Product Fuzzy Graphs

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Summary

In this paper, we introduce product fuzzy graphs and prove several results which are analogous to fuzzy graphs. We conclude by giving a necessary and sufficient condition for a product partial fuzzy sub graph to be the multiplication of two product partial fuzzy sub graphs.

Key words: *Fuzzy Graphs, Product fuzzy graphs.*

1. Introduction

Fuzzy graphs were introduced by Rosenfeld [5]. Since then lots of works on fuzzy graphs have been carried out. We have replaced 'minimum' in the definition of fuzzy graph by 'product' and call the resulting structure product fuzzy graph. We show that many of the results which are found in [4], [3] and [6] hold good for product fuzzy graphs. We note that the usual definition of cartesian product of two fuzzy graphs [4] cannot be directly extended to product fuzzy graphs since the resulting structure fails to be a product fuzzy graph. This has resulted in the introduction of multiplication of product fuzzy graphs. We give a necessary and sufficient condition for a product partial fuzzy sub graph to be the multiplication of two product partial fuzzy sub graphs.

2. Definitions and Main results

Definition 2.1 Let G be a graph whose vertex set is V, μ be a fuzzy subset of V and ρ be a fuzzy subset of V × V. We call (μ, ρ) a product partial fuzzy sub graph of G (in short, a product fuzzy graph) if $\rho(x, y) \le \mu(x) \times \mu(y)$ for all x, y ε V.

Remark: If (μ, ρ)is a product fuzzy graph, then since μ (x) and μ (y) are less than or equal to 1, it follows that $\rho(x, y) \le \mu(x) \times \mu(y) \le \mu(x) \wedge \mu(y)$ for all x, y ε V. Hence (μ, ρ) is a fuzzy graph. Thus every product fuzzy graph is a fuzzy graph.

Remark: If (μ, ρ) is a product partial fuzzy sub graph of G whose vertex set is V, we will assume that $\mu(v) \ne 0$ for all v ε V and ρ is symmetric.

Example 2.1

Let V = {a, b, c}, μ be the fuzzy subset of V defined as μ (a) = 1/4, μ (b) = 1/2 and μ (c) = 3/4. Let ρ be the fuzzy subset of V × V defined as ρ (a, b) = 1/10, ρ (b, c) = 2/8 and ρ (a, c) = 2/16. It is easy to see that (μ, ρ) is a product fuzzy graph and hence a fuzzy graph.

Following example shows that a fuzzy graph need not be a product fuzzy graph

Example 2.2

Let V = {a, b, c}, μ be the fuzzy subset of V defined as μ (a) = 1/4, μ (b) = 1/2 and μ (c) = 3/4. Let ρ be the fuzzy subset of V x V defined as ρ (a, b) = 0.2, ρ (b, c) = 0.4 and ρ (a, c) = 0.2. It is easy to see that (μ, ρ) is a fuzzy graph. However, it is not a product fuzzy graph. Note that μ (a) x μ (b) = 1/4 x 1/2 = 1/8 which is less than ρ (a, b).

Definition 2.2 A product fuzzy graph (μ, ρ) is said to be complete if $\rho(x, y) = \mu(x) \times \mu(y)$ for all x, y ε V.

Proposition 2.1: Let (μ, ρ) be a complete product fuzzy graph where μ is normal. Then $\rho^n(x, y) = \rho(x, y)$ for all x, y ε V and for all positive integers n where for n ≥ 2,

$$\rho^n(x, y) = \vee_{z \varepsilon V} \{\rho^{n-1}(x, z) \times \rho(z, y)\}$$

Proof: We will use induction on n. Firstly, if n = 2, then for all x, y ε V, we have

$$\rho^2(x, y) = \vee_{z \varepsilon V} \{\rho(x, z) \times \rho(z, y)\}$$
$$= \vee_{z \varepsilon V} \{(\mu(x) \times \mu(z)) \times (\mu(z) \times \mu(y))\}$$
$$= \vee_{z \varepsilon V} \{\mu(x) \times \mu(y) \times \mu(z)^2\}$$

