel correlation is known to be one of the most undesired impairments that lead to MIMO channel capacity degradation [2], [13], [14], [15]. However, this is not always the case as we will see in this paper; where channel correlation is an advantage when the channel is known at the transmitter and when signal to noise ratio SNR is low. Channel correlation improves the channel capacity at low SNRs if the transmitter knows the channel matrix.

In this paper, the channel eigenmodes, the power allocated to them and the capacity of these eigenmodes

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will be studied and analyzed when the channel is known at the transmitter.
This paper is organized as follows. In Section II, we will discuss the MIMO system model, the channel model and the correlation model used. MIMO channel capacity will be derived and developed in Section III. Section V is devoted to the simulation results and finally, the conclusions are drawn.

2. MIMO System Model

Since MIMO is a narrowband technology [16], a narrowband, flat fading Rayleigh correlated channel and a single user, with M_t transmit and M_r receive antennas will be considered. The channel matrix H is assumed to be perfectly known at the receiver and the transmitter. The total transmit power from all antennas is E_s, where E_s is independent of the number of antennas at the transmit side. This system is described as follows

$$y = Hx + n \quad (1)$$

where $x = [x_1\ x_2 \ldots x_{Mt}]^T$ is the $M_t \times 1$ complex vector representing the transmitted signal with the power constraint

$$tr\,(E\,(xx^H)) \leq E_s, \quad (2)$$

$y = [y_1\ y_2 \ldots y_{Mr}]^T$ is the $M_r \times 1$ complex vector representing the received signal and $n = [n_1\ n_2 \ldots\ n_{Mr}]^T$ is the $M_r \times 1$ complex vector representing the additive white Gaussian noise vector (AWGN) with a zero mean and covariance matrix $\delta^2{}_n I_{Mr}$ where I_{Mr} is the $M_r \times M_r$ identity matrix. $(.)^T$, $(.)^H$, $tr(.)$, and $E(.)$ denote transposition, conjugate transpose, trace, and expectation, respectively. H is the $M_r \times M_t$ MIMO channel matrix, whose entries h_{ij} represent the complex channel response of the channel between j^{th} transmit antenna and the i^{th} receive antenna.

2.1 MIMO Channel Model

Kronecker model will be used here in this paper to describe the Rayleigh correlated channel. In this model the channel spatial correlation $R_H = E\,[vec\,(H)\ vec\,(H^H)]$ [13], where $vec\,(H)$ denotes the $M_t M_r \times 1$ vector formed by stacking the columns of H. When the channel is rich with multipath and no LOS component exists, the transmit antennas correlation and receive antennas correlation can be considered independent. In such case, the channel correlation matrix R_H can be decomposed into two correlation matrices, the transmit correlation matrix R_t and the receive correlation matrix R_r, so as $R_H = R_t^T \otimes R_r$, where $\otimes$ is the kronecker product. Hence, the Rayleigh correlated channel can be written as $H = R_r^{1/2}\, H_{i.i.d}\, R_t^{1/2}$, this channel model is called Kronecker model. Where R_r is the receive correlation matrix, and R_t is the transmit correlation matrix. $H_{i.i.d}$ is the uncorrelated white channel matrix.

2.2 Correlation Model

The exponential correlation model will be adopted in this paper [17].
For this model, the components of the correlation matrices (R_r and R_t) are given by

$$r_{ij} = \begin{cases} r^{j-i}, & i \leq j \\ r_{ji}^{\ *}, & i > j \end{cases} \qquad |r| \leq 1$$

where r is the complex correlation coefficient of neighboring antenna. This model is suitable for studying the effects of correlation on the channel capacity, although it is not accurate for some real world scenarios. However, this model is physically reasonable, where the correlation between the adjacent antennas is larger than the correlation between none-adjacent antennas [17].

