Characterization of Fuzzy Regular Languages

Dr. V. Ramaswamy Author1[†] **and** **Mrs. Girijamma. H. A. Author2**[††],

Professor & Head,
Department of Information Science and Engineering,
B.I.E.T, Davangere, Karnataka, India

Assistant Professor,
Department of Computer Science and Engineering ,
RNSIT, Bangalore, Karnataka, India.

Summary
In this paper, we introduce the concept of fuzzy regular language and show that if L is a fuzzy regular language, then every α – cut L_α $(\alpha \in (0, 1])$ is a regular language. We also give a characterization of fuzzy regular languages.

Key words
Monoid, Non deterministic automaton, equivalence class, fuzzy regular language, fuzzy automaton.

Introduction
Consider a finite nonempty set A. A fuzzy automaton over A is a 4-tuple M = (Q, f, I, F) where Q is a finite nonempty set, f is a fuzzy subset of Q x A x Q, I and F are fuzzy subsets of Q.
Thus f : Q x A x Q $\rightarrow$ [0,1], I , F : Q $\rightarrow$[0,1].
Let S be a free monoid with identity element e generated by A.
If $s \in S$, then $s = a_1 a_2 \ldots a_n$ where $a_i \in A$. Here n is called the length of s and we write | s | = n.
We extend f to a function f* : Q x S x Q $\rightarrow$ [0,1] which is defined as follows.

f *(q, e, p) = 1 if q = p
0 otherwise.

$$f^*(q, sa, p) = \bigvee_{r \in Q} [f^*(q, s, r) \wedge f(r, a, p)] \quad (s \in S, a \in A)$$

Theorem: For any two elements $s, t \in S$ and for all $p, q \in Q$,

$$f^*(p, st, q) = \bigvee_{r \in Q} [f^*(p, s, r) \wedge f^*(r, t, q)].$$

Proof: Straight forward.

Hereafter, we will assume that S is a free monoid generated by a finite non empty set A.

Definition: A fuzzy subset L of S is said to be a fuzzy regular language if L= L(M) where M is a fuzzy automaton over S.
In what follows, we will assume that L is a fuzzy regular language.
Since L is a fuzzy regular language, we have L= L (M) where M =(Q, f*, I, F) is a fuzzy automaton over S. L is a fuzzy subset of S defined as $\forall s \in S$, L(s) = I o fs* o F where o denotes max - min composition and f_s^* : Q x Q $\rightarrow$ [0,1] is defined as f_s^* (p, q) = f* (p, s, q) for all $p, q \in Q$.
Let $\alpha \in (0,1]$ and consider $D_\alpha(M) = (Q, d_\alpha, I_\alpha, F_\alpha)$ where $d_\alpha : Q \times S \rightarrow 2^Q$ is defined as $d_\alpha(q, s) = \{ p \in Q | f^*(q, s, p) \geq \alpha \}$, $I_\alpha = \{ p \in Q \mid I(p) \geq \alpha \}$ and $F_\alpha = \{p \in Q \mid F(p) \geq \alpha\}$.
Define a relation R_α as for all $s, t \in S$, $s R_\alpha t$ if and only if $f^*(p, s, q) \geq \alpha$ only when $f^*(p, t, q) \geq \alpha$ for all $p, q \in Q$.

Lemma: R_α is a congruence relation.

Proof: R_α is reflexive because $f^*(p, s, q) \geq \alpha$ only when $f^*(p, s, q) \geq \alpha$ obviously holds for all $p, q \in Q$. If $s R_\alpha t$, then $f^*(p, s, q) \geq \alpha$ if and only if $f^*(p, t, q) \geq \alpha$ for all $p, q \in Q$. This means for all $p, q \in Q$, $f^*(p, t, q) \geq \alpha$ if and only if $f^*(p, s, q) \geq \alpha$ proving that $t R_\alpha s$ and hence R_α is also symmetric.
Suppose $s R_\alpha t$ and $t R_\alpha u$. Then for all $p, q \in Q$, $f^*(p, s, q) \geq \alpha$ if and only if $f^*(p, t, q) \geq \alpha$ which can happen if and only if $f^*(p, u, q) \geq \alpha$. Hence $s R_\alpha u$ proving that R_α is transitive. Hence R_α is an equivalence relation.
Assume that $s R_\alpha t$ and $w \in S$. We will prove that $sw R_\alpha tw$ and $ws R_\alpha wt$.
To prove $sw R_\alpha tw$, we have to prove that for all $p, q \in Q$, $f^*(p, sw, q) \geq \alpha$ if and only if $f^*(p, tw, q) \geq \alpha$. Since $s R_\alpha t$, we have $f^*(p, s, q) \geq \alpha$ if and only if $f^*(p, t, q) \geq \alpha$. Suppose $f^*(p, sw, q) \geq \alpha$. We