Since $\mu(z)^2 \le 1$ for all z,

Manuscript received January 5, 2009
Manuscript revised January 20, 2009

$\mu(x) \times \mu(y) \times \mu(z)^2 \leq \mu(x) \times \mu(y)$ for all z.
Hence $\vee \{ \mu(x) \times \mu(y) \times \mu(z)^2 \} \leq \mu(x) \times \mu(y)$ so that

$$\rho^2(x, y) \leq \mu(x) \times \mu(y) = \rho(x, y).$$

If μ is normal, then
$\mu(t) = 1$ for some t. Then

$$\begin{aligned}\rho^2(x, y) &= \bigvee_{z \in V} \{ \mu(x) \times \mu(y) \times \mu(z)^2 \} \\ &\geq \mu(x) \times \mu(y) \times \mu(t)^2 \\ &= \mu(x) \times \mu(y) \quad \text{since } \mu(t) = 1 \\ &= \rho(x, y) \quad \text{since } (\mu, \rho) \text{ is complete}\end{aligned}$$

This together with $\rho^2(x, y) \leq \rho(x, y)$ proves that

$$\rho^2(x, y) = \rho(x, y).$$

Now assuming that $\rho^k(x, y) = \rho(x, y)$, we will prove that

$$\rho^{k+1}(x, y) = \rho(x, y).$$

We have

$$\begin{aligned}\rho^{k+1}(x, y) &= \bigvee_{z \in V} \{ \rho^k(x, z) \times \rho(z, y) \} \\ &= \bigvee_{z \in V} \{\rho(x, z) \times \rho(z, y)\} \text{ by inductive hypothesis} \\ &= \rho^2(x, y) \\ &= \rho(x, y) \text{ by what we have already proved}\end{aligned}$$

We will now give an example to show that if μ is not normal, then the above result need not be true.

Example 2..3: Let V = {a, b, c}, μ be the fuzzy subset of V defined as $\mu(a) = 1/4$, $\mu(b) = 1/2$ and $\mu(c) = 3/4$. Let ρ be the fuzzy subset of $V \times V$ defined as $\rho(a, a) = 1/16$, $\rho(a, b) = 1/8$, $\rho(b, b) = 1/4$, $\rho(b, c) = 3/8$, $\rho(a, c) = 3/16$ and $\rho(c, c) = 9/16$. Clearly, (μ, ρ) is a complete product fuzzy graph and μ is not normal. However,

$$\begin{aligned}\rho^2(a, b) &= [\rho(a, a) \times \rho(a, b)] \vee [\rho(a, b) \times \rho(b, b)] \vee [\rho(a, c) \times \rho(c, b)] \\ &= (1/16 \times 1/8) \vee (1/8 \times 1/4) \vee (3/16 \times 3/8) \\ &= 9/128 \neq \rho(a, b).\end{aligned}$$

Definition 2..3: The *complement* of a product fuzzy graph (μ, ρ) is (μ^c, ρ^c) where

$$\begin{aligned}\mu^c = \mu \text{ and } \rho^c(x, y) &= \mu^c(x) \times \mu^c(y) - \rho(x, y) \\ &= \mu(x) \times \mu(y) - \rho(x, y).\end{aligned}$$

It follows that (μ^c, ρ^c) itself is a product fuzzy graph. Also

$$\begin{aligned}(\rho^c)^c(x, y) &= \mu(x) \times \mu(y) - \rho^c(x, y). \\ &= [\mu(x) \times \mu(y)] - [\mu(x) \times \mu(y) - \rho(x, y)] \\ &= \rho(x, y).\end{aligned}$$

Definition 2.4: Let (μ_1, ρ_1) be a product partial fuzzy sub graph of $G_1 = (V_1, X_1)$ and (μ_2, ρ_2) be a product partial fuzzy sub graph of $G_2 = (V_2, X_2)$. Let X' denote the set of all arcs joining the vertices of V_1 and V_2. We further assume that $V_1 \cap V_2 = \Phi$. Then the *join* of (μ_1, ρ_1) and (μ_2, ρ_2) is defined as $(\mu_1 + \mu_2, \rho_1 + \rho_2)$ where

$$\begin{aligned}(\mu_1 + \mu_2)(u) &= \mu_1(u) \text{ if } u \in V_1 \\ &= \mu_2(u) \text{ if } u \in V_2\end{aligned}$$

$$\begin{aligned}(\rho_1 + \rho_2)(u, v) &= \rho_1(u, v) \text{ if } (u, v) \in X_1 \\ &= \rho_2(u, v) \text{ if } (u, v) \in X_2 \\ &= \mu_1(u) \times \mu_2(v) \text{ if } (u, v) \in X'\end{aligned}$$

