On Weakly Compact Sets in L-Fuzzy Topological Spaces

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Abstract:
In this paper, the concept of weakly compact set in L-fuzzy topological space have been introduced. The characterization of weakly compact set are researched. Also it is pointed out that weakly compactness is an L-good extension of usual weak compactness.

Keyword: fuzzy lattice, L-fuzzy topological spaces, weakly compact set, fuzzy mapping.

1. Preliminaries

In this paper, \( L=\mathbb{L}(\leq, \lor, \land, ^{'}\leq) \) always denotes a fuzzy lattice. 1 and 0 denotes the greatest and the least elements of \( L \) respectively. Let \( X \) be a nonempty set, \( L^x \) is the set of all L-fuzzy sets on \( X \) and \( \text{M}(L) \) the set of all nonzero irreducible elements of \( L \). For \( \Omega \subset L^x \), we define that

\[
\Omega' = \{ A : A \in \Omega \}, \quad \vee \Omega = \vee \{ A : A \in \Omega \},
\]

\[
\land \Omega = \land \{ A : A \in \Omega \},
\]

\[
\text{FS}(\Omega) = \{ \Phi \subset \Omega : \Phi \text{ is finite subfamily of } \Omega \}.
\]

For \( \alpha \in L \), \( A_\alpha = \{ x \in X : A(x) \geq \alpha \} \). For each \( A \subset L^x \), \( \overline{A}, A^0, A' \) will denote the closure, the interior and the pseudo-complement of \( A \) respectively. \( A \) is called LF semi-open (LF semi-closed) if there exist a \( \delta \in \delta(G \in \delta') \) such that \( B \subset A \subset \overline{B}(G^0 \subset A \subset G) \).

\( A \) is called LF regular open (closed) if \( A = \overline{A}^0 (A = \overline{A}^0) \). Let \( SO(L^x), SC(L^x), RO(L^x), RC(L^x) \) be the family of LF semi-open, LF semi-closed, regular open, regular closed sets in \( (L^x, \delta) \) respectively.

Definition 1.1 [1] (i) \( p \in L \) is called a union-irreducible element of \( L \), if for arbitrary \( a, b \in L \) we have \( p \leq a \lor b \) \( \Rightarrow p \leq a \text{ or } p \leq b \). The set of all the non-zero union-irreducible elements of \( L \) is denoted by \( \text{M}(L) \). (ii) \( p \in L \) is called a prime element of \( L \), if for arbitrary \( a, b \in L \) we have \( a \land b \leq p \Rightarrow a \leq p \text{ or } b \leq p \).

Put \( p(L) = \{ p \in L : p \text{ is prime element of } L \text{ and } p \neq 1 \} \).

It is easy to check that \( p \in M(L) \text{ iff } p' \in p(L) \).

Put \( M(L, X) = \{ x_\alpha : x \in X, \alpha \in M(L) \} \), then we easy to check that \( M(L, X) \) is just the set of all non-zero union-irreducible elements (molecule) of \( L^x \).

According to Wang [11] in a completely distrutive lattice, each element \( \alpha \) has a greatest minimal set which we will denote by \( \beta(\alpha) \). We will put
\[ \beta'(\alpha) = \beta(\alpha) \cap M(L) \] and \[ \alpha^*(r) = (\beta'(r'))'. \]

\textbf{Definition 1.2} [2] Let \((L', \delta')\) be an L-fits, \(A \in L\) and \(\alpha \in M(L)\), \(\Omega \subset \delta'\) is called an almost \(\alpha - RF\) of \(A\), if for each \(x_\alpha \in A\), there is \(p \in \Omega\), such that \(p^0 \in \Omega(x_\alpha)\). \(\Omega\) is called an almost \(\alpha^- - RF\) of \(A\), if there is \(\lambda \in \beta'(\alpha)\) such that \(\Omega\) is a \(\lambda - RF\) of \(A\).

\textbf{Definition 1.3} [2] Let \((L', \delta')\) be an L-fits, \(A \in L\) and \(r \in p(L)\), (i) \(\Omega \subset \delta\) is called an r-cover of \(A\) if for each \(x \in X\), there is \(B \in \Omega\) satisfying \(B(x) \leq r\). \(\Omega\) is called an \(r^- - cover\) of \(A\) if there is \(t \in \alpha^*(r)\) such that \(\Omega\) is a \(t - cover\) of \(A\). (ii) \(\Omega \subset \delta\) is called an almost \(r^- - cover\) of \(A\) if for each \(x \in X\), there is \(t \in \alpha^*(r)\) such that \(\Omega\) is an almost \(t - cover\) of \(A\).

\textbf{Definition 1.4} [4] Let \((L', \delta')\) be an L-fits and \(A \in L\), \(A\) is called strong fuzzy compact set, if for each \(\alpha \in M(L)\) and every \(\alpha - RF\) of \(A\), there exist \(\Phi \in FS(\Omega)\) such that \(\Phi\) is an \(\alpha - RF\) of \(A\).

\textbf{Definition 1.5} [4] Let \(\Omega \in L\), \(r \in p(L), \Omega\) is said to have finite \(r\)-intersection property in \(A \in L\) if for each \(\Phi \in FS(\Omega)\) there exist \(x \in A\), such that \((\wedge \Phi)(x) \geq r'.\)

\textbf{Definition 1.6} [5] Let \((L', \delta')\) be an L-fits, \(A \in L\) and \(\alpha \in L\), \(\Omega\) is called the family which has almost \(\alpha\) -intersection property in \(A\), if for each \(\Phi \in FS(\Omega)\) there is \(x \in A\), such that \((\wedge \Phi)(x) \geq \alpha\).

