The Wiener Index of Some Particular Graph

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Summary
The wiener index W (G) of a connected graph G is the sum of the distances between all pairs of vertices of G. In this paper, we give theoretical results for calculating the wiener index for some composed graphs (star-graphs, path-graphs, fan-graph, etc.).

Key words:
Graph, distance, wiener index.

1. Introduction

A graph G is a pair G = (V,E) consisting of a finite set V and a set E of two-element subsets of V. The elements of v are called vertices. An element e = {a, b} of E is called an edge with end vertices a and b. In a graph G, a path is a sequence of vertices and edges $P = v_0, e_1, v_1, e_2, \dots, v_{n-1}, e_n, v_n$ sach that $e_i = \{v_{i-1}, v_i\}$. A graph G is called connected if any two of its vertices may be connected by a path [7]. A graph can have multiple number of edges between two vertices. A graph is simple if two vertices there is at most one edge. In this paper , we are interested in simple graphs and connected. The distance d(u, v) between the vertices *u* and *v* of the graph *G* is equal to the length of the shortest path that connects *u* and *v* [1, 7]. The Wiener index W(G) of a connected graph G is the sum of all the distances between pairs of vertices of G.

$$W(G) = \sum_{u \in V(G)} \sum_{v \in V(G)} d(u, v)$$

the wiener index of a vertex v in G is defined as:

$$w(v, G) = \sum_{u \in V(G)} d(u, v)$$

This index was introduced by the chemist Wiener [2] in the study of relations between the structure of organic compounds and their properties. It has since been studied extensively by both chemists and mathematicians, especially for graphs; see the survey [3, 7, 8] for many results and references. The Wiener index is, apart from a constant factor, the geometric mean of the extremely values, which are given for the star E_m and the path P_m respectively

$$(m-1)^2 = W(E_m) \leq W(T_m) \leq W(P_m) = \binom{m+1}{3}$$

It has same result for a planar graph C_m

$$W(E_m) \leq W(C_m) \leq W(P_m)$$

Where E_m is the maximal planar simple graph [2].

2. The main result

In this section, we will give the formulas for the wiener index of composit graph. The interested readers for more information on topological indices of graph operations can be referred to the papers [3 ,4, 5, 6] and their references.

Let $G_1 \cdot G_2$ be the graph composed by graph G_1, G_2 that possess respectively m_1 , m_2 vertices connected by a vertex s (see Fig. 1). We denote by G_i the graph G_{m_i} (with m_i vertex)

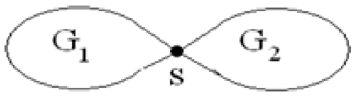

Fig. 1 The graph $G_1 \cdot G_2$

$V(G_1 \cdot G_2) = V(G_1) \cup V(G_2)$.

$|V(G_1 \cdot G_2)| = |V(G_1)| + |V(G_2)| - 1$.

If $u \in G_1,\ v \in G_2$

Then: d (u, v) = d (u, s) + d (s, v)

Lemma 1:

The wiener index of $G_1 \cdot G_2$ *is:*

$$W(G_1.G_2) = W(G_1) + W(G_2) + (m_1 - 1)w(s, G_2) + (m_2 - 1)w(s, G_1)$$

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We generalize the previous result

Let G_N be the graph composed by n graphs $G_1 \cdot G_2 \cdot \cdots \cdot G_n$ that possess respectively $m_1, m_2, \ldots, m_n$ vertices connected by a vertex s (see Fig . 2).

$$N = \sum_{i=1}^{n} m_i - n + 1$$

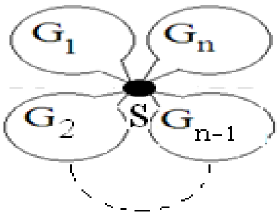

Fig. 2: Star- graphs

Lemma 2:

The wiener index of star - graph $G_N = G_1 \cdot G_2 \cdot \cdots \cdot G_n$ *is:*

$$W(G_N) = \sum_{i=1}^{n} w(G_i) +$$
$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} [(m_j - 1)w(s, G_i) + (m_i - 1)w(s, G_j)]$$