Proposition 2.2: $(\mu_1 + \mu_2, \rho_1 + \rho_2)$ is a product partial fuzzy sub graph of $G = (V, X)$ where $V = V_1 \cup V_2$ and $X = X_1 \cup X_2 \cup X'$.

Proof: We have to prove that
$(\rho_1 + \rho_2)(u, v) \leq (\mu_1 + \mu_2)(u) \times (\mu_1 + \mu_2)(v) \quad \forall (u, v) \in X$.
If $(u, v) \in X_1$, then u and v belong to V_1 so that
$(\rho_1 + \rho_2)(u, v) = \rho_1(u, v)$ and $(\mu_1 + \mu_2)(u) \times (\mu_1 + \mu_2)(v)$
$= \mu_1(u) \times \mu_1(v)$.
Hence the inequality follows from the fact that (μ_1, ρ_1) is a product partial fuzzy sub graph. Similarly, we can prove if $(u, v) \in X_2$. If $(u, v) \in X'$, then $u \in V_1$ and $v \in V_2$. Now
$(\rho_1 + \rho_2)(u, v) = \mu_1(u) \times \mu_2(v)$ whereas
$(\mu_1 + \mu_2)(u) \times (\mu_1 + \mu_2)(v) = \mu_1(u) \times \mu_2(v)$.
This completes the proof.

Proposition 2.3: $(\mu_1 + \mu_2, \rho_1 + \rho_2)$ is complete if and only if (μ_1, ρ_1) and (μ_2, ρ_2) are both complete.

Proof: First assuming that (μ_1, ρ_1) and (μ_2, ρ_2) are both complete, we will prove that $(\mu_1 + \mu_2, \rho_1 + \rho_2)$ is complete. If $(u, v) \in X_1$, then both u and v belong to V_1. Now $(\rho_1 + \rho_2)(u, v) = \rho_1(u, v) = \mu_1(u) \times \mu_1(v)$ since (μ_1, ρ_1) is complete. Again, $(\mu_1 + \mu_2)(u) \times (\mu_1 + \mu_2)(v) = \mu_1(u) \times \mu_1(v)$. Similarly, we can argue if $(u, v) \in X_2$. Suppose $(u, v) \in X'$. Then $u \in V_1$ and $v \in V_2$. Now
$(\rho_1 + \rho_2)(u, v) = \mu_1(u) \times \mu_2(v)$ whereas $(\mu_1 + \mu_2)(u) \times (\mu_1 + \mu_2)(v) = \mu_1(u) \times \mu_2(v)$. We have thus shown that $(\rho_1 + \rho_2)(u, v) = (\mu_1 + \mu_2)(u) \times (\mu_1 + \mu_2)(v)$ in all cases proving that $(\mu_1 + \mu_2, \rho_1 + \rho_2)$ is complete.

Conversely, assuming that $(\mu_1 + \mu_2, \rho_1 + \rho_2)$ is complete, we will establish that (μ_1, ρ_1) and (μ_2, ρ_2) are both complete. To prove (μ_1, ρ_1) is complete, we have to prove that for all $(u, v) \in X_1$, $\rho_1(u, v) = \mu_1(u) \times \mu_1(v)$. But this follows from the fact that $(\mu_1 + \mu_2, \rho_1 + \rho_2)$ is complete since $(\rho_1 + \rho_2)(u, v) = \rho_1(u, v)$ whereas
$(\mu_1 + \mu_2)(u) \times (\mu_1 + \mu_2)(v) = \mu_1(u) \times \mu_1(v)$. Similarly, we can prove that (μ_2, ρ_2) is also complete.

