Algebraic Structures on Dominating Sets of Graphs

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Summary
In this paper we observed some algebraic structures on the collection of dominating sets and independent sets of an undirected non weighted graph. We prove that the collection of all dominating sets and in-dependent sets forms a group under some specific binary operators. Finally we conclude with some fundamental algebraic properties over these sets.
Key words:
Dominating set, Independent set, Groups, Homomorphism.

1. Introduction

Let G = (V, E) be an undirected non weighted graph where V (G) is the set of vertices and E(G) ⊆ {{u, v}|u, v ∈ V (G), u = v}. Order of G and size of G are |V (G)| and |E(G)| respectively. A set of vertices D is a dominating set for a graph G if every vertex is either in D or adjacent to a vertex which is in D. A set of vertices I is an independent set for a graph if no two vertices in I are adjacent to each other in G. In this section we formally define some of the basic algebraic structures and its properties.

Definition 1.1. A nonempty set of elements G is said to form a group if in G there is defined a binary operation, called the product and denoted by '· ' such that

(1) a, b ∈ G implies that a · b ∈ G(closed).

(2) a, b, c ∈ G implies that a · (b · c) = (a · b) · c (associative law).

(3) There exists an element e ∈ G such that a·e = e·a = a for all a ∈ G (the existence of identity element in G).

(4) For every a ∈ G there exists an element a^{-1} ∈ G such that $a \cdot a^{-1} = a^{-1} \cdot a = e$ (the existence of inverse in G)

Definition 1.2. A nonempty set of elements G is said to form a semigroup if in G there is defined a binary operation, called the product and denoted by '· ' such that

(1) a, b, c ∈ G implies that a · (b · c) = (a · b) · c (associative law).

Definition 1.3. A group G is called a cyclic group if there is an element a∈G, such that every element in G can be expressed as a power of a. In that case a is called the generator of G. We express this by writing G = (a).

Definition 1.4. A nonempty subset H of a group G is said to be a subgroup of G if, H itself forms a group under the same binary operation of G.

Fact 1.5. If G is a group, then (i) The identity element of G is unique. (ii) For every a ∈ G has a unique inverse in G.

Fact 1.6. A group G of prime order must be cyclic and every element of G other than identity can be taken as a generator.

Here all the operations are assumed to carry the same meaning as the set theoretic operations.

2. Structure of Dominating Sets

Let G be a graph. Let D and I be the set of all dominating sets and independent sets of the graph G respectively.

Lemma 2.1. D is a semi group under the binary operation set union ∪.

Proof. Let '· ' be the binary operation over D. If A, B be any two dominating sets in D, then A · B is defined as A · B=A∪B. Clearly D is nonempty, as V (G) in D. Let A, B, C be any three dominating sets ∈ D. Now consider (A · B) · C = (A ∪ B) · C = (A ∪ B) ∪ C = A ∪ B ∪ C = A ∪ (B ∪ C) = A∪(B·C)=A·(B·C). Associatively holds good for union over D. Hence D is a semi group.

Remark 2.2. D can not be a semigroup under the binary operations set difference − , set intersection ∩, symmetric difference △.

Lemma 2.3. I is a semigroup under the binary operation set intersection ∩.

Proof. Let '· ' be the binary operation over I. If A, B be any two independent sets in I, then A · B is defined as A · B = A ∩ B. Clearly I is nonempty, as every single vertex alone is in I. Let A, B, C be any three dominating sets in I. Now consider (A·B)·C = (A∩B)·C = (A∩B)∩C = A∩B ∩ C = A ∩ (B ∩ C)=A ∩ (B·C)=A·(B·C). Associatively holds good for intersection over I. Hence I is a semi group.

Remark 2.4. I can not be a semi group under the binary operations set union ∪ and set symmetric difference △.

Corollary 2.5. ID does not follow any algebraic structure under the binary operations set intersection ∩, set union ∪, set difference − and set symmetric difference △.

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3. The numbering scheme

Let us arrange the elements of D and I in the following manner. Let the first element of D and I be the sets V (G) and ∅ respectively. Let us arrange the rest elements of D and I in a lexicographic ordering. Let D' and I' be the sets after this rearrangement of elements. Let $|D'| = m$ and $|I'| = n$. Let D^i denotes the index of i in the set D' . We assume the index of first element in the set D' is 0, second element in D' is 1 and so on. Let $D'[i]$ denotes the ith element of D'.

