Tree Lyapunov Second Method and Certain Matrix Inequalities for Some Class of Deyla-Difference System of Cellular Neural Networks

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Summary
In this paper, we derive a sufficient condition for asymtotic stability of the zero solution of delay- difference system of cellular neural networks in terms of certain matrix inequalities by using a discrete version of the Lyapunov second method. The result is applied to obtain new asymptotic stability codition in terms of matrix inequalities for some class of delay- difference system of cellular neural networks such as delay- difference system of cellular neural networks with multiple delays in terms of certain matrix inequalities. Our results can be well suited for computational purposes.

Key words:
Asymptotic stability ;Delay-difference system;Cellular neural networks;Lyapunov function;Matrix inequalities.

1. Introduction

In recent decades, cellular neural networks have been extensly studied in many aspects and successfully applied to many fields such as pattern identifying, voice recognizing, system controlling, signal processing systems, static image treatment, and solving nonlinear algebraic system , etc. Such applications are based on the existence of equilibrium points, and qualitative properties of systems. In electronic implementation, time delays occur due to some reasons such as circuit integration, switching delays of the amplifiers and communication delays is of particular importance to manufacturing high quality microelectronic celluar neural networks.

While stability analysis of continuous-time neural networks can employ the sability theory of differentialsystem [13],it is much harder to study the stability of discrete-time neural networks[7] with time delays [2] or impulses [9]. The techniqes currently available in the literature for discrtr-time systems are mostly based on the construction Lyapunov second method [10].For Lyapunov second method, it is well known that no general rule exists to guide the construction of a proper Lyapuonv function.

For a given system. In fact, the construction of the Lyapunov function becomes a very difficult task.

In this paper , we consider delay – difference system of cellular neural networks of the form

S(d+1)=- MS(d)+AL(S(d))+BL(S(d-k)) + g,
(1)

$S(d) \subseteq R^n$ is the neuron state vector, $k \geq 0, m = diag\{m_1, D, m_n\}, m_i \geq 0, i = 1,2,...,n$ is the n × n constant relaxation matrix, A_i , B_i , $i = 1, 2, \ldots, n$ are the n × n constant weight matrices, $g = (g_1, D, g_n) \in R^n$ is the constant external input vector and $L(z) = [L_1(z_1), K, L_n(z_n)]^T$ with $l_i \in C^1[R, (-1,1)]$ where L_i is the neuron activations and monotonically increasing for each $i = 1, 2, \ldots, n$ The asymptotic stability of the zero solution of the dealy-differential system of Hopfield neural networks has been developed durring the past several years. We refer to monographs by Burton [3] an Ye [13] and the references cited therein. Much less is known regarding the asymptotic stability of the zero solution of the dealy-differential system of cellular neural networks. Therefore , the purpose of thise paper is to establish sufficient condition for the asymptotic stability of the zero solution of (1)in terms of certain matrix inequlilites.

2. PRELIMINARIES

We will define the following notations . R^+ is the set of all nonnegative real numbers , Z^+ is the set of all nonnegative integers , R^n is the n finite dimensional euclidean space with the euclidean norm $\|.\|$ and the scalar product between and is defined by $x^T y$; $R^{n \times m}$ denotes the set of all $(n \times m)$ matrices and A^T denotes the transpose of the matrix A ; Matrix $V \in R^{n \times m}$ is positive semidefinite $(V \geq 0)$ if $Vx \geq 0, \forall x \in R^n$. If $x^T V x > 0 (x^T V v < 0, resp)$. It is easy to verify that $V > 0, (V < 0, resp)$ if

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$$\exists \alpha > 0 : x^T V x \geq \alpha \|x\|^2, \forall x \in R^n,$$
$$(\exists \alpha > 0 : x^T V x \leq -\alpha \|x\|^2, \forall x \in R^n, resp).$$

Fact 2.1 For any positive scalar and vectors and , the following inequality holds:

$$x^T y + y^T x \leq \lambda x^T x + \lambda^{-1} y^T y .$$

Let us denote $Q_\alpha = \{x \in \mathbb{R}^n : \|x\| < \alpha\}$

Lemma2.1 [10] The zero solution of difference system is asymptoti stability if there exists a positive definite function $Q(x) : R^n \longrightarrow R^+$ such that

$$\exists \beta > 0 : \Delta Q(x(d)) = Q(x(d+1)) - Q(x(d)) \leq -\beta \|x(d)\|^2,$$

Along the solution of the system . In case the above condition holds for all $x(d) \in Q_\alpha$, we say that the zero solution is locally asymptotically stable.