Proposition 2.4 Let (μ_1, ρ_1) and (μ_2, ρ_2) be product partial fuzzy sub graphs of G_1 and G_2 respectively, then the following hold.

i. $(\mu_1 + \mu_2 ,\ \rho_1 + \rho_2)^c = (\mu_1{}^c \cup \mu_2{}^c, \rho_1{}^c \cup \rho_2{}^c)$

ii $((\mu_1 \cup \mu_2)^c, (\rho_1 \cup \rho_2)^c) = (\mu_1{}^c + \mu_2{}^c,\ \rho_1{}^c + \rho_2{}^c)$.

Proof of (i): If u ε V_1, then $(\mu_1 + \mu_2)^c(u) = (\mu_1 + \mu_2)(u) = \mu_1(u)$ whereas $\max(\mu_1{}^c(u), \mu_2{}^c(u)) = \max(\mu_1(u), \mu_2(u)) = \mu_1(u)$. Similarly, we can argue if u ε V_2. Suppose (u, v) ε X_1. Then u, v ε V_1 and

$$(\rho_1+\rho_2)^c(u, v)=(\mu_1+\mu_2)(u) \times (\mu_1+\mu_2)(v) - (\rho_1+\rho_2)(u, v)$$
$$= \mu_1(u) \times \mu_1(v) - \rho_1(u, v) = \rho_1{}^c(u, v)$$
$$= \max(\rho_1{}^c(u, v), \rho_2{}^c(u, v)).$$

Similarly, we can prove if (u, v) ε X_2. Suppose (u, v) ε X'. Then u ε V_1 and v ε V_2 and

$$(\rho_1+\rho_2)^c(u, v) = (\mu_1+\mu_2)(u) \times (\mu_1+\mu_2)(v) - (\rho_1+\rho_2)(u, v)$$
$$= \mu_1(u) \times \mu_2(v) - \mu_1(u) \times \mu_2(v)$$
$$= 0 = \max(\rho_1{}^c(u, v), \rho_2{}^c(u, v)).$$

We will now prove (ii).

If u ε V_1, then $(\mu_1 \cup \mu_2)^c(u) = (\mu_1 \cup \mu_2)(u) = \mu_1(u) = \max(\mu_1(u), \mu_2(u)) = \max(\mu_1{}^c(u), \mu_2{}^c(u))$
$=(\mu_1{}^c + \mu_2{}^c)(u)$.Similarly, if u ε V_2.
This proves that $(\mu_1 \cup \mu_2)^c = \mu_1{}^c + \mu_2{}^c$. To prove $(\rho_1 \cup \rho_2)^c = \rho_1{}^c + \rho_2{}^c$, consider (u, v) ε X_1.Then u, v ε V_1 and $(\rho_1 \cup \rho_2)^c(u, v) = (\mu_1 \cup \mu_2)(u) \times (\mu_1 \cup \mu_2)(v) - (\rho_1 \cup \rho_2)(u, v) = \mu_1(u) \times \mu_1(v) - \rho_1(u, v) = \rho_1{}^c(u, v)$. If (u, v) ε X_2, then u, v ε V_2 and $(\rho_1 \cup \rho_2)^c(u, v) = (\mu_1 \cup \mu_2)(u) \times (\mu_1 \cup \mu_2)(v) - (\rho_1 \cup \rho_2)(u, v) = \mu_2(u) \times \mu_2(v) - \rho_2(u, v) = \rho_2{}^c(u, v)$. If (u, v) ε X′, then u ε V_1, v ε V_2 and $(\rho_1 \cup \rho_2)^c(u, v) = (\mu_1 \cup \mu_2)(u) \times (\mu_1 \cup \mu_2)(v) - (\rho_1 \cup \rho_2)(u, v) = \mu_1(u) \times \mu_2(v) = \mu_1{}^c(u) \times \mu_2{}^c(v)$ (note that $\rho_1(u, v) = \rho_2(u, v) = 0$). All these prove that $(\rho_1 \cup \rho_2)^c = \rho_1{}^c + \rho_2{}^c$.