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will prove that $f^*(p, tw, q) \geq \alpha$. We have

$$\alpha \leq f^*(p, sw, q) = \vee_{r \in Q} [f^*(p, s, r) \wedge f^*(r, w, q)].$$

Hence $f^*(p, s, r) \wedge f^*(r, w, q) \geq \alpha$ for some $r \in Q$. This means $f^*(p, s, r) \geq \alpha$ from which it follows that $f^*(p, t, r) \geq \alpha$ since $s R_\alpha t$. Also $f^*(r, w, q) \geq \alpha$. Now

$$f^*(p, tw, q) = \vee_{z \in Q} [f^*(p, t, z) \wedge f^*(z, w, q)] \geq \alpha.$$

Similarly, we can prove that if $f^*(p, tw, q) \geq \alpha$ then $f^*(p, sw, q) \geq \alpha$ This proves that sw R_α tw. Using exactly a similar argument, we can prove that ws R_α wt proving further that R_α is a congruence relation.

Let $E_\alpha = \{[s]_\alpha \mid s \in S\}$ where $[s]_\alpha$ denotes the equivalence class of s in R_α. Define a binary operation $*_\alpha$ on E_α as $[s]_\alpha *_\alpha [t]_\alpha = [st]_\alpha$.

Lemma: $(E_\alpha, *_\alpha)$ is a monoid.

Proof: We first have to prove that $*_\alpha$ is well defined. Suppose $u \in [s]_\alpha$ and $v \in [t]_\alpha$.

We have to prove that $[st]_\alpha = [uv]_\alpha$. i.e., st R_α uv. We have s R_α u and t R_α v. Suppose $f^*(p, st, q) \geq \alpha$ where $p, q \in Q$. We have to prove that $f^*(p, uv, q) \geq \alpha$. We have

$$\alpha \leq f^*(p, st, q) = \vee_{r \in Q} [f^*(p, s, r) \wedge f^*(r, t, q)].$$

Hence $f^*(p, s, r) \wedge f^*(r, t, q) \geq \alpha$ for some $r \in Q$ which means $f^*(p, s, r) \geq \alpha$ and $f^*(r, t, q) \geq \alpha$. Since s R_α u and t R_α v, we obtain $f^*(p, u, r) \geq \alpha$ and $f^*(r, v, q) \geq \alpha$ so that

$$f^*(p, uv, q) = \vee_{p_1 \in Q} \{ f^*(p, u, p_1) \wedge f^*(p_1, v, q) \} \geq \alpha.$$

Similarly, we can prove that if $f^*(p, uv, q) \geq \alpha$, then $f^*(p, st, q) \geq \alpha$. Hence st R_α uv and $[st]_\alpha = [uv]_\alpha$. It is easy to see that $*_\alpha$ is associative and $[e]_\alpha$ is the identity element. Hence $(E_\alpha, *_\alpha)$ is a monoid.

Theorem: For every $\alpha \in (0,1]$, the α - cut L_α is a regular language.

Proof: We will prove that $L_\alpha = L(D_\alpha(M))$. This will mean that L_α is the language accepted by a non deterministic automaton and hence a regular language. Let $s \in L_\alpha$. Then $L(s) \geq \alpha$ i e $I \circ f_s^* \circ F = \vee [(f_s^* \circ F)(p) \wedge I(p)] \geq \alpha$ which means $(f_s^* \circ F)(p) \wedge I(p) \geq \alpha$ for some $p \in Q$. Hence $(f_s^* \circ F)(p) \geq \alpha$ and $I(p) \geq \alpha$ so that $p \in I_\alpha$. Again $\alpha \leq (f_s^* \circ F)(p) = \vee [f_s^*(p, r) \wedge F(r)]$ and hence $F(r) \wedge f_s^*(p, r) \geq \alpha$ so that $F(r) \geq \alpha$ implying that $r \in F_\alpha$ and $f_s^*(p, r) \geq \alpha$ implying that $f^*(p, s, r) \geq \alpha$.

Thus $r \in d_\alpha(p, s)$. We have thus proved that $\exists\, p \in I_\alpha$ such that $d_\alpha(p, s) \cap F_\alpha \neq \phi$ proving that $s \in L(D_\alpha(M))$. Thus $L_\alpha \subseteq L(D_\alpha(M))$.

Conversely, let $s \in L(D_\alpha(M))$. Then there exists $p \in I_\alpha$ such that $d_\alpha(p, s) \cap F_\alpha \neq \phi$.

Let $q \in d_\alpha(p, s) \cap F_\alpha$.

