The Algorithm of the Calculation of Homology Group on Oriented Connected Simplicial Complexes

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Summary
In this paper we will discuss the homology group of an oriented connected simplicial complexes. Some methods for calculation of homology groups. Finally, we will introduce an algorithms for these calculation.

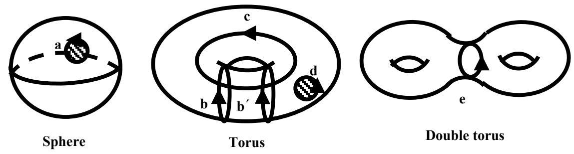
Key words:
Homology, algorithm

1. Introduction

Roughly speaking the homology groups for a simplicial complex tell us about the holes in the associated polyhedron.

The idea behind homology can conveniently be illustrated by the following description: suppose that a surface is subdivided into regions, curves and vertices consider any curve in the subdivision for the surface, it may have end points which are vertices of the subdivision, but on the other hand it may be closed so that it has no end point. In the later case it is called a 1-cycle. Any given 1-cycle may or may not be the boundary of one of the regions into which the surface is subdivided. If a 1-cycle is the boundary of the region, it is called a bounding 1-cycle. Two 1-cycles are called ``homologous" if they together form a bounding 1-cycle, that is if together they bound a region. For example in Figure(1) a, b, b', c, d, e are 1-cycles. Now a and d are each the boundary of regions, such a region is shaded, but it would be possible to take ``complementary" region occupying the remainder of the surface.

On the other hand b, b' and c do not bound a region. The fact that the simple closed curve b fails to bound region reveals the presence of a hole in the torus“ b goes round the hole". Similarly c reveals the presence of another hole. Now b and b' between them bound a region, thus they are homologous. This reveals the fact that b and b' go round the same hole in the torus



Figure(1)

The relation ``homologous" is an equivalence relation in the set of all 1-cycles, which can therefore be separated into classes such that any two 1-cycles in a given class are homologous. Defining addition of homology classes in the natural way, we obtain a group know as the 1-dimensional homology group or first homology group, [4]. The size of this group measures, roughly speaking, the number of 1-dimensional holes that the object has. The sphere has none and the tours has two, their first homology group are respectively 0 and Z Z.

Homology group in other dimensions can also be defined in a similar manner and the ideas can be extended to a wider class of spaces than surfaces.

The significance of the homology group lies in the fact that they are topologically invariance. This means that homeomorphic spaces have isomorphic homology groups.

To define homology groups in general it is necessary to introduce the concepts of chains, cycles and boundaries.

2. Calculation of homology groups

Here we will determine the zero, first and second homology groups of an oriented connected simplicial complexes. There are many methods for calculation of homology groups and we will discuss two of them and introduce their algorithm.

a) Calculation of homology groups by using chains

This method depends on the chains of simplicial complexes and is based on writing the 0,1 and 2-chains of the simplicial complex and form it we can obtain the cycles and boundaries and hence by using the formula

$$H_p(k) = Z_p(K)/B_p(K),$$

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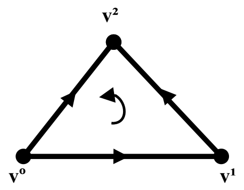
the homology groups can be obtained. To be precise we will discuss in the following examples.

Examples:

(i) Let K be the closure of a 2-simplex (v^o v^1 v^2) with orientation shown in Figure(2).



Figure(2)

The 1-chain in K is a sum of the form

$$c = \mu_o(v^o v^1) + \mu_1(v^1 v^2) + \mu_2(v^2 v^o),$$

where μ_o, μ_1, μ_2 are integers, then

$$\partial c = (-\mu_o - \mu_2)v^o + (\mu_o - \mu_1)v^1 + (\mu_o + \mu_2)v^2 \qquad (1)$$

thus c is a 1-cycle iff $-\mu_o - \mu_2 = 0, \mu_o - \mu_1 = 0, \mu_o + \mu_2 = 0$,

then $\mu_o = \mu_1 = -\mu_2 = \mu,$ where

μ is any integer, thus the 1-cycles is a chain of the form

$$\mu(v^o v^1) + \mu(v^1 v^2) + \mu(v^2 v^o) \qquad (2)$$

Thus $Z_1(K)$ is isomorphic to Z.