We present the following technical lemmas , which be used in the proof of our main result.

Lemma2.2 [6] For any constant symmetric matrix

$M \in \mathbb{R}^{n \times m}$ $M = M^T > 0$ $l \in \mathbb{R}^+ / \{0\}$

vector function $l_\in \mathbb{R}^+ / \{0\}$, vector function

$W : [0, l] \rightarrow \mathbb{R}^n$, we have

$$l \sum_{i=0}^{l-1} (W^T(i) M W(i)) \geq \left(\sum_{i-0}^{l-1} W(i) \right)^T M \left(\sum_{i-0}^{l-1} W(i) \right).$$

3 MAIN RESUULTS

In this section , we consider the sufficient condition for asymptotic stability of the zero solution of (1) in terms of certain matrix inqualities .without loss of generality, we

can assume that $s^* = 0, s(0) = 0$ and g =0 (for otherwise, we let $x = S - S^*$ and define

$L(x) = L(x + S^*) - L(S^*)$.

The new form of (1)is now given by

$$x(d-1) = -Mx(d) + AL(x(d)) + BL(x(d-k)).$$

(2)

Throughout this paper we assume the neuron activations $l_i(x_i), i = 1, 2, \ldots, n$ is bounded and monotonically nondecreasing on R , and $l_i(x_i)$ is Lipschitz continuous, that is , there exist constant

$q_i > 0, i = 1, 2, \ldots, n$ such that

$$|l_i(v_1) - l_i(v_2)| \leq q_i |v_1 - v_2|, \forall v_1, v_2 \in \mathbb{R} .$$

(3)

By condition (3) , $l_i(x_i)$ satisfy

$$|l_i(x_i)| \leq l_i |x_i|, i = 1, 2, \ldots, n.$$

(4)

Theorem 3.1 The zero solution of the delay-difference system (2) is asymptotically stable if there exist symmetric positve definite matrices P,G,W and $J = diag[j_1, \ldots, j_n] > 0$ satisfying the following matrix inequalities:

$$\varphi = \begin{pmatrix} (1,1) & 0 & 0 \\ 0 & (2,2) & 0 \\ 0 & 0 & (3,3) \end{pmatrix} < 0$$

(5)

$(1,1) = g^T Pg - p + kG + W \alpha A^T PBB^T PA + \alpha_2 JA^T PBB^T PAJ + JA^T PAJ + \alpha^{-1} JJ$

,

$(2,2) = JB^T PBL + \alpha_1^{-1} JJ + \alpha_2^{-1} JJ - W$,

$(3,3) = -kG$.

Proof Consider the Lyapunov function

$Q(y(d)) = Q_1(y(d)) + Q_2(y(d)) + Q_3(y(d))$,

where

$$Q_1(y(d)) = x^T(d) P x(d),$$
$$Q_2(y(d) = \sum_{i=d-k}^{d-1} (k - d + i) x^T(i) G x(i),$$
$$Q_3(y(d)) = \sum_{i=d-k}^{d-1} x^T(i) W x(i),$$

P,G,W being symmetric positive definite solutions (5) and $y(d) = [x(d), x(d-k]$

Then difference of $Q(y(d))$ along trajectory of solutions of (2) is given by

$$\Delta Q(y(d)) = \Delta Q_1(y(d)) + \Delta Q_2(y(d)) + \Delta Q_3(y(d))$$

. where

$$\Delta Q_1(y(d)) = Q_1(x(d+1)) - Q_1(x(d))$$
$$= \left[-Mx(d) + AL(x) + BL(x(d-k))\right]^T P$$
$$\times\left[-Mx(d) + AL(x(d)) + BL(x(d-k))\right] - x^T(d)Px(d)$$
$$= x^T(d)\left[M^T PM - P\right]x(d) - x^T(d)M^T PAL(x))$$
$$-L^T(x(d))A^T PMx(d) - x^T(d)M^T PBL(x(d-k))B^T PMx(d)$$
$$+L^T(x(d))A^T PBL(x(d-k)) + L^T(x(d-k))B^T PAL(x(d))$$
$$+L^T(x(d))A^T PAL(x(d))$$
$$+L^T(x(d-k))B^T PBL(x(d-k)),$$
$$\Delta Q_2(y(d)) = \Delta\left(\sum_{i=d-k}^{d-1}(k-d+i)x^T(i)Gx(i)\right)kx^T(d)Gx(d)$$
$$-\sum_{i=d-k}^{d-1}x^T(i)Gx(i)$$
and
$$\Delta Q_3(y(d)) = \Delta\left(\sum_{i=d-k}^{d-1}x^T(i)Wx(i)\right)$$
$$= x^T(d)Wx(k) - x^T(d-k)Wx(d-k)$$