\textbf{Definition 1.7} [3] Let \((L', \delta')\) be an L-fits, \(B \in L\) is called LF regular semi-open if there exist a \(Q \in RO(L')\) satisfying \(Q \subset B \subset \overline{Q}\); \(B\) is called LF regular semi-closed if there exist a \(P \in RC(L')\) satisfying \(P^0 \subset B \subset P\).

\textbf{Definition 1.8} [6] Let \((L', \delta')\) and \((L', \varepsilon')\) be two L-fits and \(f: L' \rightarrow L'\) a fuzzy mapping \(f\) is semi-continuous if \(f^{-1}(B) \in SO(L')\) for each \(B \in \varepsilon\); \(f\) is almost continuous if \(f^{-1}(B) \in \delta\) for each \(B \in RO(L')\); \(f\) is weakly continuous if \(f^{-1}(B) \subset (f^{-1}(B))^0\) for each \(B \in \varepsilon\).

\section{Weakly Compact Set}

\textbf{Definition 2.1} Let \((L', \delta')\) be an L-fits, \(A \in L\), \(A\) is called weakly compact set, if for each countably \(\alpha - RF\) \(\Omega\) of \(A\).

Of \(A\), there exists \(\Phi \in FS(\Omega)\) such that \(\Phi\) is an almost \(\alpha^- - RF\) of \(A\). Particularly, when \(1\) is weakly...
compact set, we call \((L', \delta')\) as a weakly compact space.

**Theorem 2.2** An L-fuzzy set \(A\) in \((L', \delta')\) is a weakly compact set iff for any \(r \in p(L)\) and every countably \(r\)-cover \(\Omega\) of \(A\), there exists \(\Phi \in \text{FS}(\Omega)\) such that \(\Phi\) is an almost \(r^*\)-cover of \(A\).

**Theorem 2.3** Let \((L', \delta')\) be an L-fts, \(A \in L'\). Then \(A\) is a weakly compact set iff for any \(r \in M(L)\) and every countably family \(\Omega \subset \delta'\) which has the \(r\)-intersection property in \(A\), there exists \(A \in \Omega\) such that \(A\) is a finite subfamily \(\{A_i, \ldots, A_n\}\) such that \(A = \{\chi_A^i : i=1, 2, \ldots, n\}\) is a finite almost \(r^*\)-cover of \(1_x\), i.e. for each \(x \in X\) there is \(\chi_A^i \in \Phi\) satisfying \((\chi_A^i)^\gamma(x) \neq 0\). Since \((\chi_A^i)^\gamma = \chi_A^i\) and \(x \in A, \bigcup_{i=1}^n A_i = X\).

This implies that \((X, \tau)\) is weakly compact space. Conversely, for each \(r \in p(L)\), suppose that \(\phi \in \Omega_{\tau}\) is any countably \(r\)-cover of \(1_x\). Then for each \(x \in X\), there exists \(P \in \phi\) such that \(P(x) \notin r\). Hence \(l_r(\phi) = \bigcup \{P \in \phi\} \subset \tau\) is a countably cover of \((X, \tau)\). From the weakly compactness of \((X, \tau)\), there exists \(\Phi \in \text{FS}(\Omega)\) such that \(l_r(\phi)\) is a cover of \((X, \tau)\). Hence for each \(x \in X\), there exists \(P \in \phi\) such that \(x \in l_r(P) \in l_r(\phi)\). Therefore \((L', \Omega_{\tau})\) is the L-fuzzy topological space.

**Theorem 2.6** An L-fuzzy set \(A\) in \((L', \delta)\) is a weakly compact set iff every countably \(\alpha-RORF\) of \(A\) has a finite subfamily \(\Phi\) that is an \(\alpha-RORF\) of \(A\) for each \(\alpha \in M(L)\).

**Theorem 2.7** Let \((L', \delta)\) be an L-fts, \(A \in L'\). A is called RS-compact, if for each \(\alpha \in M(L)\) and each \(\alpha-RSC\)

\(\Omega\) of \(A\), there exists \(\Phi \in \text{FS}(\Omega)\) such that \(\Phi\) is an
RS-compact. 

Theorem 2.8 Let $(X, \tau)$ be a crisp topological space and $(L^x, \delta^x)$ be the corresponding L-fts, then $(X, \tau)$ is RS-compact iff $(L^x, w(\tau))$ is RS-compact.

Theorem 2.9 An L-fuzzy set $A$ in L-fts $(L^x, \delta^x)$ is RS-compact iff for each $r \in p(L)$ and every $r$-regular semi-open cover $\Omega$ of $A$, there exists $\Phi \in FS(\Omega)$ such that $\Phi$ is a $r$-regular semi-open cover of $A$.

Theorem 2.10 Let $(L^x, \delta^x)$ and $(L^y, \varepsilon^y)$ be two L-fts and $f: L^x \rightarrow L^y$ be an almost continuous L-fuzzy mapping. If $A$ is an L-fuzzy weakly compact set in $L^x$, then $f(A)$ is an L-fuzzy weakly compact set in $L^y$.

Theorem 2.11 Let $(L^x, \delta^x)$ and $(L^y, \varepsilon^y)$ be two L-fts and $f: L^x \rightarrow L^y$ be an weakly continuous fuzzy mapping. If $A$ is an L-fuzzy weakly compact set in $L^x$, then $f(A)$ is an L-fuzzy weakly compact set in $L^y$.

Theorem 2.12 Let $(L^x, \delta^x)$ and $(L^y, \varepsilon^y)$ be two L-fts and $f: L^x \rightarrow L^y$ be an semi-continuous fuzzy mapping. If $A$ is an L-fuzzy weakly compact set in $L^x$, then $f(A)$ is an L-fuzzy weakly compact set in $L^y$.

References