3. MIMO Channel Capacity

The theoretical capacity of this system is expressed by the following formula [1]

$$C = E_H\left[\log_2 \det\left(I_{M_r} + \frac{\rho}{M_t} HQH^H\right)\right] \quad (3)$$

where, $Q = E[xx^H]$ is the input covariance matrix, and ρ (SNR) = (E_s / N_0), E_s is the total transmit power, N_0 is the noise power in each antenna at the receive side.
In equation (3), the mean is taken over the random channel. The capacity depends on the number of antennas at both sides, input covariance matrix Q, and the channel statistics. When channel H is Rayleigh distributed, its mean will be zero (no LOS component exists) and its covariance is 1.

Q represents the covariance matrix of the transmitted vector. This matrix is diagonal and its elements are all real numbers. The trace of this matrix should not exceed the number of transmit antennas. In other words, $tr\,(Q) = M_t$. There are two cases for this matrix. When the transmitter does not have a prior knowledge about the channel matrix (i.e. uninformed transmitter), this matrix will be equal to the identity matrix $Q = I_{M_t}$, meaning that the transmitter will divide the total transmitted power E_s equally among its antennas, and when the instantaneous channel matrix is available at the transmitter (informed transmitter), Q matrix can be optimized for optimum capacity. We will consider the latter case (Informed Transmitter case) in what follows,

3.1 Informed Transmitter

There is a possibility that transmitter learns the channel state information (CSI or channel matrix H) before it transmits the data vector. For instance, in TDD (Time Division Duplexing) systems, the channel matrix can be fed back to the transmitter from the receiver. In such an event, the capacity can be increased by resorting to the so-called waterfilling principle [18], by assigning various levels of transmit power to various transmitting antennas. This power is assigned on the bases that the better the channel is, the more power it gets and vice versa.

3.1.1. Waterfilling Algorithm

When H is known at the transmitter, waterfilling algorithm can be used to maximize the channel capacity by allocating more power to the eigenvalues that are in a good condition and less or none at all to the bad eigenvalues [8].

Given $H = USV^H$ (Singular Value Decomposition theorem or SVD), the system expressed by equation (1) can be rewritten as

$$y = USV^H x + n \tag{8}$$

U is a matrix containing the eigenvectors of the receiver, V is a matrix containing the transmitter eigenvectors and the matrix S is a diagonal matrix containing the singular values (σ_i, where $\sigma_i = \sqrt{\lambda_i}$) of the matrix H. U and V matrices are unitary, satisfying $UU^H = U^H U = I_{Mr}$, and $VV^H = V^H V = I_{M_t}$.

The transmitted vector is multiplied by a matrix V prior to transmission to cancel the effect of the matrix V^H contained in H. In the same way, received vector is multiplied by a matrix U^H to cancel the effect of the matrix U contained in H.

$x' = Vx,\ y' = U^H y,\ n' = U^H n,$

Substituting these values in equation (8), will produce the following

$$y' = Sx' + n' \tag{9}$$

The system modeled by equation (9) is representing a group of parallel SISO channels; their power gains are the none zero diagonal elements of the matrix S.

The capacity of the MIMO channel is the sum of the individual parallel SISO channel capacities and is given by

$$C = E_H\left[\sum_{i=1}^{rr} \log_2\left(1 + \frac{\rho\gamma_i}{M_t}\lambda_i\right)\right] \tag{10}$$

γ_i is the amount of power transmitted over the eigenvalues λ_i such that

$$\sum_{i=1}^{rr} \gamma_i = M_t \tag{11}$$

Channel capacity maximization implies that transmitter accesses the individual sub-channels (the eigenvalues) and allocates variable power levels to them. Using Lagrangian method, the optimal energy allocated to each eigenvalue is

$$\gamma_i^{opt} = \left(\mu - \frac{M_t}{\rho\lambda_i}\right)_+ ,i=1,2,\ldots,rr \tag{13}$$

and

$$\sum_{i=1}^{rr} \gamma_i^{opt} = M_t \tag{14}$$

where, μ is a constant representing the water level and $(x)_+$ implies

$$(x)_+ = \begin{cases} x & if \quad x \geq 0 \\ 0 & if \quad x \prec 0 \end{cases} \tag{15}$$

Now, the optimal energy allocation is found iteratively through waterfilling algorithm as described below.