Let G_1 and G_2 be two graphs whose vertex sets are V_1 and V_2 respectively. We will define a new graph $G = G_1 \times G_2$ (called the product graph of G_1 and G_2) whose vertex set is $V_1 \times V_2$ and whose edge set is a subset of $(V_1\times V_2) \times (V_1 \times V_2)$.Let (μ_1, ρ_1) be a product partial fuzzy sub graph of G_1 and (μ_2, ρ_2) be a product partial fuzzy sub graph of G_2. If v_1 ε V_1 and v_2 ε V_2, then define $(\mu_1 \times \mu_2)(v_1, v_2) = \mu_1(v_1) \times \mu_2(v_2)$. Also define $(\rho_1 \times \rho_2)((u_1, u_2), (v_1, v_2)) = \rho_1(u_1, v_1) \times \rho_2(u_2, v_2)$ for all u_1, v_1 ε V_1 and for all u_2, v_2 ε V_2. $\mu_1 \times \mu_2$ is thus a fuzzy subset of $V_1 \times V_2$ and $\rho_1\times \rho_2$ is a fuzzy subset of $(V_1 \times V_2)$ x $(V_1\times V_2)$.

Proposition 2.5 $(\mu_1\times \mu_2, \rho_1 \times \rho_2)$ is a product partial fuzzy sub graph of $G_1\times G_2$.

Proof: For all u_1, v_1 ε V_1 and u_2, v_2 ε V_2, we have

$$(\rho_1 \times \rho_2)((u_1, u_2), (v_1, v_2)) = \rho_1(u_1, v_1) \times \rho_2(u_2, v_2)$$
$$\leq[\mu_1(u_1) \times \mu_1(v_1)] \times [\mu_2(u_2) \times \mu_2(v_2)]$$
$$= [\mu_1(u_1) \times \mu_2(u_2)] \times [\mu_1(v_1) \times \mu_2(v_2)]$$
$$= (\mu_1 \times \mu_2)(u_1, u_2) \times (\mu_1 \times \mu_2)(v_1, v_2)$$

Definition 2.5 The product partial fuzzy sub graph $(\mu_1 \times \mu_2,\ \rho_1 \times \rho_2)$ is referred to as the *multiplication* of the product partial fuzzy sub graph (μ_1, ρ_1) of G_1 and the product partial fuzzy sub graph (μ_2, ρ_2) of G_2.

Proposition.2.6: Let (μ_1, ρ_1) be a product partial fuzzy sub graph of G_1 and (μ_2, ρ_2) be a product partial fuzzy sub graph of G_2 .Then $(\mu_1 \times \mu_2,\ \rho_1 \times \rho_2)$ is complete if and only if both (μ_1, ρ_1) and (μ_2, ρ_2) are complete.

Proof: First assuming that (μ_1, ρ_1) and (μ_2, ρ_2) are both complete, we will prove that $(\mu_1 \times \mu_2, \rho_1 \times \rho_2)$ is complete. If u_1, v_1 ε V_1 and u_2, v_2 ε V_2, then

$$(\rho_1 \times \rho_2)((u_1, u_2), (v_1, v_2)) = \rho_1(u_1, v_1) \times \rho_2(u_2, v_2)$$
$$= [\mu_1(u_1) \times \mu_1(v_1)] \times [\mu_2(u_2) \times \mu_2(v_2)]$$
$$= [\mu_1(u_1) \times \mu_2(u_2)] \times [\mu_1(v_1) \times \mu_2(v_2)]$$
$$= (\mu_1 \times \mu_2)(u_1, u_2) \times (\mu_1 \times \mu_2)(v_1, v_2)$$

This proves that $(\mu_1 \times \mu_2, \rho_1 \times \rho_2)$ is complete. Conversely, assuming that $(\mu_1 \times \mu_2, \rho_1 \times \rho_2)$ is complete, we will prove that (μ_1, ρ_1) and (μ_2, ρ_2) both are complete. We will first show that at least one of (μ_1, ρ_1) and (μ_2, ρ_2) is complete. Suppose both (μ_1, ρ_1) and (μ_2, ρ_2) are not complete. Then there exist u_1, v_1 ε V_1 and u_2, v_2 ε V_2 for which the following inequalities hold.

$$\rho_1(u_1, v_1) < \mu_1(u_1) \times \mu_1(v_1)$$

$$\rho_2(u_2, v_2) < \mu_2(u_2) \times \mu_2(v_2)$$

Now consider

$((u_1, u_2), (v_1, v_2))$ ε $(V_1 \times V_2) \times (V_1 \times V_2)$.

