On Weakly Compact Sets in L-Fuzzy Topological Spaces

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Abstract:
In this paper, the concept of weakly compact set in L-fuzzy topological space have been introduced. The characterization of weakly compact set are researched. Also it is pointed out that weakly compactness is an L-good extension of usual weak compactness.

Keyword:
fuzzy lattice, L-fuzzy topological spaces, weakly compact set，fuzzy mapping.

1. Preliminaries

In this paper, $L=L(\leq,\vee,\wedge,')$ always denotes a fuzzy lattice. 1 and 0 denotes the greatest and the least elements of L respectively. Let X be a nonempty set, L^x is the set of all L-fuzzy sets on X and $M(L)$ the set of all nonzero irreducible elements of L. For $\Omega\subset L^x$, we define that

$\Omega'=\{A':A\in\Omega\}$, $\vee\Omega=\vee\{A:A\in\Omega\}$,

$\wedge\Omega=\wedge\{A:A\in\Omega\}$,

$FS(\Omega)=\{\Phi\subset\Omega:\Phi \textit{ is finite subfamily of } \Omega\}$ .For $\alpha\in L$, $A_\alpha=\{x\in X:A(x)\geq\alpha\}$. For each $A\subset L^x$, $\overline{A}, A^0, A'$ will denote the closure, the interior and the pseudo-complement of A respectively.

A is called LF semi-open (LF semi-closed) if there exist a $B\in\delta(G\in\delta')$ such that $B\subset A\subset\overline{B}(G^0\subset A\subset G)$.

A is called LF regular open (closed) if $A=\overline{A}^0(A=\overline{A^0})$. Let SO($L^x$),SC($L^x$),RO($L^x$),RC($L^x$) be the family of LF semi-open, LF semi-closed, regular open, regular closed sets in (L^x,δ) respectively.

Definition1.1 [1] (i) $p\in L$ is called a union-irreducible element of L, if for arbitrary a, b$\in L$ we have $p\leq a\vee b$ $\Rightarrow p\leq a \textit{ or } p\leq b$. The set of all the non-zero union-irreducible elements of L is denoted by M (L). (ii) $p\in L$ is called a prime element of L, if for arbitrary a, b$\in L$ we have $a\wedge b\leq p\Rightarrow a\leq p \textit{ or } b\leq p$.

Put

$p(L)=\{p\in L: p \textit{ is prime element of } L \textit{ and } p\neq 1\}$.

It is easy to check that $p\in M(L) \textit{ iff } p'\in p(L)$.

Put $M(L,X)=\{x_\alpha: x\in X,\alpha\in M(L)\}$, then we easy to check that $M(L,X)$ is just the set of all non-zero union-irreducible elements (molecule) of L^X . According to Wang [11] in a completely distrubitive lattice, each element α has a greatest minimal set which we will denote by $\beta(\alpha)$. We will put

Manuscript received June 5, 2010
Manuscript revised June 20, 2010

$\beta^*(\alpha)=\beta(\alpha)\bigcap M(L)$ and $\alpha^*(r)=(\beta^*(r'))'$.

Definition1.2 [2] Let (L^x,δ) be an L-fts, $\mathrm{A}\in\mathrm{L}^x$ and $\alpha\in M(\mathrm{L}),\ \Omega\subset\delta'$ is called an almost $\alpha-RF$ of A, if for each $x_\alpha\in A,$ there is $p\in\Omega,$ such that $p^0\in\Omega(x_\alpha).$ Ω is called an almost $\alpha^- - RF$ of A, if there is

$\lambda\in\beta^*(\alpha)$ such that Ω is a $\lambda-RF$ of A.