The iteration count p is set to 1, and then the constant μ in equation (13) is calculated based on the following formula,

$$\mu = \frac{M_t}{(rr - p + 1)}\left[1 + \frac{1}{\rho}\sum_{i=1}^{rr-p+1}\frac{1}{\lambda_i}\right] \tag{16}$$

Using the obtained value of μ from equation (16), the power allocated to the i[th] eigenvalue can be calculated using

$$\gamma_i = \left(\mu - \frac{M_t}{\rho\lambda_i}\right) ,\ i= 1,2,\ldots,rr-p+1. \tag{17}$$

If the power allocated to the eigenvalue with the lowest gain is negative i.e. the term $\frac{M_t}{\rho\lambda_i}$ is greater than μ (the eigenvalue is bad), this eigenvalue is discarded and by setting $\gamma_{rr-p+1}^{opt} = 0$ and the algorithm is rerun with incrementing the iteration account by 1. This algorithm is repeated until all good eigenvalues are allocated the optimal power. The capacity of MIMO channels when the channel is known at the transmitter is at least equal to that obtained when the channel is unknown at the transmitter. Once the optimal power allocation across the spatial eigenvalues is determined, the optimized input covariance matrix Q is now obtained,

$$Q^{opt} = diag\{\gamma_1^{opt}, \gamma_2^{opt}, \ldots, \gamma_{rr}^{opt}\} \tag{18}$$

and equation (3) will take the new form

$$C = E_H\left[\log_2 \det\left(I_{M_r} + \tfrac{\rho}{M_t} HQ^{opt}H^H\right)\right] \tag{19}$$

At low SNR (ρ), waterfilling algorithm allocates all the available transmit power to the strongest eigenmode λ_{max} (λ_{max} = max (λ_i)). So equation (19) will reduce to the following equation at low SNR,

$$C = E_H\left[\log_2\left(1+\rho\lambda_{\max}\right)\right] \quad (20)$$

Where $\rho = E_s / N_0$.

When the channel H is orthogonal i.e. there is no correlation (unrealistic condition) all its eigenmodes (λ_i) will be equal (λ_i = 1, i = 1,2,3…min(M_t, M_r)), so in this case there is no λ_{max} since all the eigenmodes are equal and the condition number of the channel H is equal to 1 ($\lambda_{max} = \lambda_{min}$). However, Real channels are correlated, which means that the eigenmodes will not be equal and the condition number of the channel will not be equal to 1 (condition number of a channel H = $\lambda_{max} / \lambda_{min}$) because λ_{max} will get larger and λ_{min} will get smaller when the correlation increases. When channel correlation approaches 1, λ_{max} will approach $M_r * M_t$ and λ_{min} approaches zero. This means according to eq. (20) that capacity of the channel will increase when the channel correlation increases.

4. Simulation Results

In this paper, we study MIMO channel capacity over the SNR range from -20 dB to 20 dB, and channel correlation factor (*r*) takes the values from 0 to 0.9 at step 0.1. Channel matrix *H* is considered perfectly known at the transmitter and the receiver. Monte-Carlo simulation technique is used to estimate the channel capacity which is calculated at each SNR point by generating 10,000 channel matrices and taking the average over them.

Figure 1 plots the power allocated to each channel eigenmode and shows how this allocated power change with SNRs. For a (2,2) channel, we can see that, when SNR < -5 for instance, all the transmit power is allocated only to one eigenmode (the largest eigenmode, λ_{max}). This is referred to as the dominant eigenmode transmission. But when SNR > -5 dB, the transmitter begins to divide the power between the channel eigenmodes according to their gains, where the eigenmode with higher gain gets more power so that the channel capacity is maximized. At high SNRs, the transmitter divide the power almost equally among the eigenmodes. For (2,2) channel, when SNR is above 30 dB, gamma1 = gamma2, which means the transmit power is divided equally between the two eigenmodes of the channel. For (2,4) channel, the transmitter begins dividing the power equally between the eigenmodes when SNR = 20 dB and above, and for (2,10) channel, at SNR =10 dB the transmitter begins deviding the power equally among the channel eigenmodes. For this reason, the channel capacity at high SNRs is equal whether the channel matrix is known or unknown to the transmtter. Figures 2 and 3, show the power allocations for (3,3) and (4,4) channels.