We have

$$(\rho_1 \times \rho_2)((u_1, u_2), (v_1, v_2)) = \rho_1(u_1, v_1) \times \rho_2(u_2, v_2)$$
$$< [\mu_1(u_1) \times \mu_1(v_1)] \times [\mu_2(u_2) \times \mu_2(v_2)]$$
$$= [\mu_1(u_1) \times \mu_2(u_2)] \times [\mu_1(v_1) \times \mu_2(v_2)]$$
$$= (\mu_1 \times \mu_2)(u_1, u_2) \times (\mu_1 \times \mu_2)(v_1, v_2)$$

This is a contradiction since $(\mu_1 \times \mu_2, \rho_1 \times \rho_2)$ is complete. We have thus proved that at least one of (μ_1, ρ_1) and (μ_2, ρ_2) is complete. Without loss of generality, we will assume that (μ_1, ρ_1) is complete and show that (μ_2, ρ_2) is also complete. For any u_1, v_1 ε V_1 and u_2, v_2 ε V_2, we have $(\rho_1 \times \rho_2)((u_1, u_2), (v_1, v_2)) = (\mu_1 \times \mu_2)(u_1, u_2) \times (\mu_1 \times \mu_2)(v_1, v_2)$ (since $(\mu_1 \times \mu_2,\ \rho_1 \times \rho_2)$ is complete) ie.

$\rho_1(u_1, v_1) \times \rho_2(u_2, v_2 = [\mu_1(u_1) \times \mu_2(u_2)] \times [\mu_1(v_1) \times \mu_2(v_2)]$
(by definition) $= [\mu_1(u_1) \times \mu_1(v_1)] \times [\mu_2(u_2) \times \mu_2(v_2)]$
$= \rho_1(u_1, v_1) \times [\mu_2(u_2) \times \mu_2(v_2)]$
(since (μ_1, ρ_1) is complete)

We first note that ρ_1 $(u_1, v_1) \neq 0$. For, if ρ_1 $(u_1, v_1) = 0$, then
$(\rho_1 \times \rho_2)$ $((u_1, u_2), (v_1, v_2)) = \rho_1$ $(u_1, v_1) \times \rho_2$ $(u_2, v_2) = 0$.
ie. $(\mu_1 \times \mu_2)$ $(u_1, u_2) \times (\mu_1 \times \mu_2)$ (v_1, v_2)
$= \mu_1(u_1) \times \mu_2(u_2) \times \mu_1(v_1) \times \mu_2(v_2) = 0$ which means at least one of $\mu_1(u_1)$, $\mu_2(u_2)$, $\mu_1(v_1)$ and $\mu_2(v_2)$ is 0. But this cannot happen (Remark 2.3). Cancelling ρ_1 (u_1, v_1) on both sides, we obtain ρ_2 $(u_2, v_2) = \mu_2(u_2) \times \mu_2(v_2)$ which proves that (μ_2, ρ_2) is complete.

Following example shows that one of (μ_1, ρ_1) and (μ_2, ρ_2) can be complete without $(\mu_1 \times \mu_2, \rho_1 \times \rho_2)$ being complete.

Example 2.4 Let $V_1 = \{ u, v \}$ and $V = \{ x, y \}$.
Define μ_1 (u) = 0.2, μ_1 (v) = 0.3, ρ_1 (u, u) = 0.04, ρ_1 (u, v) = ρ_1 (v, u) = 0.06 and ρ_1 (v, v) = 0.09. Also define μ_2 (x) = 0.5, μ_2(y) = 0.4, ρ_2 (x, x) = 0.2, ρ_2 (x, y) = ρ_2 (y, x) = 0.1 and ρ_2 (y, y) = 0.1. We can easily see that (μ_1, ρ_1) is complete whereas (μ_2, ρ_2) is not. Now $(\mu_1 \times \mu_2)$ (u, x) = μ_1 (u) $\times$ μ_2 (x) = 0.2 x 0.5 = 0.1 and $(\mu_1 \times \mu_2)$ (u, y) = μ_1 (u) $\times$ μ_2 (y) = 0.2 $\times$ 0.4 = 0.08. Hence $(\mu_1 \times \mu_2)$ (u, x) $\times$ $(\mu_1 \times \mu_2)$ (u, y) = 0.1 $\times$ 0.08 = 0.008. However,
$(\rho_1 \times \rho_2)$ ((u, x), (u, y)) = ρ_1 (u, u) $\times$ ρ_2 (x, y) = 0.04 $\times$ 0.1 = 0.004. This shows that $(\mu_1 \times \mu_2, \rho_1 \times \rho_2)$ is not complete.