Proof: we denote by: $V^*(G_i) = V(G_i) \backslash \{s\}$

$$W(G_N) = \sum_{u \in V(G_N)} \sum_{v \in V(G_N)} d(u, v)$$

$$= \sum_{u \in V^*(G_{n-1})} \sum_{v \in V^*(G_n)} d(u, v) +$$

$$\sum_{u \in V^*(G_{n-2})} \sum_{v \in V^*(G_{n-1})} d(u, v) + \sum_{u \in V^*(G_{n-2})} \sum_{v \in V^*(G_n)} d(u, v) +$$

$$\sum_{u \in V^*(G_i)} \sum_{v \in V^*(G_{i+1})} d(u, v) + \sum_{u \in V^*(G_i)} \sum_{v \in V^*(G_n)} d(u, v) + \cdots +$$

$$\sum_{u \in V^*(G_1)} \sum_{v \in V^*(G_2)} d(u, v) + \cdots + \sum_{u \in V^*(G_1)} \sum_{v \in V^*(G_n)} d(u, v).$$

Then

$$W(G_N) = \sum_{i+1}^{n} w(G_i) +$$

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\sum_{u \in V^*(G_i)} \sum_{v \in V^*(G_j)} d(u, v) \right)$$

Or $d(u, v) = d(u, s) + d(s, v)$ then

$$\sum_{u \in V^*(G_i)} \sum_{v \in V^*(G_j)} d(u, v) =$$

$$(m_j - 1) \sum_{u \in V^*(G_i)} d(u, s) + (m_i - 1) \sum_{v \in V^*(G_j)} d(s, v)$$

$$= (m_j - 1)w(s, G_i) + (m_i - 1)w(s, G_j).$$

The hence result. ▢

Particular case:

If the graphs G_i have the same number of vertices m $(m_i = m$ *for* $i = 1, \ldots, n)$, we have:

$$\begin{cases} w(s, G_i) = w(s, G_j) \\ G_i = G_j = G_m \\ N = nm - n + 1 \end{cases} \quad \text{for } i, j \in \{1, 2, \ldots, n\}$$

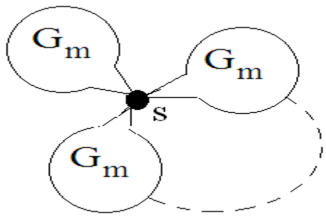

Fig. 3: Star graphs G_N: G_m . G_m . … . G_m

Lemma 3:

The wiener index of star graph $G_N = G_m \cdot G_m \cdot \cdots \cdot G_m$ *is:*

$$W(G_N) = n\, w(G_m) + n(n-1)(m-1)w(s, G_m)$$

Proof: We use lemma 2.

Let G_N be the graph composed by graph G_1 , G_2 , G_3 that possess respectively m_1, m_2, m_3 vertices connected by two vertices s_1, s_2 (see Fig . 4) . We denote by G_1 - G_2 - G_3 the graph G_N $(with\ N = m_1 + m_2 + m_3 - 2)$.

Fig .4 the graph G_1 - G_2 - G_3

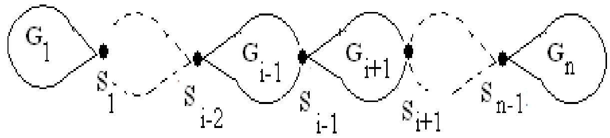

Fig. 5 The path-graphs

Lemma 4:

The wiener index of $G_N = G_1$ - G_2 - G_3 *is:*

$$W(G_N) = W(G_1) + W(G_2) + W(G_3) +$$
$$(m_1 - 1)[\,W(s_1, G_2) + W(s_2, G_3)\,] +$$
$$(m_2 - 1)[\,W(s_1, G_1) + W(s_2, G_3)\,] +$$
$$(m_3 - 1)[\,W(s_1, G_1) + W(s_2, G_2)\,] +$$
$$(m_1 - 1)(m_3 - 1)\, d(s_1, s_2).$$

Proof :

$$W(G_1 - G_2 - G_3) = \sum_{u\in(G_N)} \sum_{v\in(G_N)} d(u,v)$$

$$= W(G_1) + W(G_2) + W(G_3) +$$

$$\sum_{\substack{u\in V(G_1)\\ v\in V(G_2)}} (d(u,s_2) + d(s_1,v)) +$$

$$\sum_{\substack{u\in V(G_1)\\ v\in V(G_3)}} \big(d(u,s_1) + d(s_1,s_2) + d(s_2,v)\big) +$$

$$\sum_{\substack{u\in V(G_2)\\ v\in V(G_3)}} (d(u,s_2) + d(s_2,v))$$

The hence result. ▢

We generalize the lemma 4:

Let G_N be tree formed by the trees $G_1, G_2, \ldots, G_n$ that possess respectively $m_1, m_2, \ldots, m_n$ vertices, those graphs are connected by the vertices $s_1, s_2, \cdots, s_{n-1}$(see Fig .5). We denote the graph $G_1 - G_2 - \cdots - G_n$ by G_N *where* $N = \sum_{i=1}^{n} m_i - (n-1)$

Lemma 5:

The wiener index of path graph $G_N = G_1 - G_2 - \cdots - G_n$ is:

$$W(G_N) = \sum_{i=1}^{n} w(G_i) +$$
$$\sum_{i=1}^{n-1} [(m_{i+1} - 1)w(s_i, G_i) + (m_i - 1)w(s_i, G_{i+1})] +$$
$$\sum_{i=1}^{n-2} \sum_{j=i+2}^{n} [\,(m_j - 1)w(s_i, G_i) +$$
$$(m_i - 1)w(s_{j-1}, G_j) + (m_i - 1)(m_j - 1)d(s_i, s_{j-1})\,]$$

Proof:

$V(G_i)$ is the set of the vertices of G_i and

$V^*(G_i) = V(G_i)\backslash\{s_i\}$

$$W(G_N) = \sum_{i=1}^{n} w(G_i) + \sum_{u\in V^*(G_{n-1})} \sum_{v\in V^*(G_n)} d(u,v) +$$

$$\sum_{u\in V^*(G_{n-2})} \sum_{v\in V^*(G_{n-1})} d(u,v) + \sum_{u\in V^*(G_{n-2})} \sum_{v\in V^*(G_n)} d(u,v) + \cdots +$$

$$\sum_{u\in V^*(G_i)} \sum_{u\in V^*(G_{i+1})} d(u,v) + \cdots + \sum_{u\in V^*(G_i)} \sum_{v\in V^*(G_n)} d(u,v)$$
$$+ \cdots +$$

$$\sum_{u\in V^*(G_1)} \sum_{u\in V^*(G_2)} d(u,v) + \cdots + \sum_{u\in V^*(G_1)} \sum_{v\in V^*(G_n)} d(u,v)$$

$$\sum_{u\in V^*(G_1)} \sum_{u\in V^*(G_2)} d(u,v) + \sum_{u\in V^*(G_1)} \sum_{v\in V^*(G_3)} d(u,v) + \cdots +$$

$$\sum_{u\in V^*(G_1)} \sum_{u\in V^*(G_n)} d(u,v)$$

We have for $u \in V^*(G_i)$ and $v \in V^*(G_j)$:

$$d(u,v) = d(u,s_i) + d(s_i,v)$$

$$\sum_{u\in V^*(G_i)} \sum_{v\in V^*(G_{i+1})} d(u,v) =$$

$$(m_{i+1}-1)w(s_i,G_i) + (m_i-1)w(s_i,G_{i+1})$$

And we have

for $u \in V^*(G_i)$ and $v \in V^*(G_j)$, $j \geq i+2$:

$$d(u,v) = d(u,s_i) + d(s_i,s_{j-1}) + d(s_{j-1},v)$$

$$\sum_{u\in V(G_i)^*} \sum_{v\in V(G_j)^*} d(u,v) =$$

$$(m_j-1)w(s_i,G_i) + (m_i-1)w(s_{j-1},G_j) +$$

$$(m_i-1)(m_j-1)d(s_i,s_{j-1})$$

The hence the result. ▢

Particular case:
If $m_i = m$ for i = 1, … , n then

$$(1)\begin{cases} w(s_i,G_m) = w(s_1,G_m) & \text{for } i = 1,\dots,n-1 \\ d(s_1,s_2) = d(s_i,s_{i+1}) & \text{for } i = 1,\dots,n-2 \\ N = nm - n + 1 \end{cases}$$

G_N is the graph $G_m - G_m - \cdots - G_m$ (see Fig. 6)