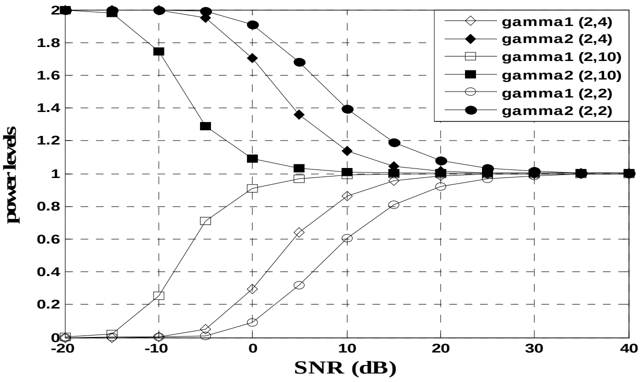

Figure1: The power levels allocated to each eigenmode of a (2,2), (2,4) and (2,10) channels, correlation = 0.

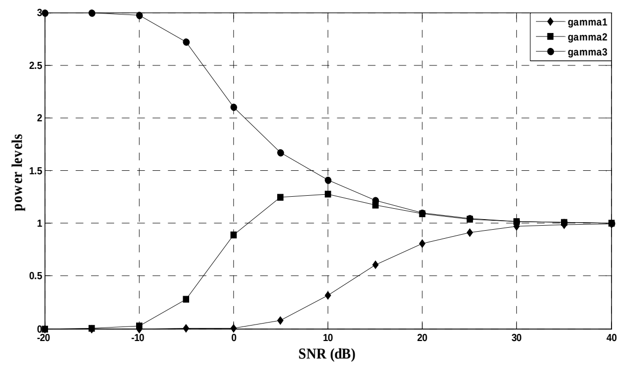

Figure2: The power levels allocated to each eigenmode of a (3,3) channel correlation = 0.

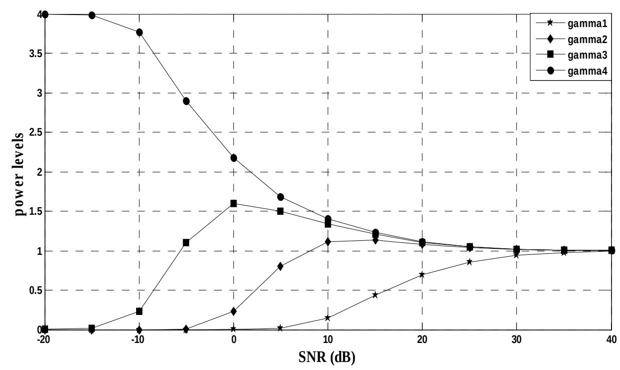

Figure3: The power levels allocated to each eigenmode of a (4,4) channel correlation = 0.

Figure 4 depicts the (2,2) channel eigenmodes against channel correlation when SNR = 0 dB. Channel eigenmodes are independent of the SNR. We can see in figure 4 how the two eigenmodes change with the channel correlation. The largest eigenmode (lambda2) increases with the channel correlation while the other eigenmodes decrease. When the channel correlation approaches 1, lambda2 approaches 4 (M_r * M_t), and lambda1 approaches zero. Figure 5 shows the power allocated to the eigenmodes in figure4. When channel correlation increases the transmitter allocates more power to lambda2 and less to lambda1. When the channel correlation becomes 0.6, the transmitter allocates all the transmit power to lambda2 and nothing to lambda1 because lambda1 becomes near zero at this correlation.