Definition1.3 [2] Let (L^x,δ) be an L-fts, $\mathrm{A}\in\mathrm{L}^x$ and $r\in p(\mathrm{L}),$ (i) $\Omega\subset\delta$ is called an r-cover of A if for each $x\in X$, there is $B\in\Omega$ satisfying $B(x)\not\leq r.$ Ω is called an r^*-cover of A if there is $t\in\alpha^*(\mathrm{r})$ such that Ω is a t-cover of A. (ii) $\Omega\subset\delta$ is called an almost r-cover of A if for each $x\in X$, there is $B\in\Omega$ satisfying $\overline{B}(x)\not\leq r.$ Ω is called an almost r^*-cover of A if for each $x\in X$, there exist $t\in\alpha^*(\mathrm{r})$ such that Ω is an almost t-cover of A.

Definition1.4 [4] Let (L^x,δ) be an L-fts and $\mathrm{A}\in\mathrm{L}^x$, A is called strong fuzzy compact set, if for each $\alpha\in M(\mathrm{L})$ and every $\alpha-RF$ of A, there exist $\Phi\in FS(\Omega)$ such that Φ is an $\alpha-RF$ of A.

Definition1.5 [4] Let $\Omega\in L^x$, $r\in p(L)$, Ω is said to have finite r-intersection property in $A\in L^x$ if for each $\Phi\in FS(\Omega)$ there exist $x\in A_{r'}$ such that $(\wedge\Phi)(x)\geq r'$.

Definition1.6 [5] Let (L^x,δ) be an L-fts, $\mathrm{A}\in\mathrm{L}^x$ and $\alpha\in\mathrm{L}$, Ω is called the family which has almost α-intersection property in A, if for each $\Phi\in FS(\Omega)$ there is $x\in A_\alpha$ such that $(\wedge\Phi^0)(x)\geq\alpha$.

Definition1.7 [3] Let (L^x,δ) be an L-fts, $\mathrm{B}\in\mathrm{L}^x$ is called LF regular semi-open if there exist a $Q\in RO(L^x)$ satisfying $Q\subset B\subset\overline{Q}$; B is called LF regular semi-closed if there exist a $P\in RC(L^x)$ satisfying $P^0\subset B\subset P.$

Definition1.8 [6] Let (L^x,δ) and $(\mathrm{L}^y,\varepsilon)$ be two L-fts and *f:* $\mathrm{L}^x\to\mathrm{L}^y$ a fuzzy mapping : (i) *f* is semi-continuous if $f^{-1}(B)\in SO(L^x)$ for each $B\in\varepsilon$; (ii) *f* is almost continuous if $f^{-1}(B)\in\delta$ for each $B\in RO(L^y)$; (iii) *f* is weakly continuous if $f^{-1}(B)\subset(f^{-1}(\overline{B}))^0$ for each $B\in\varepsilon$.

2. Weakly Compact Set

Definition2.1 Let (L^x,δ) be an L-fts, $\mathrm{A}\in\mathrm{L}^x$.A is called weakly compact set, if for each countably $\alpha-\mathrm{RF}\,\Omega$

Of A, there exists $\Phi\in\mathrm{FS}(\Omega)$ such that Φ is an almost $\alpha^- - \mathrm{RF}$ of A. Particularly, when 1_x is weakly

compact set, we call (L^x, δ) as a weakly compact space..

Theorem 2.2 An L-fuzzy set A in (L^x, δ) is a weakly compact set iff for any $r \in p(L)$ and every countably $r-$cover Ω of A, there exists $\Phi \in FS(\Omega)$ such that Φ is an almost r^+-cover of A..

Theorem 2.3 Let (L^x, δ) be an L-fts, $A \in L^x$. Then A is a weakly compact set iff for any $r \in M(L)$ and every countably family $\Omega \subset \delta'$ which has the r-intersection property in A, there exists $x \in A_i$ such that $(\wedge\bar{\Omega})(x) \geq r$.

Theorem 2.4 Let (L^x, δ) be an L-fts, $A \in L^x$. Then A is a weakly compact set iff for each regular closed countably $\alpha - RF\Omega$ of A, there exists $\Phi \in FS(\Omega)$ such that Φ is an $\alpha^- - RF$ of A..

Theorem2.5 Let ($L^x, w_L(\tau)$) be the L-fuzzy topological space topologically generated by a crisp space (X, τ). Then ($L^x, w_L(\tau)$) is L-fuzzy weakly compact set iff (X, τ) is weakly compact space.