(6)

Where (4) and Fact 2.1 are utilized in (6), respectively. Note that

$$-x^T(d)M^T PAS(x(d)) - L^T(x(d))A^T PMx(d) \le$$
$$\alpha x^T(d)M^T PAA^T PMx(d) + \alpha^{-1}L^T(x(d))L(x(d))$$
$$-x^T(d)M^T PBS(x(d-k)) - L^T(x(d-k))B^T PMx(d) \le$$
$$\alpha^{-1}{}_1 x^T(d)M^T PBB^T PMx(d) + \alpha_1^{-1}L^T(x(d-k))L(x(d-k))$$
$$L^T(x(d))A^T PBL(x(d-k)) + L^T(x(d-k))B^T PAL(x(d)) \le \alpha_2 L^T(d)PBB^T PAL(d)$$
$$+\alpha_2^{-1}L^T(x(d-k))L(x(d-k))$$
$$L^T(x(d-k))B^T PBL(x(d-k)) \le x^T(d-k)JAB^T PBJx(d-k),$$
$$L^T(x(d))A^T PAL(x(d)) \le x^T(d)JA^T PAJx(d),$$
$$\alpha_2 L^T(d)A^T PAL(x(d)) \le \alpha_2 x^T(d)JA^T PBB^T PAJx(d),$$
$$\alpha_1^{-1}L^T(x(d-k)L(x(d-k)) \le \alpha_1^{-1}x^T(|k-d|)JJx(|k-d|)$$
$$\alpha 2^{-1}L^T(x(d-k)L(x(d-k)) \le \alpha_2^{-1}x^T(|k-d|)JJx(|k-d|)$$

and

$$\alpha^{-1}L^T(x(d))L(x(d)) \le \alpha^{-1}x^T(d)JJx(d),$$

Hence

$$\Delta Q_1 \le x^T(d)\left[g^T pg^T - P\right]x(d) + \alpha x^T(d)A^T PBB^T PAx(d) + \alpha_1 x^T(d)g^T PBB^T Pgx(d)$$
$$+x^T(d-k)JB^T PBJx(d-k) + x^T(d)JA^T PAJx(d) + \alpha_2 x^T(d)JA^T PBB^T PAJx(d)$$
$$+\alpha_1^{-1}x^T(d-k)JJx(d-k) + \alpha_2^{-1}x^T(d-k)JJx(d-k) + \alpha^{-1}x^T(d)JJx(d).$$

Then we have

$$\Delta Q \le$$
$$x^T(d)\left[g^T pg^T - P + kG + W + \alpha A^T PBB^T PA + \alpha_1 J^T PBB^T PJ + \alpha_2 JA^T PBB^T PAJ + JA^T PAJ + \alpha^{-1}JJ\right]x(d)$$
$$+x^T(d-k)\left[JB^T PBJ + \alpha_1^{-1}JJ + \alpha_2^{-1}JJ - W\right]x(d-k)$$
$$-\sum_{i=d-k}^{d-1}x^T(i)Gx(i).$$

Using lemma 2.2 ,we obtain

$$\sum_{i=d-k}^{d-1}x^T(i)Gx(i) \ge \left(\frac{1}{k}\sum_{i=d-k}^{d-1}x(i)\right)^T(kG)\left(\frac{1}{k}\sum_{i=d-k}^{d-1}x(i)\right)$$

From the above inequlaity it follows that:

$$\Delta Q \le$$
$$x^T(d)\left[g^T pg^T - P + kG + W + \alpha A^T PBB^T PA + \alpha_1 J^T PBB^T PJ + \alpha_2 JA^T PBB^T PAJ + JA^T PAJ + \alpha^{-1}JJ\right]x(d)$$
$$+x^T(d-k)\left[JB^T PBJ + \alpha_1^{-1}JJ + \alpha_2^{-1}JJ - W\right]x(d-k)$$
$$-\left(\frac{1}{k}\sum_{i=d-k}^{d-1}x(i)\right)^T(kG)\left(\frac{1}{k}\sum_{i=d-k}^{d-1}x(i)\right).$$