Proposition 2.7: Let $V_1 = \{v_{11}, v_{12},\ldots\ldots\ldots v_{1n}\}$ and $V_2 = \{v_{21}, v_{22},\ldots\ldots v_{2m}\}$ be the vertex sets of G_1 and G_2 respectively. Further, let (μ, ρ) be the multiplication of a product partial fuzzy sub graph of G_1 and a product partial fuzzy sub graph of G_2.Then the following equations have solutions in [0, 1].
i. $x_i \times y_j = \mu$ (v_{1i}, v_{2j}) (i = 1,2,…n, j = 1,2,…,m)
ii $z_{ik} \times w_{jl} = \rho$ $((v_{1i}, v_{2j}), (v_{1k}, v_{2l}))$ (i, k = 1,2,.....n, j, l = 1,2,…,m)

Proof: Let (μ_1, ρ_1) be a product partial fuzzy sub graph of G_1, (μ_2, ρ_2) be a product partial fuzzy sub graph of G_2 and $(\mu, \rho) = (\mu_1 \times \mu_2, \rho_1 \times \rho_2)$. Then
$\mu = \mu_1 \times \mu_2$ and $\rho = \rho_1 \times \rho_2$. If v_{1i} ε V_1 and v_{2j} ε V_2, then

$$\begin{aligned}\mu (v_{1i}, v_{2j}) &= (\mu_1 \times \mu_2) (v_{1i}, v_{2j})\\ &= \mu_1 (v_{1i}) \times \mu_2 (v_{2j})\\ &= x_i \times y_j \text{ where } x_i = \mu_1 (v_{1i}) \; \varepsilon \; [0, 1]\\ &\qquad \text{and } y_j = \mu_2 (v_{2j}) \; \varepsilon \; [0, 1].\end{aligned}$$

If v_{1i} , v_{1k} ε V_1 and v_{2j} , v_{2l} ε V_2 , then

$$\begin{aligned}\rho ((v_{1i}, v_{2j}), (v_{1k}, v_{2l})) &= (\rho_1 \times \rho_2) ((v_{1i}, v_{2j}), (v_{1k}, v_{2l}))\\ &= \rho_1 (v_{1i} , v_{1k}) \times \rho_2 (v_{2j} , v_{2l})\\ &= z_{ik} \times w_{jl} \text{ where } z_{ik} = \rho_1 (v_{1i} , v_{1k}) \; \varepsilon \; [0, 1]\\ &\qquad \text{and } w_{jl} = \rho_2 (v_{2j} , v_{2l}) \; \varepsilon \; [0, 1].\end{aligned}$$

Theorem 2.1: Let G be the product of two graphs G_1 and G_2 and let (μ, ρ) be a product partial fuzzy sub graph of G where ρ is normal. Let $V_1 = \{v_{11}, v_{12},\ldots v_{1n}\}$ be the vertex set of G_1 and $V_2 = \{ v_{21}, v_{22},\ldots v_{2m} \}$ be the vertex set of G_2 . Suppose the following equations have solutions in [0, 1].

$x_i \times y_j = \mu$ (v_{1i}, v_{2j}) (i = 1,2,…n, j = 1,2,…,m)
$z_{ik} \times w_{jl} = \rho$ $((v_{1i},v_{2j}) ,(v_{1k},v_{2l}))$ (i, k=1,2,..n,j,l= 1,2,…,m)

Then (μ, ρ) is the multiplication of a product partial fuzzy sub graph of G_1 and a product partial fuzzy sub graph of G_2.