Now $p \in I_\alpha$ means $I(p) \geq \alpha$, $q \in d_\alpha(p, s)$ means $f^*(p, s, q) \geq \alpha$. i.e $f_s^*(p, q) \geq \alpha$. Now $q \in F_\alpha$ means $F(q) \geq \alpha$. Hence $f_s^*(p, q) \wedge F(q) \geq \alpha$ so that

$$(f_s^* \circ F)(p) = \vee_{r \in Q} [f_s^*(p, r) \wedge F(r)] \geq \alpha.$$

Again $I(p) \wedge (f_s^* \circ F)(p) \geq \alpha$ means

$$I \circ f_s^* \circ F = \vee_{t \in Q} [I(t) \wedge (f_s^* \circ F)(t)] \geq \alpha.$$

Hence $L(s) \geq \alpha$ proving that $s \in L_\alpha$.

Thus $L(D_\alpha(M)) \subseteq L_\alpha$. This together with $L_\alpha \subseteq L(D_\alpha(M))$ proves that $L_\alpha = L(D_\alpha(M))$.

Theorem: $L_\alpha = \cup [s]_\alpha$ where the union is taken over all equivalence classes of s for which there exists $p \in I_\alpha$ such that $d_\alpha(p, s) \cap F_\alpha \neq \phi$.

Proof: Suppose $t \in L_\alpha$. Since $L_\alpha = L(D_\alpha(M))$, there exists $p \in I_\alpha$ such that $d_\alpha(p, t) \cap F_\alpha \neq \phi$. Clearly $t \in [t]_\alpha$. Conversely, assume that $t \in \cup [s]_\alpha$ where the union is taken over all equivalence classes of s for which there exists $p \in I_\alpha$ such that $d_\alpha(p, s) \cap F_\alpha \neq \phi$. Then $t \in [s]_\alpha$ where $d_\alpha(p, s) \cap F_\alpha \neq \phi$ for some $p \in I_\alpha$. Hence $s \in L(D_\alpha(M)) = L_\alpha$ which means $L(s) \geq \alpha$.

Let $q \in d_\alpha(p, s) \cap F_\alpha$. Then $f^*(p, s, q) \geq \alpha$ so that $f^*(p, t, q) \geq \alpha$ since s R_α t. Hence $q \in d_\alpha(p, t)$. Also $q \in F_\alpha$ and hence $d_\alpha(p, t) \cap F_\alpha \neq \phi$ where $p \in I_\alpha$. This means $t \in L(D_\alpha(M)) = L_\alpha$.

For every $\alpha \in (0,1]$ and $x \in S$, define $\alpha_L(x) = \alpha$ if $L(x) \geq \alpha$ and 0 otherwise. We note that each α_L is a fuzzy set.

Result: $L = \cup_{\alpha \in (0,1]} \alpha_L$ where $\cup$ denotes fuzzy union.

Proof: Let s ∈ S and assume that L (s) = β ∈ (0, 1]. Then β_L (s) = β so that β ≤ max α_L (s) where the maximum is taken over all α ∈ (0,1].

This proves that L (s) = β ≤ $\bigcup_{\alpha \in [0,1]} \alpha_L$ (s)

Now take any γ ∈ (0, 1]. If γ ≤ β = L (s), then γ_L (s) = γ ≤ β. If γ > β = L (s), then γ_L (s) = 0 ≤ β. Thus γ_L (s) ≤ β for any γ ∈ (0, 1] so that max α_L (s) ≤ β where the maximum is taken over all α ∈ (0,1]. Hence

$$\bigcup_{\alpha \in (0,1]} \alpha_L (s) \leq \beta \leq \bigcup_{\alpha \in (0,1]} \alpha_L (s)$$

Thus $\bigcup_{\alpha \in (0,1]} \alpha_L$ (s) = β = L (s).

References

[1] George J. Klir, Bo Yuan, Fuzzy sets and fuzzy logic Theory and Applications, Prentice – Hall of India Pvt. Ltd, 1997.

[2] Jiri Mockor, Fuzzy and Non deterministic Automata, Research Report No. 8, Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, Chech Republic.

Dr. V Ramaswamy obtained his Ph.D degree from Madras University, in 1982, He is working as Professor and Head in the Department of Information Science and Engineering. He has more the 25 years of teaching experience including his four years of service in Malaysia. He is guiding many research scholars and has published many papers in national and international conference and in many international journals. He has visited many universities in USA and Malaysia.

Mrs. Girijamma .H. A. received her B.E in Computer Science & Engineering in 1993 &. M.Tech in Computer Science & engineering in 2001 from vishveshveshwariaha technological university (VTU).Registered for Ph.D in "Studies in Fuzzy Automata" under the guidance of Dr. V. Ramaswamy in Kuvempu university (India) in 2004.Having more than 12 years of experience of teaching in company & industry, currently serving as an Assistant Professor in the department of computer science & engineering in RNS Institute Of Technology, Bangalore (India).