Theorem 3.1. D' is a group under the operation '· ' and identity V (G), where '· ' is defined as follows. If A, B ∈ D' then $A \cdot B = D'_A *_m D'_B$. If $D'_A = a < m$ and $D'_B = a < m$ and $D'_C = c < m$, then $A \cdot B = D'_A *_m D'_B$, where $a + b \equiv c (\mod m)$.

Proof. Let A, B, C ∈ D' . Now consider $A \cdot B = D'_A *_m D'_B$. If $D'_A = a < m$ and $D'_B = a < m$ and $D'_C = c < m$, then $A \cdot B = D'_A *_m D'_B$, where $a + b \equiv c (\mod m)$. Since $D'_C \in D'$, D' is closed under $*_m$.
Consider $A \cdot B \cdot C = (A \cdot B) \cdot C = (D'_A *_m D'_B) \cdot C = D'_P \cdot C = D'_P *_m D'_C = D'_Q \in D'$, where and $a + b \equiv p (\mod m)$ and $p+c=q (\mod m)$, since we know that $*_m$ is associative. Hence D' is associative.
Let $A \in D'$. Consider $V (G) \cdot A = D'_{V(G)} *_m D'_A = D'_A$. Since $D'_{V(G)} = 0$ and let $D'_A = a$ then $0 + a \equiv a (\mod m)$.
A·V(G)= A .So existence of identity holds. Consider $A \cdot D'_{m-D'_A} = D'[0] = V(G)$. This gives the existence of inverse. Hence the theorem holds. □

Theorem 3.2. I' is a group under the operation '·' and identity∅ , where '· ' is defined as follows. If A, B ∈ I' then $A \cdot B = I'_A *_n I'_B$.

.***Proof.*** The proof is similar to the proof of 3.1. □

Remark 3.3. The collection of all independent dominating sets does not follow any algebraic structure under the operation $*_m$ (m is the cardinality of the set), where as D and I forms a cyclic group under the same operation.

Lemma 3.4. If O(D) = p, where p is a prime number then every element of D excluding the identity element is a generator for D.

Proof. We know that D is a finite abelian group. Given that O(D) = p and p is a prime number. So number of elements of order p = $\phi(p) = p - 1$, where $\phi(p)$ denote the Euler's phi function. Hence it has p -1 generators. Hence the claim.

□

Remark 3.5. If O(D) = k, then we can write k as a product of prime factors (say) $p_1, p_2, \dots p_n$. In fact we can onstruct normal subgroups $D_{p_1}, D_{p_2}, \dots D_{p_n}$ such that each element in $D_{p_1}, D_{p_2}, \dots D_{p_n}$ will act as a generator in their irrespective groups.

4. Homomprphism of dominating sets

Definition 4.1 Let D_1 and D_2 be two groups formed by the dominating sets of a graph G(V, E). A homomorphism ϕ from D_1 to D_2 is a mapping from D_1 to D_2 that preserves the group operation i.e. $(A \cdot B) = \phi(A) \diamond \phi(B)$, where '$\diamond$ ' is the binary group operation on D_2.

Theorem 4.2. Let ϕ be a homomorphism from a group D_1 to a group D_2 and let H be a subgroup of D_1. Then
(1) $\phi(H) = \{\phi(A) | A \in H\}$ is a subgroup of D_2.
(2) If H is cyclic $\phi(H)$ is cyclic.
(3) If H is abelian $\phi(H)$ is abelian.
(4) If H is normal in D_1 then $\phi(H)$ is normal in $\phi(D_1)$.

Theorem 4.3. Let ϕ be a homomorphism from a group D_1 to a group D_2 and let A be an element of D_1 . Then
(1) ϕ carries the identity element D_1 to the identity of D_2.
(2) $\phi(A^n) = (\phi(A))^n$

Remark 4.4. If there exists a homomorphism between two groups of dominating sets then we can say both the dominating sets dominate the same graph.

5. Conclusion

In this paper we observe the collection of all dominating sets and independent sets form a group under the index integer modulo operation. In fact they form a cyclic group. Moreover given any two groups of dominating sets we can determine whether they dominate the same graph or not.

References

[1] J. A. Gallian, Contemporary Abstract Algebra Narosa Publication, 2003.
[2] M. Artin, Algebra, Prentice-Hall, 1991.
[3] I.N. Herstein, Abstract Algebra, Prentice-Hall India, 1996.