Proof: Define μ_1: $V_1 \to [0, 1]$ as μ_1 $(v_{1i}) = x_i$,
μ_2: $V_2 \to [0, 1]$ as μ_2 $(v_{2j}) = y_j$,
ρ_1: $V_1 \times V_1 \to [0, 1]$ as ρ_1 $(v_{1i}, v_{1k}) = z_{ik}$ and
ρ_2: $V_2 \times V_2 \to [0, 1]$ as ρ_2 $(v_{2j}, v_{2l}) = w_{jl}$ (i, k = 1,2,…n, j,l=1,2,…m). We have to prove the following.
i. (μ_1, ρ_1) is a product partial fuzzy sub graph of G_1.
ii. (μ_2, ρ_2) is a product partial fuzzy sub graph of G_2.
iii. $\mu = \mu_1 \times \mu_2$.
iv. $\rho = \rho_1 \times \rho_2$.

If v_{1i}, v_{1k} ε V_1, then for all v_{2j}, v_{2l} ε V_2, we have

$$\begin{aligned}\rho ((v_{1i}, v_{2j}), (v_{1k}, v_{2l})) &\leq \mu (v_{1i}, v_{2j}) \times \mu (v_{1k}, v_{2l})\\ &= (x_i \times y_j) \times (x_k \times y_l)\\ &= (x_i \times x_k) \times (y_j \times y_l)\\ &\leq x_i \times x_k \text{ since } y_j , \; y_l \leq 1\\ &= \mu_1 (v_{1i}) \times \mu_1 (v_{1k})\end{aligned}$$

We have thus proved the following.

$$z_{ik} \times w_{jl} \leq \mu_1 (v_{1i}) \times \mu_1 (v_{1k}) \text{ for all j, l} \qquad (1)$$

Since ρ is normal, ρ $((v_{1p}, v_{2s}), (v_{1q}, v_{2t})) = 1$ for some p, q, s and t. This means $z_{pq} \times w_{st} = 1$ implying that $z_{pq} = w_{st} = 1$ since z_{pq} , w_{st} ε [0, 1]. Replacing j by s and l by t in (1), we obtain

$$\rho_1 (v_{1i} , v_{1k}) = z_{ik} = z_{ik} \times w_{st} \leq \mu_1 (v_{1i}) \times \mu_1 (v_{1k})$$

This proves that (μ_1, ρ_1) is a product partial fuzzy sub graph of G_1. Similarly, we can prove that (μ_2, ρ_2) is a product partial fuzzy sub graph of G_2. If v_{1i} ε V_1 and v_{2j} ε V_2 , then $(\mu_1 \times \mu_2)$ $(v_{1i}, v_{2j}) = \mu_1$ $(v_{1i}) \times \mu_2$ (v_{2j}) = $x_i \times y_j = \mu$ (v_{1i}, v_{2j}) proving that $\mu = \mu_1 \times \mu_2$.

If v_{1i} , v_{1k} ε V_1 and v_{2j} , v_{2l} ε V_2 , then

$$\begin{aligned}(\rho_1 \times \rho_2) ((v_{1i}, v_{2j}), (v_{1k}, v_{2l})) &= \rho_1 (v_{1i} , v_{1k}) \times \rho_2 (v_{2j} , v_{2l})\\ &= z_{ik} \times w_{jl}\\ &= \rho ((v_{1i}, v_{2j}), (v_{1k}, v_{2l})).\end{aligned}$$

This proves that $\rho = \rho_1 \times \rho_2$.

3. Conclusion:

We are able to obtain sufficient condition (Theorem 2.1) for a product partial fuzzy sub graph G to be the multiplication of a product partial fuzzy sub graph of G_1 and a product partial fuzzy sub graph of G_2 under the assumption that ρ is normal. We are trying to prove this theorem without this assumption or give an example to show that the result need not be true without this assumption. If each of $V_1$and V_2 contains 3 elements, then $V_1 \times V_2$ will contain 9 elements and $(V_1 \times V_2) \times (V_1 \times V_2)$ will contain 81 elements. Hence algorithmic solution is sought.

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