The 2-chain in K is of the form c=λ(v^o v^1 v^2), where λ is an integer, then

$$\partial c = \partial(\lambda(v^o v^1 v^2)) = \lambda(v^o v^1) + \lambda(v^1 v^2) - \lambda(v^o v^2) \qquad (3)$$

Thus ∂c=0 iff λ=0 i.e. $Z_2(K)$, then $H_2(K) = 0.$

From equation (2),(3) we notice that the 1-cycles and 1-boudaring cycles has the same form, thus

$$Z_1(K) = B_1(K) \text{ i.e., } H_1(K) = 0.$$

Now, since K is connected then $H_o(K) \cong Z.$. Thus the homology group for the simplex (v^o v^1 v^2) are

$H_o(K) \cong Z,$
$H_1(K) = 0,$
$H_2(K) = 0.$

(ii) Consider the complex K of Figure (3), whose underlying space is the boundary of a square with edges $e^1, e^2, e^3, e^4.$.

The general 1-chain c is of the form $\sum \lambda_i e^i$. . Computing ∂c , we can see that its coefficients at the vertex v is $(\lambda_1 - \lambda_2)$.

A similar argument, applied to the other vertices shows that c is

a cycle if and only if

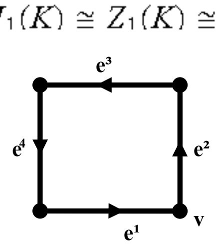
$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4,$$

one concludes that Z(K) is infinite cyclic and generated by the chain

$$e^1 + e^2 + e^3 + e^4.$$

Since there is no 2-simplex, $B_1(K)$ is trivial and hence

$$H_1(K) \cong Z_1(K) \cong Z.$$



Figure(3)

3. The algorithm for this method

Step(1): If n = 0 then $H_n(K) \cong Z$, otherwise go to Step(2).

Step(2): Select the smallest subscript n, for n ≥ 0, such that n-chain c has not already been found. If no such subscript is found go to **step(4).**

Step(3): Perform the following:

(3.1) writing the form of n-chain c.

(3.2) obtain $\partial_n c$.

(3.3) obtain the form of n-cycle, and evaluate

$$Z_n(K), B_n(K).$$

(3.4) calculate the homology group $H_n(K)$ by using the formula

$$H_n(K) = Z_n(K)/B_n(K).$$

(3.5) Return to step (2).

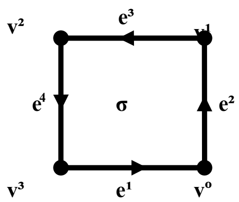
Step(4): If there is no n-chain then ,

$$B_n(K) = 0 \text{ and } H_n(K) = Z_n(K)/0 \cong Z_n(K).$$

4. Calculation of homology groups by using cell complexes

A better method is to use cell complexes instead of simplicial complexes. A cell complex is a similar in nature to a simplicial complexes, but consists of cells instead of simplexes in calculating the homology groups.

Examples :
(i) Consider the square, a cellular decomposition of it consists of four 0-cells, four 1-cells and a 2-cell, as shown in Figure(4).

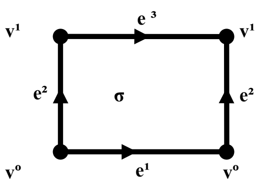


Figure(4)

The only one 2-cell σ whose boundary is $e^1 + e^2 + e^3 + e^4 \neq 0$, so there are no non-zero cycles hence $H_2(K) \cong 0$. Now, the 1-cell e^1 is not a 1-cycle since its boundary is $v^1 - v^o$, similar e^2, e^3, e^4. But $e^1 + e^2 + e^3 + e^4$ is a 1-cycle , which is the boundary of σ i.e. $H_1(K) \cong 0$. Finally $H_\circ(K)$ is an infinite cyclic group. Thus the homology groups for the square are

$H_\circ(K) \cong Z,$
$H_1(K) \cong 0,$
$H_2(K) \cong 0.$

(ii) Consider the cylinder obtained by identifying the sides of a rectangle as shown in Figure (5).



Figure(5)

Here there is only one 2-cell σ whose boundary is $e^1 + e^2 - e^3 - e^4 \neq 0,$, and so there is no non-zero 2-cycles, hence $H_2(K) \cong 0.$ To find $H_1(K),$, notice that e^1 is a 1-cycle which is not a boundary of a 2-cell, similar e^3 , but $e^1 + e^2$ is not a 1-cycle since its boundary is $v^1 - v^o$. Similar $e^2 + e^3$, Thus $H_1(K) \cong Z.$

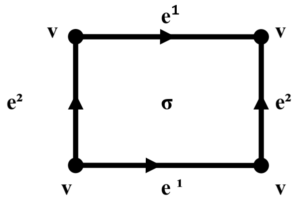
Finally, the 0-cycles of K form the group

$$Z_\circ(K) \cong v^\circ + v^1 \text{ since } \partial(e^2) = v^1 - v^\circ$$

i.e., $v^1 \sim v^o$ and $H_\circ(K) \cong Z.$. Thus the homology groups for the cylinder are

$H_\circ(K) \cong Z,$
$H_1(K) \cong Z,$
$H_2(K) \cong 0.$

(iii) Consider the tours obtained by identifying the sides of a rectangle as shown in Figure(6).