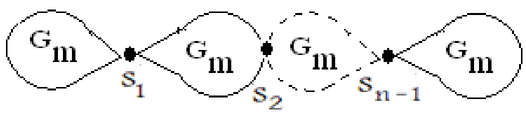

Fig. 6: The path graph: $G_m - G_m - \ \dots\ G_m$

Lemma 6:

The wiener index of path graph $G_N = G_m - G_m - \ \dots\ G_m$ is:

$$W(G_N) = n\,w(G_m) + n(m-1)(n-1)w(s_1,G_m)$$

$$+\frac{n(n-2)(n-1)(m-1)^2}{6}\,d(s_1,s_2)$$

Proof: We use (1) in lemma 5:

$$W(G_N) = n\,w(G_m) + n(m-1)(n-1)w(s_1,G_m)$$

$$+2(m-1)(s_1,G_m)\sum_{i=1}^{n-2}\sum_{j=i+1}^{n} 1 +$$

$$(m-1)^2\sum_{i=1}^{n-2}\sum_{j=i+2}^{n} d(s_i,s_{j-1})$$

where

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} 1 = \sum_{i=1}^{n-2}((n-1)-i) = \frac{(n-1)(n-2)}{2}$$

We poses
$d = d(s_i,s_{i+1})$,

We have $d(s_i,s_{i+2}) = 2d$, …

$d(s_i,s_{n-1}) = (n-1-i)d$, then:

$$\sum_{j=i+1}^{n} d(s_i,s_{j-1}) = d(1+2+\cdots+(n-1-i))$$

$$= \frac{d}{2}[n^2+n+i(1-2n)+i^2]$$

$$\sum_{i=1}^{n-2}\sum_{j=i+2}^{n} d(s_i,s_{j-1}) = \frac{d}{2}(n^2+n)(n-2) +$$

$$\frac{d}{2}(1-2n)\sum_{i=i}^{n-2} i + \frac{d}{2}\sum_{i=1}^{n-2} i^2 = \frac{n(n-1)(n-2)d}{6}.$$

Then

$$W(G_N) = n\,w(G_m) + n(m-1)(n-1)w(s_1,G_m)$$

$$+\frac{n(n-2)(n-1)(m-1)^2}{6}\,d(s_1,s_2).$$

3. Application

3.1 Case of a cycle C_m

Let G_N the graph formed by cycle C_m where m is the number of vertices (see Fig . 7)

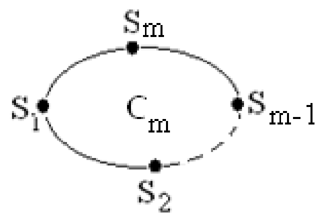

Fig. 7 the cycle C_m

The wiener index of C_m is:

$$(2)\ \ W(s_i, C_m) = \begin{cases} \frac{m^2}{4} & if\ \ m\ \ is\ odd, \quad m \geq 2 \\ \frac{m^2-1}{4} & if\ \ m\ \ is\ even, \quad m \geq 3 \end{cases}$$

$$(3)\ \ W(C_m) = \begin{cases} \frac{m^3}{8} & if\ \ m\ \ is\ odd, \quad m \geq 2 \\ \frac{m(m^2-1)}{8} & if\ \ m\ \ is\ even, \quad m \geq 3 \end{cases}$$

Let G_N where by the cycle whose branches are chains $G_N = C_m - C_m - \ldots - C_m$ (n times) (see Fig. 8).

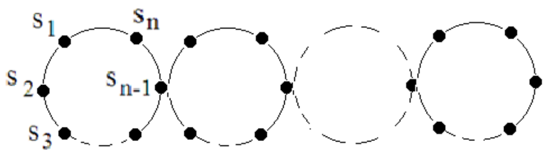

Fig. 8 the cycle path graph $C_m - C_m - \cdots - C_m$

Theorem 1:

The wiener index of cycle path graph $G_N = C_m - C_m - \cdots - C_m$ is:

$$W(G_N) = \frac{nm}{24}(3m^2 + 6m(n-1)(m-1) + 2(n-1)(n-2)(m-1)^2),\ n\ is\ odd,\quad n \geq 2, m \geq 2$$

$$W(G_N) = \frac{nm(m^2-1)}{8} + \frac{n(n-1)(m-1)(m^2-1)}{4} + \frac{nm(n-1)(n-2)(m-1)^2}{12},\quad n\ is\ even, n \geq 2, m \geq 3$$