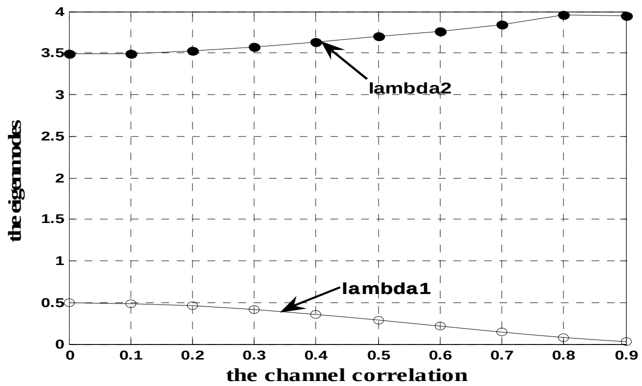

Figure4: The (2,2) channel matrix eigenmodes at SNR = 0 dB.

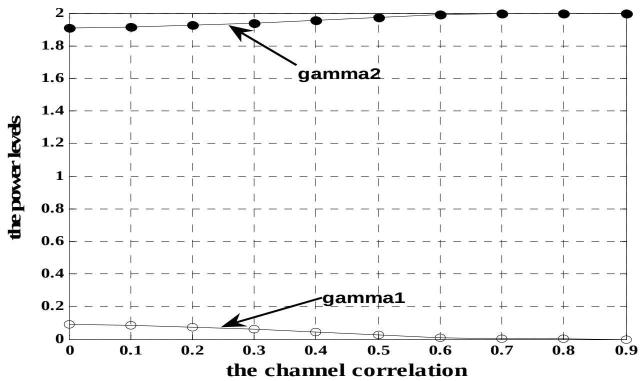

Figure5: The power levels allocated to each eigenmode of a (2,2) at SNR = 0 dB.

Figures 6 and 7 also show the eigenmodes and the power levels allocated to them for a (5,5) channel. It is clear from figure 6 that lambda5 increases when the channel correlation increases and the other eigenmodes of the channel decrease. In the same way like the case of (2,2) channel, when correlation approaches 1, the channel will have only one eigenmode which is lambda5 (lambda5 = 25, when correlation = 1) and all the other eigenmodes become zero.

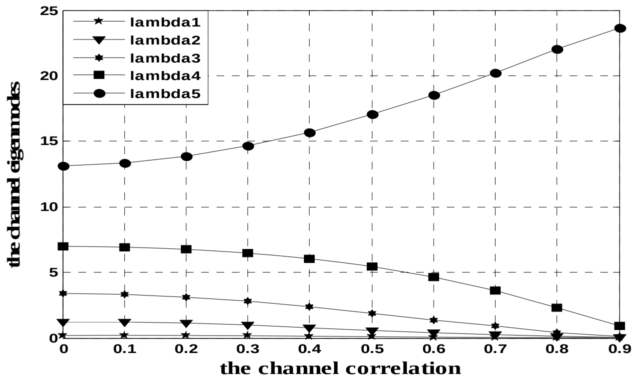

Figure6: The (5,5) channel matrix eigenmodes at SNR = 0 dB.

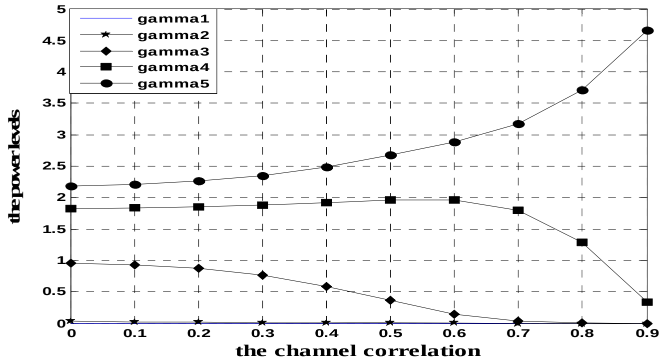

Figure7: The power levels allocated to each eigenmode of a (5,5) at SNR = 0 dB.

Figures 8, 9 and 10 show the capacity of each eigenmode of a (2,2) channel and the total capacity of the channel at (0, 0.5, 0.9) correlations. We can see that, when SNR is less than 0 dB the total capacity of the channel is equal to capacity of lambda2, and lambda1 is not contributing to the total capacity of the channel.

