On Independence Problem of P_2-Graph

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Summary

Let G be a graph and r be a positive integer $r \geq 1$. Then the vertices of the *r-path graph* $P_r(G)$ are the set of all paths of length r in G. Two vertices in $P_r(G)$ are adjacent if and only if the intersection of the corresponding paths forms a path of length r-1, and their union forms a path or cycle of length r+1. The 1-history (or simply history) of a vertex in $P_r(G)$ is a path p of length r in G. In this paper, the definition of a history is used in a new interpretation of the domination problem to study properties of the maximal independent sets in 2-path graphs, and then to provide an algorithm for finding the maximum independence domination number in 2-path graph of a tree T. The complexity of the algorithm is $\theta(n^2)$, where n is the number of vertices in T.

Key words:
Path graph, maximum independent set, graph history, networks.

1. Introduction

In this paper, the usual graph terminology will be followed. For example, a graph G = (V,E) consists of a set V of vertices and a set E of edges. Each edge connects two vertices. In the case of undirected graph G, the order of vertices that represent an edge is not important. In other words, the edge ab is the same as ba. Unlike undirected graph, the directed graph differentiates between the edges ab and ba. Since the orientation of ab is from a to b (it is written $\overrightarrow{ab}$), where ba has the orientation headed in a. No opposite orientation of the same edge is allowed. $|V(G)|$ and $|E(G)|$ denote the number of vertices and edges of a graph G, respectively. $|V(G)|$ is sometimes known as the order of a graph G. The result of the path operator P_r, $r \geq 1$, on a graph G is the *r-path graph* $P_r(G)$, where the vertex set $V(P_r(G))$ consists of all paths of length r in G, and the edge set $E(P_r(G))$ consists of all pairs p_1p_2 where $p_1 \cap p_2$ is a path of length r-1 and $p_1 \cup p_2$ is a path or a cycle of length r+1. For r = 1 we get the line graph $L(G) = P_1(G)$.

Path graphs were introduced by Broersma and Hoede in [3] as a generalization of line graphs. The Characterization, diameter, and centers of Path Graphs were covered in [4], [1], and [5], respectively. The Dynamics of the path graph were studied in [2]. In [7], The *history* of a subgraph of $P_r(G)$ was introduced as in the following definition.

Definition A. [7] Let G be a graph and let H' be a subgraph of $P_r(G)$. Then, the history of H' is the subgraph $P_r^{-1}(H')$ of G defined as $P_r^{-1}(H') = \bigcup_{v \in V(H')} P_r^{-1}(v)$.

The definition of a history of a vertex in r-path graph is implicitly included in the above definition. So that the history of a vertex v in $P_r(G)$ is a path p of length r in G. Domination and k-histories of iterated r-path graph [6] was conducted in this field.

A set $S \subseteq V(G)$ is said to be *dominating independent* when it is dominating and independent. In other words, for every

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vertex $v \in V(G)$, either $v \in S$ or there exists a vertex $u \in S$ such that u is adjacent to v, and no two vertices of S are adjacent. S is said to be *maximal* when none of its proper supersets is independent. The *maximum* cardinality of independent dominating set is called independence domination number (or simply independence number since every maximal independent set is also dominating) and denoted $\beta(G)$.

The study of a path graph operator showed the importance of such operator especially in interconnection *networks* design and analysis. Algorithms to identify sub-networks using graph concepts are presented in [8]. In addition, the recursive definition of a graph operator seems to be a very promising direction to get results regarding topological properties of iterated graphs. Such that connectivity, cycles, diameter and distances, and present a wide range of open problems not only in relation to their topology, but also regarding routing problems, symmetry, graph dynamics, algebraic properties, and finally their possible applications in other fields.

In section 2, some basic properties of maximal independent sets of vertices in $P_2(G)$ using the history definition of 2-path graph will be presented. Section 3 is devoted to show that in the case of a tree graph, the maximum of independence number can be determined in much less complexity than the general graphs.

2. Preliminary Results

In this section, the problem of the construction of a maximal independent dominating set in the path graph is reduced to finding a special orientation of the original graph.