$$= \left(x^T(d), x^T(d-k), (\frac{1}{k}\sum_{i=d-k}^{d-1}x(i))^T\right)\begin{pmatrix}(1,1) & 0 & 0\\ 0 & (2,2) & 0\\ 0 & 0 & (3,3)\end{pmatrix}\begin{pmatrix}x(d)\\ x(d-k)\\ (\frac{1}{k}\sum_{i=d-k}^{d-1}x(i))\end{pmatrix}$$
$$= y^T(k)\varphi y(d),$$

where

$$(1,1) = g^T Pg - p + kG + W \alpha A^T PBB^T PA + \alpha_2 JA^T PBB^T PAJ + JA^T PAJ + \alpha^{-1}JJ$$
$$(2,2) = JB^T PBL + \alpha_1^{-1}JJ + \alpha_2^{-1}JJ - W,$$
$$(3,3) = -kG,$$
$$y(d) = \begin{pmatrix}x(d)\\ x(d-k)\\ (\frac{1}{k}\sum_{i=d-k}^{d-1}x(i))\end{pmatrix}.$$

And By the condition (5), $\Delta Q(y(d))$ is negative definite, namely there is a number $\chi > 0$ such that $\Delta Q(y(d)) \le -\chi\|y(d)\|^2$, and hence, the asymptotic stability of the system immediately follows from lemma 2.1 . This completes the proof.

Remark 3.1 Theorem 3.1 gives a sufficient condition for the asymptotic stability of delay – difference system (2) via matrix inequalities. These conditions are described in terms of certain diagonal matrix inequalities, which can realized by using the linear matrix inequality algorithom proposed in [4] But Hu and wang [9] these conditions are described in terms of certain symmetric matrix inequalites,which can be realized by using the schur complement lemma and linear matrix inequlities, which can be realzed by using the schur complement lemma and linear matrix inequlity algorithm proposed in [4] .

4. APPLICATIONS

In this section, we apply the main result of thise paper , which provides a sufficient condition for the asymptotic of dealy-difference system of cellular neural networks with multiple dealays in terms of certain matrix inequaities.

We consider dealy-difference system of cellular neural networks with multiple dealays of the form

$$S(d+1) = -MS(d) + AL(S(d)) + \sum_{i=1}^{n}B_i L(S(d-k_i)) + g, \quad (7)$$

$S(d) \subseteq R^n$ is the neuron state vector,

$0 \le k_1 \le D \le k_n \ge 0, M = diag\{m_1, D, m_n\}, m_i \ge 0, i = 1,2,...,n$ is the n ⌘ n constant relaxation matrix, A, B_i, $i = 1,2,...,n$ are the n × n constant weight matrices, $g = (g_1, D, g_n) \in R^n$ is the constant external input vector and $L(z) = [L_1(z_1), K, L_n(z_n)]^T$ with $l_i \in C^1[R, (-1,1)]$ where L_i is the neuron activations and monotonically increasing for each $i = 1,2,...,n$.

We consider the sufficient condition for asymptotic stability of the zero solution S^* of (7) in terms of certain matrix inequalities. Without loss of generality , we can assume that $S^* = 0, S(0) = 0$ and g=0 (for otherwise, we let $x = S - S^*$ and define $L(x) = L(x + S^*) - L(S^*))$. Then new form of (7) is now given by

$$x(d-1) = -Mx(d) + AL(x(d)) + \sum_{i=1}^{n} B_i L(S(d-k_i)).$$

(8)

Theorem 4.1 The zero solution of dealy-difference system (8) is asymptotically stable if there exist symmetric positive definite matrices $P, G_i, W_i, i = 1,2,...,n$ and $J = diag[j_1, D, j_n] > 0$ satisfying the following matrix inequalities:

(0,0)	**0**	**0**	**J**	**0**	**0**	**0**	**0**	**J**	**0**
0	(1,1)	(1,2)	J	(1,n)	0	0	0	J	0
0	(2,1)	(2,2)	J	(2,n)	0	0	0	J	0
M	M	M	0	M	M	M	M	M	M
= 0	(n,1)	(n,2)	J	(n,n)	0	0	0	J	0
0	0	0	J	0	(n+1,n+1)	0	0	J	0
0	0	0	0	0	0	(n+2,n+2)	0	J	0
M	M	M	M	M	M	0	0	0	M
0	0	0	0	0	0	M	0	0	0
0	0	0	0	0	0	0	J	0	(2n,2n)

$\varphi < 0$

(9)