Proof : We use (2) and (3) in lemma 6:

Let G_N be the graphs composed by a cycle C_m as a star. G_N *is the graph* $C_m . C_m . \ldots . C_m$.(n times) (See Fig. 9),

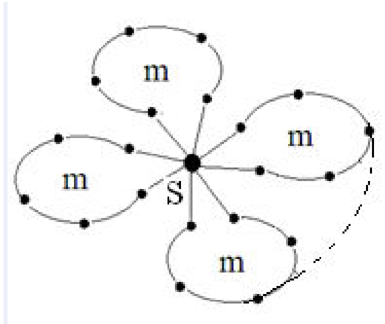

Fig. 9: the cycle graph stars $C_m . C_m . \ldots . C_m$

Theorem 2:

The wiener index of cycle graph stars $G_N = C_m . C_m . \ldots . C_m$ is:

$$W(G_m) = \frac{nm^2}{4}\left(\frac{m}{2} + (m-1)(n-1)\right), \quad n\ \text{is odd},\quad n \geq 2, \quad m \geq 2$$

$$W(G_m) = \frac{nm(m^2-1) + 2(n-1)(m-1)(m^2-1)}{8}, \quad n\ \text{is even}, \quad n \geq 2, \quad m \geq 3$$

Proof: We use lemma 3.

3.2 Case of a fan F_m

Let F_m be the fan (see Fig. 11)

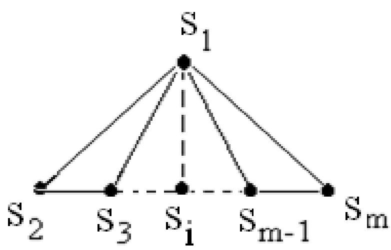

Fig. 11 the fan F_m

$$W(s_i, F_m) = \begin{cases} m-1 & \text{for } i = 1 \\ 2m-4 & \text{for } i = 1 \text{ and } m \\ 2m-3 & \text{for } i = 3, \ldots, m-1 \end{cases}$$

The wiener index of F_m[9] is:

$$W(F_m) = m^2 - 3m + 3 \qquad , m \geq 3$$

Let G_N be the fan formed by n stars F_m connected by a vertex s. G_N is $F_m . F_m . \ldots . F_m$ (see Fig. 12) .

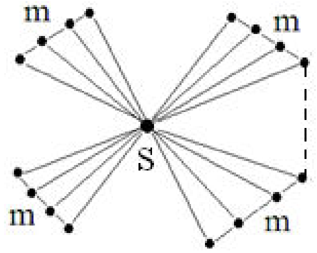

Fig. 12 Fan-graph $F_m . F_m . \dots . F_m$

Theorem 3:

The wiener index of Fan-graph $G_N = F_m . F_m . \dots . F_m$ is:

$$W(F_m) = n(m^2 - 3m + 3) + n(n-1)^2\,(m-1)$$

$$m \geq 3, \quad n \geq 1$$

Proof: We use lemma 3.

Let G_N be star-fan composed by m-2 fan $F_3 . F_4 . F_5 . \dots . F_m$ and connected by a vertex s
We denote by G_n the graph $F_3 . F_4 . F_5 . \dots . F_m$
(see Fig.13).

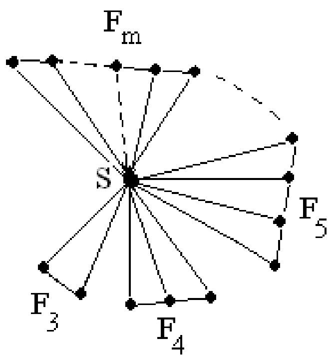

Fig.13 the graph $F_3 . F_4 . F_5 . \dots . F_m$

Theorem 4:

The wiener index of G_N is:

$$W(F_m) = \frac{m^4 + 6m^3 + 6m^2 + m}{4}, \quad m \geq 3$$

.

Proof: We use lemma 2.

Conclusion

In this article we give theoretical results for calculating the wiener index for some composed graphs (star-graphs, path-graphs, fan-graph, etc.).

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