Proof Assume that ($L^x, w_L(\tau)$) is L-fuzzy weakly compact space and Ω is an countably open cover of X. Put $\chi_\Omega = \{\chi_A : A \in \Omega\} \subset w_L(\tau)$. For each $r \in p(L)$, χ_Ω is a countably $r-$cover of 1_X. Indeed, for each $x \in X$, there exists $A \in \Omega$ such that $x \in A$ and so $\chi_A(x) = 1 \not\leq r$. Hence χ_Ω is a countably $r-$cover of 1_X. From the theorem 3.2, Ω has a finite subfamily $\{A_1, \cdots, A_n\}$ such that $\Phi = \{\chi_{A_i} : i = 1, 2, \cdots, n\}$ is a finite almost r^+-cover of 1_X, i.e.for each $x \in X$ there is $\chi_{A_i} \in \Phi$ satisfying $(\chi_{A_i})^-(x) \neq 0$ .Since $(\chi_{A_i})^- = \chi_{\bar{A}_i}$ and $x \in \bar{A}_i, \bigcup_{i=1}^{n} \bar{A}_i = X$.

This implies that (X, τ) is weakly compact space. Conversely, for each $r \in p(L)$, suppose that $\phi \in w_L(\tau)$ is any countably r-cover of 1_X . Then for each $x \in X$, there exists $P \in \phi$ such that $P(x) \not\leq r$ .Hence $l_r(\phi) = \{l_r(P) : P \in \phi\} \subset \tau$ is a ciuntably cover of (X, τ). From the weakly compactness of (X, τ), there exists $\Phi \in FS(\Omega)$ such that $\overline{l_r(\phi)}$ is a cover of (X, τ). Hence for each $x \in X$, there exists $P \in \phi$ such that $x \in \overline{l_r(P)} \in \overline{l_r(\phi)}$. Therefore ($L^x, w_L(\tau)$) be the L-fuzzy topological space.

Theorem 2.6 An L-fuzzy set A in (L^x, δ) is a weakly compact set iff every countably $\alpha - RORF\ \Omega$ of A has a finite subfamily Φ that is an $\alpha^- - RORF$ of A for each $\alpha \in M(L)$.

Theorem 2.7 Let (L^x, δ) be an L-fts, $A \in L^x$. A is called RS-compact, if for each $\alpha \in M(L)$ and each $\alpha - RSOR$

Ω of A, there exists $\Phi \in FS(\Omega)$ such that Φ is an

$\alpha - \mathrm{RSOR}$ of A.. (L^x, δ) is called RS-compact.

Theorem 2.8 Let (X, τ) be a crisp topological space and (L^x, δ) be the corresponding L-fts, then (X, τ) is RS-compact iff ($L^x, w(\tau)$) is RS-compact.

Theorem 2.9 An L-fuzzy set A in L- fts (L^x, δ) is RS-compact iff for each $r \in p(L)$ and every r-regular semi-open cover Ω of A, there exists $\Phi \in \mathrm{FS}(\Omega)$ such that Φ is a r-regular semi-open cover of A.

Theorem 2.10 Let (L^x, δ) and (L^y, ε) be two L-fts and *f:* $L^x \to L^y$ be an almost continuous L-fuzzy mapping. If A is an L-fuzzy weakly compact set in L^x, then *f*(A) is an L-fuzzy weakly compact set in L^y.

Theorem 2.11 Let (L^x, δ) and (L^y, ε) be two L-fts and *f:* $L^x \to L^y$ be an weakly continuous fuzzy mapping. If A is an L-fuzzy weakly compact set in L^x, then *f*(A) is an L-fuzzy weakly compact set in L^y.

Theorem 2.12 Let (L^x, δ) and (L^y, ε) be two L-fts and *f:* $L^x \to L^y$ be an semi-continuous fuzzy mapping. If A is an L-fuzzy weakly compact set in L^x, then *f*(A) is an L-fuzzy weakly compact set in L^y.

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