Where

$$(0,0) = g^T Pg - P + \sum_{i=1}^{n} kG_i + W_i + \alpha \sum_{i=1}^{n}\sum_{r=1}^{n} A^T PB_i B_r^T PA +$$

$$\alpha_1 \sum_{i=1}^{n}\sum_{r=1}^{n} g^T PB_i B_J^T Pg + \alpha_2 \sum_{i=1}^{n}\sum_{r=1}^{n} J^T A^T PB_i B_r^T PAJ + JA^T PAJ + \alpha^{-1} JJ,$$

$$(i,r) = JB_i^T PB_r J + \alpha_1^{-1} JJ + \alpha_2^{-1} JJ - W_i, \forall i = r = \{1,2,D,n\}$$

$$(i,r) = JB_i^T PB_r J + \alpha_1^{-1} JJ + \alpha_2^{-1} JJ, \forall i \ne r = \{1,2,D,n\}, and$$

$$(i,r) = -kG_i, \forall i = r = \{n+1, n+2, D, 2n\}$$

Proof Consider the Lyapunov function $Q(y(d)) = Q_1(y(d)) + Q_2(y(d)) + Q_3(y(d))$, where

$$Q_1(y(d)) = x^T(d) Px(d),$$

$$Q_2(y(d)) = \sum_{i=1}^{n} \sum_{r=d-k}^{d-1} (k - d + i) x^T(i) Gx(i),$$

$$Q_3(y(d)) = \sum_{i=1}^{n} \sum_{r=d-k}^{d-1} x^T(i) Wx(i),$$

$P, G_i, W_i, i = 1,2,...,n$ being symmetric positive definte solution of (9) and

$$y(d) = [x(d), x(d-k_1), D, x(d-k_n)].$$

The diffrence of $Q(y(d))$ along trajectory of solution of (8) is given by

$$\Delta Q(y(d)) = \Delta Q_1(y(d)) + \Delta Q_2(y(d)) + \Delta Q_3(y(d))$$

,where

$$\Delta Q_1(y(d)) = \Delta Q_1(x(d+1)) - \Delta Q_1(x(d)) + \Delta Q_3(y(d))$$

$$= \left[-Mx(d) + AL(x(d))x(d) + \sum_{i=1}^{n} B_i L(x(d-k_i))\right]^T P$$

$$\times \left[-Mx(d) + AL(x(d))x(d) + \sum_{i=1}^{n} B_i L(x(d-k_i))\right] -$$

$$x^T(d)Px(d)$$

$$= x^T(d)\left[M^T PM - P\right]x(d)$$

$$-x^T LPA(X(d))M^T(X(d))PA^T L(x(d))$$

$$-\sum_{i=1}^{n} x^T(d) LPB_i M(X(d-k_i)) - \sum_{i=1}^{n} M^T(x(d-k_i))B_i^T PLx(d)$$

$$+\sum_{i=1}^{n} L^T(x(d))A^T PB_i L(x(d-k_i)) + \sum_{i=1}^{n} L^T(x(d-k_i))B_i^T PAL(x(d))$$

$$+\sum_{i=1}^{n} L^T(x(d))A^T PB_i L(x(d-k_i)) + \sum_{i=1}^{n} L^T(x(d-k_i))B_i^T PAL(x(d))$$

$$+\sum_{i=1}^{n} L^T(x(d))A^T PAL(x(d)) + \sum_{i=1}^{n}\sum_{i=1}^{n} L^T(x(d-k_i))B_i^T PB_r L(x(d-k_r))$$

$$\Delta Q_2(y(d)) = \Delta\left(\sum_{i=1}^{n}\sum_{r=d-k_i}^{d-1} (k_i - d + r) x^T(r) Gx(r)\right)$$

$$= \sum_{i=1}^{n} k_i x^T G_i x(d) - \sum_{i=1}^{n}\sum_{r=d-k_i}^{d-1} x^T(i) Gx(i)$$

and

$$\Delta Q_3(y(d)) = \Delta\left(\sum_{i=1}^{n}\sum_{r=d-k_i}^{d-1} x^T(r) Wx(r)\right)$$

$$= \sum_{i=1}^{n} x^T(d) W_i x(d) - \sum_{i=1}^{n} x^T(d-k_i) W_i x(d-k_i)$$

The rest of the proof is similar to that of **Theorem** 3.1 need hold.

5.CONCLUSIONS

In this paper, based on a discrete analog of the Lyapunov second method,we have established a sufficint condition for the asymptotc stability of deay-difference system of cellular neural networks in terms of certain matrix inqualities.The result is appllied to obtain new stability condition in terms of certain matrix inqualities for some class of deay-difference system of cellular neural networks with multiple delays in terms of certain matrix inequalities.

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