Let G be a graph. We shall investigate properties of maximal independent sets of vertices in $P_2(G)$. Recall that each vertex v from the path graph $P_2(G)$ corresponds to a path of the length two in G. Suppose that S is a maximal set of independent vertices in $P_2(G)$. A directed graph $\vec{G}$ can be created with the same vertex set as G. Let us call the edge that is going to be oriented by an *arc*. The set of arcs in $\vec{G}$ depends on S. Each arc of $\vec{G}$ is obtained from an edge of G by assigning the orientation in the following way. Let abc be a history of a vertex $v \in S$. We create arcs $\vec{ab}$ and $\vec{cb}$ in $\vec{G}$. So both arcs are oriented into the middle vertex of the history of v. This procedure will never create two arcs that have the same vertices but the opposite orientation. Suppose that edge pq in G is in the intersection of histories of two vertices u and v from the set S. If p is the middle vertex of $P_2^{-1}(v)$ and q is the middle vertex of $P_2^{-1}(u)$ then vertices v and u are adjacent, which is a contradiction with independency of S. We can observe the following properties of the orientation defined by the maximal independent set S of $P_2(G)$.

a) The input degree of each vertex is either zero or is greater than one.

b) In-degrees of end vertices of an edge that does not correspond to any arc are equal to 0. In G does not exist a pair of adjacent edges such that none of them corresponds to an arc in $\vec{G}$.

The properties a) and b) allow completing $\vec{G}$ by adding arcs $\vec{ab}$ corresponding to edges ab that are not included in history of S. The orientation of $\vec{ab}$ is chosen randomly. The condition b) guarantees that the in-degree of some vertices may change from 0 to 1 and no other changes in in-degrees will occur. Hence no new pairs of arcs having the same second vertex will be created.

Vise versa, let us consider directed graph $\vec{G}$ without pairs of opposite arcs. We create an undirected graph G with the same vertex set, and edges corresponding to arcs. Let us now consider the set $S_{\vec{G}}$ of all vertices in $P_2(G)$ with histories abc in G where $\vec{ab}$ and $\vec{cb}$ are arcs from $\vec{G}$. We claim that $S_{\vec{G}}$ is an independent set. Indeed, if two vertices in $S_{\vec{G}}$ are adjacent, then they correspond to two adjacent paths, say abc and bcd (where d is not necessarily distinct from a). This is not possible, since $\vec{G}$ then contains both edges $\vec{bc}$ and $\vec{cb}$.

Hence for a given graph G, a maximal independent vertex set S in $P_2(G)$ corresponds to the set $S_{\vec{G}}$ in $\vec{G}$. The number of vertices in $S_{\vec{G}}$ can be expressed by counting the pairs of oriented edges headed in the same vertex by the following formula.

$$\beta(P_2(G)) = \max_{\vec{G}} \left| S_{\vec{G}} \right| = \max_{\vec{G}} \sum_{v \in \vec{G}} \binom{in\deg v}{2} \qquad (1)$$

The number $\binom{in\ \deg v}{2}$ is the number of pairs of arcs with the second vertex v. Each pair corresponds to a history of a vertex in $P_2(G)$. We shall call this number the *contribution* of vertex v. As there exist exponentially many (with respect to the size of the graph) orientations of G, the formula (1) does not provide an efficient method for finding independence number $\beta(P_2(G))$

In section 3, we intend to show that in the case G is a tree, the maximum in (1) can be determined with a less time complexity than the general graph.

3. Main Results

Starting from now we will consider a fixed non-empty tree T with a completed orientation O corresponding to a maximum independent set in $P_2(T)$. We chose a vertex v (root) in T. Denote as T_i^v the subtrees of T that are rooted in v, where $1 \le i \le \deg v$.

Obviously, $\beta(P_2(T)) = \binom{in\ \deg v}{2} + \sum_{i=1}^{\deg v} \beta(P_2(T_i^v))$.