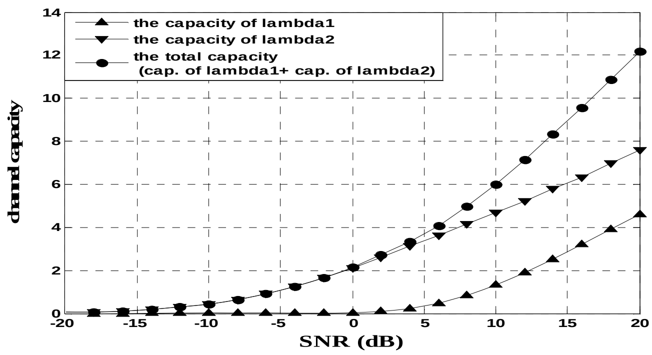

Figure8: The capacity of (2,2) channel, correlation is 0.

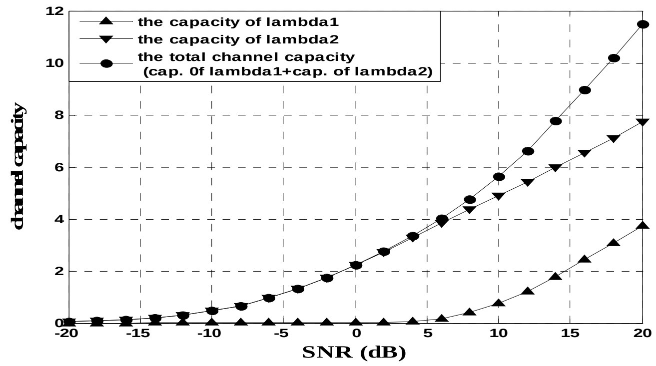

Figure9: The capacity of (2,2) channel, correlation is 0.5.

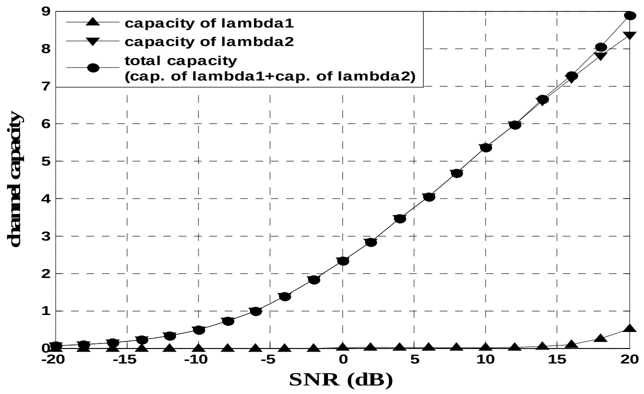

Figure 10: the capacity of (2,2) channel, correlation is 0.9.

Figures 11,12 and 13 also show the capacity of a (5,5) channel's eigenmodes and the total capacity of this channel at (0, 0.5, 0.9) correlations.

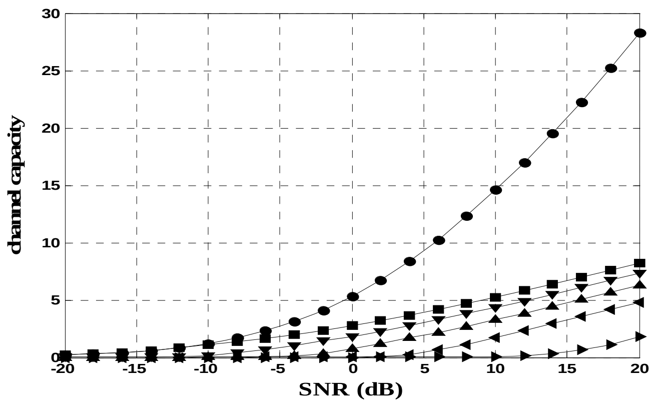

Figure11: The capacity of (5,5) channel, correlation is 0.