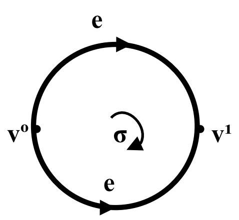
Figure(6)

There is one 2-cell σ whose boundary is $e^1 + e^2 - e^1 - e^2 = 0$, and so σ is a 2-cycle, but there are no three 3-cycles, hence σ is a non bounding 2-cycle. So for any non-zero integer λ, λ σ is a non bounding 2-cycle. Moreover, these are the only integral 2-cycles, and so $H_2(K)$ is an infinite cyclic group i.e., $H_2(K) \cong Z$. To find $H_1(K)$, notice that there are two 1-cycles e^1 and e^2 and they do not bound since the boundary of the only 2-cell is zero. It follows that $H_1(K)$ is the direct sum of two infinite cyclic group i.e., $H_1(K) \cong Z \oplus Z.$.

Finally, to find $H_\circ(K)$, notice that the only 0-cell v is a non-bounding 0-cycle, hence $H_\circ(K)$ is an infinite cyclic group, i.e., $H_\circ(K) \cong Z,$ thus the homology groups for the tours are

$H_\circ(K) \cong Z,$
$H_1(K) \cong Z \oplus Z,$
$H_2(K) \cong Z.$

(iv) Let K be the sphere S^2 with cell complex given by the standard planar diagram in Figure(7).



Figure(7)

Since there are 0-,1- and 2-cells present, we must compute

$$H_\circ(K), H_1(K) \text{ and } H_2(K).$$

The only 2-cell is σ . The 2-chains are

σ, 2σ , 3σ , ... ,-σ ,-2σ ,... and 0,

so .

$$C_2(K) \cong Z.$$

Since there are no 3-cells, so σ can not be the boundary of anything and $B_2(K) = 0$. Therefore

$$H_2(K) = Z_2(K) \backslash B_2(K) \cong Z.$$

Now

$$C_1(K) = \{e, 2e, ..., 0, -e, -2e, ...\} \cong Z,$$

since $\partial(e) = v^1 - v^o$, so e is not a 1-cycle, nor are 2e, -e , etc . Thus $Z_1(K) = 0$, and since $\partial(\sigma) = 0$, so $B_1(K) = 0$. Hence

$$H_1(K) = Z_1(K) = 0.$$

Again

$$C_\circ(K) = \{0, v^\circ, 2v^\circ, ..., -v^\circ, ..., -v^1, v^1, 2v^1, ..., v^\circ + v^1, 2v^\circ - 3v^1, etc.\},$$

so

$$C_\circ(K) = Z \oplus Z$$

Since $\partial(v^o)=0$ and $\partial(v^1)=0$ so v^o and v^1 and all combinations are 0-cycles $Z_\circ(K) = Z \oplus Z$, and hence any combinations of vertices v^o and v^1 is homologous to some multiple of v^1

$$H_\circ(K) = \{v^\circ, v^1, 2v^1, ..., -v^1, -2v^\circ, ...\} \cong Z.$$

5. The algorithm for this method

Step(1): Consider how many n- cells are there.

Step(2): Take the biggest subscript n, for n ≥ 0, such that n- cell has not already been check.

Step(3): Consider if the boundary of n- cell not equal to zero, then there is no n-cycle i.e. $Z_n(K) = 0$ and hence $H_n(K) = 0$, otherwise, perform the following:

(3.1) find $Z_n(K)$ and $B_n(K)$,

(3.1.1) check that if all of n- cell is the boundary of n-1-cell, then $Z_n(K) = B_n(K)$, and hence $H_n(K) \cong 0$.

(3.1.2) check that if there is no n- cycle which it is boundary, then

$B_n(K) = 0$, and hence $H_n(K) \cong Z_n(K)$.

(3.2) compute $H_n(K)$ by using the formula $H_n(K) = Z_n(K)/B_n(K)$.

(3.3) return to **Step(2).**

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