Denote as T_i^{v+} the subtree of T that are rooted in v and contains the directed edge that is coming to v. In similar way, denote as T_i^{v-} the subtree of T that are rooted in v and contains the directed edge that is going out from v. So the above formula can be rewritten in the following way:

$$\beta(P_2(T)) = \binom{in\deg v}{2} + \sum_{i=1}^{k} \beta(P_2(T_i^{v-})) + \sum_{j=k+1}^{\deg v} \beta(P_2(T_j^{v+})).$$

Let us call $\beta(P_2(T_i^v))$ the contribution of the T_i^v. It is easy to notice that $in \deg v = \deg v - k$. where $0 \le k \le \deg v$ and represents the number of edges that are going out from v. Now, the following lemma can be introduced.

Lemma 1. Let T be a directed tree and v be a vertex of degree at least 2. Let O be the orientation of edges of T corresponding to the maximum independent set. Then :

$$\beta(P_2(T)) = \sum_{i=1}^{k} \beta(P_2(T_i^{v-})) + \sum_{i=k+1}^{\deg v} \beta(P_2(T_i^{v+})) + \binom{\deg v - k}{2}$$

Proof. Suppose that S is the maximum independent set of $P_2(T)$ with cardinality $\beta(P_2(T))$. Edges of T can be divided into edges coming to v, edges going out from v, and the edges of subtrees that are created by removing vertex v. The first group of edges represents the contribution of the vertex v. Each edge going out from v is assigned to the subtree containing the second vertex of this edge to create T_i^{v-}. Since the orientation of T_j^{v+} have an edge coming to v, then this edge will not increase the contribution of this subtree. This means that removing the vertex v from T_j^{v+} will not affect its contribution. Orientation of edges in subtree is induced by orientation O. We claim that the number of vertices in the intersection of S and $P_2(T_i^v)$ equals to $\beta(P_2(T_i^v))$. Suppose that is not true. Then it is possible to find an independent set in $P_2(T_i^v)$ with larger number of vertices. Let O_i^v be the corresponding orientation of edges in T_i^v. As all rooted subtrees are edge disjoint, it is possible to change orientation O on edges of T_i^v to O_i^v and let unchanged in the rest. The independent set S' corresponding to the new orientation is larger than S which is a contradiction with the assumption that S is maximum. So for all subtrees, the number of independent vertices is as maximal as possible. Therefore, $\beta(P_2(T))$ equals to the sum of contribution of vertex v and the contributions of subtrees rooted in v.

Lemma1 can be used to introduce an algorithm to find the maximum independent domination number of 2-path graph of any tree. The algorithm is based on the recursive divide and conquer method, with the following recurrence relation:

$$\beta(P_2(T)) = \binom{w}{2} \quad \text{, if } T \text{ is a star } K_{1,w}.$$

$$\beta(P_2(T)) = \max_k \left(\sum_{i=1}^{k} \beta(P_2(T_i^{v-})) + \sum_{j=k+1}^{\deg v} \beta(P_2(T_j^{v+})) + \binom{\deg(v)-k}{2} \right),$$

otherwise.

If T is a star graph $K_{1,w}$, then $P_2(T)$ is a trivial graph with $\binom{w}{2}$ isolated vertices. Thus, $\beta(P_2(T)) = \binom{w}{2}$.

Now, let us show that the recursive step working properly. Let S_i^v denotes the maximum set of independent vertices in $P_2(T- v)$. By previous lemma we have the following:

$$\beta(P_2(T)) = \max_k \left(|S_i^v| + \sum_{i=1}^{k} [\beta(P_2(T_i^{v-})) - \beta(P_2(T_i^{v+}))] + \binom{\deg(v)-k}{2} \right) \quad (2)$$

The value k corresponds to the number of edges that is directed out from v in the orientation of T. If an edge is oriented towards the vertex v then it increases the contribution of vertex v. On the other hand, if an edge is directed out of vertex v towards subtree T_i^v, in this case the contribution of this edge is $\left[\beta(P_2(T_i^{v-}) - \beta(P_2(T_i^{v+})\right]$. Let us call this contribution $g(T_i^v)$.

Since the values $\beta(P_2(T_i^{v-})$ and $\beta(P_2(T_i^{v+})$ are computed recursively, we can assume these values are known. Thus, $|S_i^v| = \sum_{i=1}^{\deg v} \beta(P_2(T_i^{v+})$. It is enough to sort the difference $\left[\beta(P_2(T_i^{v-}) - \beta(P_2(T_i^{v+})\right]$ in non-increasing order and find maximum of (2).