Finally, Figure 14 shows how the capacity of a channel with $M_t = 2$, and $M_r = 2,5,15$ and at 0 dB change with the channel correlation. When $M_r = 2$, the capacity of the channel increases with the channel correlation because the (2,2) channel capacity at 0 dB depends only on lambda2, meaning that the total capacity of the channel is equal to the capacity produced by lambda2 (and the capacity of lambda1 is zero). We know from our discussion earlier that lambda2 of a (2,2) channel increases with the channel correlation, hence the capacity will also increase with channel correlation.

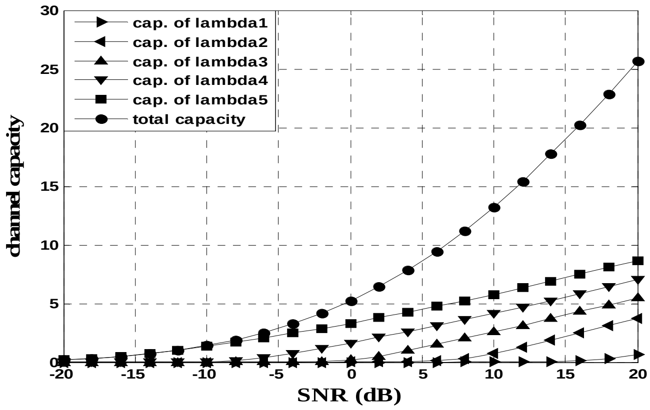

Figure12: The capacity of (5,5) channel, correlation is 0.5.

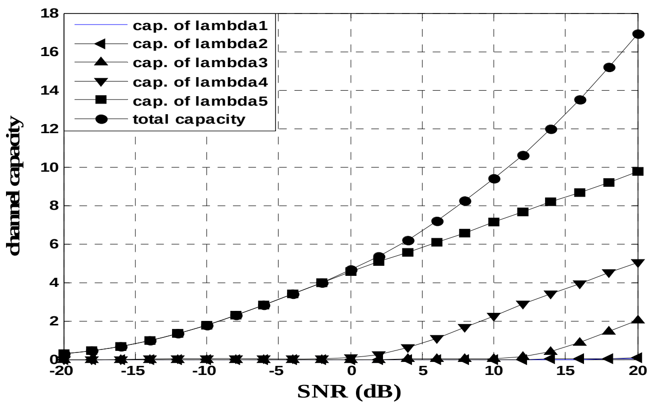

Figure13: The capacity of (5,5) channel, correlation is 0.9.

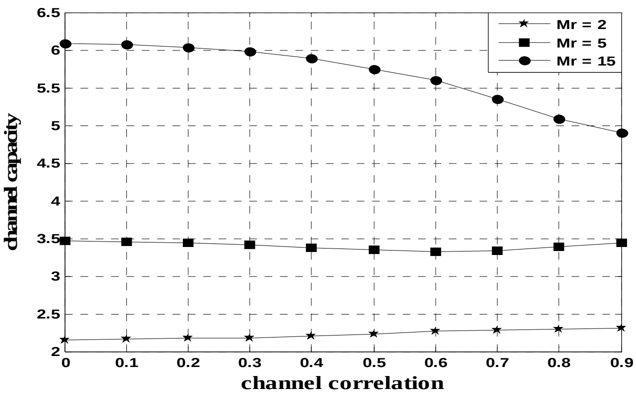

Figure14: the channel capacity of Mt = 2, Mr = 2,5,15 at SNR = 0dB.

5. Conclusions

In this paper, the eigenmodes of MIMO Rayleigh channels were studied and analyzed. First, the eigenmodes were evaluated how to change with SNR, and with the channel correlation. The power levels allocated to each eigenmode and how to change with SNR and channel correlation were also studied. The capacity of these eigenmodes was also evaluated. It was found that, the eigenmodes of any channel are independent of the SNRs, but depend on channel correlation. The maximum eigenmode of any MIMO channel increases with the channel correlation while all other eigenmodes decrease. When the channel correlation approaches 1, the maximum eigenmode approaches $M_r * M_t$, and all the other eigenmodes approach zero. Finally, when the SNR is low, the total capacity of the channel will be equal to the capacity of the largest eigenmode of this channel. Hence, the channel capacity will increase with the channel correlation.

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