Based on the above description, we express the following algorithm in pseudo code to find the maximum independent domination number of 2-path graph for any tree graph.

Algorithm 2. Input: A tree T. Output: $|S| = \beta(P_2(T))$.

Function MID_P_2-Tree(T)

{

If T is a star $K_{1,w}$ then return ($\binom{w}{2}$)

Else {

Choose a vertex v from T such that the largest component T_i^v has order no more than $n/2$;

For i =1 to deg v do

{/* Recursive call for the T-v components */

g(T_i^v) = MID_P_2_Tree(T_i^{v-}) - MID_P_2_Tree(T_i^{v+});

}

Sort the difference g(T_i^v) resulting in a sequence $T_1^v \geq T_2^v \geq \ldots \geq T_{\deg v}^v$.

Let $E_k = G_k + \binom{\deg v - k}{2}$, where G_k = g(T_1^v) +……+ g(T_k^v) for $0 \leq k \leq \deg v$;

Set $E = \max_k E_k$;

$$|S| = \sum_{i=1}^{\deg v} |S_i^v| + E \;;$$

Return $|S|$;

}

}

Theorem 3. Algorithm 2. finds maximum independence domination number of P_2-path graph of any tree T in $\theta(n^2)$ time complexity, where n is the order of T.

Proof.

The proof is straight forward and can be done by time analysis of algorithm 2.

Selecting vertex v in such a way the largest component T_i^v has no more than (n/2) vertices, will reduce the time complexity in the recursive calls.

The algorithm takes O(n) time complexity to locate vertex v. Moreover, sorting algorithm requires at least O(n log n) time. In our algorithm each recursive call sort deg v components, so the algorithm will sort in total less than $\sum_{v \in T} \deg v = 2m = 2\,(n-1)$, where m is number of edges and n is the order.

Therefore, the time complexity spent by algorithm 2 in sorting is less than 2(n-1) log (2(n-1)) = O (n log (n)).

Let W(n) be the time spent in recursive calls to get MID number of $P_2(T)$. We have

W(1) = 0.

$$\mathrm{W}(n) = n + \sum_{i=1}^{\deg v}\left(W(T_i^{v+}) + W(T_i^{v-})\right),$$ but the order of the component T_i^v is less or equal (n/2)

W(n)=

$$\left[n + \sum_{i=1}^{k} W(T_i^{v-}) + \sum_{j=k+1}^{\deg v} W(T_i^{v+})\right] \le \left[n + \sum_{i=1}^{k} W(\tfrac{n}{2}) + \sum_{j=k+1}^{\deg v} W(\tfrac{n}{2})\right]$$

$\mathrm{W}(n) = n + \sum_{i=1}^{\deg v} W(\tfrac{n}{2})$. In the worst case where deg v = 2, the two subtrees rooted in v have approximately (n/2) vertices. W(n) ≤ n + 4W(n/2), multiple 4 represents 2 by worst case of deg v, where the value 2 means that the algorithm has to compute $\beta(P_2(T_i^{v+}))$ and $\beta(P_2(T_i^{v-}))$ for each component T_i^v. By master theorem we obtain that W(n) = $\theta(n^2)$. Since the time complexity of sorting is less than $\theta(n^2)$, we conclude that the whole time complexity done by algorithm 2 is $\theta(n^2)$ which completes the proof.

4. Conclusion

In this paper, an algorithm for finding the Maximum independence number of 2-path graph of any tree T is presented. First, we have found a different interpretation of the maximum independent set of vertices in $P_2(G)$ using the concept of a history. The structure of a tree allows the recursive strategy of finding the value of independence number $\beta(P_2(T))$. The complexity of the algorithm is $\theta(n^2)$ where n is the order of the tree. Based on the results presented in the paper, the method that have been used for finding the independence number can be applied on some other types of graphs like bipartite graphs. A similar strategy can be used to find the minimum independence number of the 2-path graph of any tree.

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