Numerical Methods for Fractional Differential Equations

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Summary
Fractional differential equations are a field of mathematics study that grow out of the traditional definitions of calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. [1],[2],[5], this paper concern with the problem that tries to approximate ate the solution of fractional differential equations through and how.

Key words:
numerical methods, fractional differential equations.

1. Introduction

In recent years, the cubic spline interpolation method, as applied to the solution of differential equations employ some from approximating function such as polynomial to approximate the solution by evaluating the function for sufficient numbers of points in the domain of the solution [6], so to provide it determination of unknown coefficients that define the approximating function [3].
It has been found that using spline curves, or piece-wise polynomials, is more effective in representing the solution to the differential equations [4], [8].

2. Important of this research

Fractional differential equations (FDEs) represent an important tool in technology, science and economics and engineering applications included population models , control engineering electrical network analysis , gravity , medicine , etc, [9],[14].
Fractional differential equations consist of a Fractional differential with specified value of the unknown function at more than one given point in the domain of the solution.
Recently numerical methods have been used approximate at the solution of the (FDEs), which open the doors wide for future applications of these methods to tough real life problems involving the numerical solution.
The most common methods are cubic spline interpolation, finite difference method, [10],[11].
This research will add new numerical method (Legendre – spline interpolation method) to approximate the solution of Fractional differential equations.

3. Objectives of this research

This research aims to the following:
a) Discuss and compare the cubic spline interpolation with Legendre- spline interpolation method.
b) Propose new method s to approximate Fractional differential equations solution.
c) Use the new method to approximate the solution of partial Fractional differential equations.
d) Compare the gained results in terms of accuracy between the cubic spline with the lengendre – spline method.
e) Discuss the stability and convergent for the Legendre – spline interpolation method.
f) Discuss the perturbation of the solution of Fractional differential equations

4. Research phases :

Phase 1: background enhancement:

A comprehensive of text book that covers Fractional differential equations, their existing numerical solvers will be my first step to precede advanced researches.

Phase 2: research survey:

I will use the literature survey of research papers that are related to my work, by the end of this phase, I will suppose:

- Write a progress report that cover the existing solvers in my field of work, their advantages limitations and a collection of test cases and examples that covers the different categories of problems in my area of research.
- Write the related existing algorithm Fractional differential equations using my own programming language.

Phase 3: focusing numerical methods:

I will consider two parts on differential equation:

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- Initial value problems for ordinary differential equations.
- Numerical methods for Fractional differential equations.

Phase 4: research work:

After the completion of the first three phases, I will receive the guidance from my supervisor about what problem in certain categories to tackle and consequently:

- Start solving the problem using other existing numerical solvers.
- Solve the problems using the computer algorithm.
- A comparison will be made with the numerical result presented elsewhere.

Description of the Method :
First; the aim of cubic spline interpolation is to get an interpolation formula that is continuous in both the first and second, derivatives, by spanning $\{1, x, x^2, x^3\}$, [12],[15].
with a suitable coefficients both within the some intervals and at the interpolating nodes,[15].
On the other hand, the new method (the legendre-spline interpolation) is generatored by spanning the set

$$\left\{1, x, \frac{1}{2}(3x^2-1), \frac{1}{2}(5x^3-3x)\right\}, \text{with a suitable coefficients, i.e}$$

$$L(x)=\begin{cases} a_{11}+b_{12}x+\frac{1}{2}c_{13}(3x^2-1)+\frac{1}{2}d_{14}(5x^3-3x), & x\in[a=t_1, \ t_2] \\ a_{i1}+b_{i2}x+\frac{1}{2}c_{i3}(3x^2-1)+\frac{1}{2}d_{i4}(5x^3-3x), & x\in[t_i, \ t_{i+1}] \\ a_{n1}+b_{n2}x+\frac{1}{2}c_{n3}(3x^2-1)+\frac{1}{2}d_{n4}(5x^3-3x), & x\in[t_{n-1}, \ t_n=b] \end{cases}$$

where the $[t_{i-1}, t_i]$ *is a regular partition on* $[a, b]$, *i.e*
$a=t_1 \prec t_2 \prec \ldots \prec t_i \prec t_{i+1} \prec \ldots \prec t_n = b$,
Here, $L(x)$ *satisfy the conditions,*

$$\begin{aligned} &1-L_i(t_i)=L_{i+1}(t_i), && i=1,\ldots\ldots,n \\ &2-L_i^{I}(t_i)=L_{i+1}^{I}(t_i), && i=1,\ldots\ldots,n \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1) \\ &3-L_i^{II}(t_i)=L_{i+1}^{II}(t_i), && i=2,\ldots\ldots,n-1 \\ &4-L_1^{II}(t_1)=0, \ L_n^{II}(t_n)=0 \end{aligned}$$

now, we apply the four conditions on the function $L(x)$, *to get a system of linear equations,*
$AX=B$, *where,*

$$A=\begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} & 0 & 0 & 0 & 0 & .. & .. & .. & 0 \\ l_{21} & l_{22} & l_{23} & l_{24} & 0 & 0 & 0 & 0 & .. & .. & .. & . \\ 0 & 0 & 0 & 0 & l_{35} & l_{36} & l_{37} & l_{38} & 0 & 0 & .. & . \\ 0 & 0 & 0 & 0 & l_{45} & l_{46} & l_{47} & l_{48} & 0 & 0 & .. & . \\ .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & . \\ .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & . \\ .. & .. & .. & .. & .. & l_{ij} & l_{ij+1} & l_{ij+2} & l_{ij+3} & . & . & . \\ . & . & . & . & . & l_{i+1j} & l_{i+1j+1} & l_{i+1j+2} & l_{i+1j+3} & .. & .. & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & l_{n-1n-3} & l_{n-1n-2} & l_{n-1n-1} & l_{n-1n} \\ 0 & 0 & 0 & 0 & .. & . & . & . & l_{nn-3} & l_{nn-2} & l_{nn-1} & l_{nn} \end{bmatrix}$$

$$X=\begin{bmatrix} a_{11} \\ b_{12} \\ c_{13} \\ d_{14} \\ . \\ a_{ij} \\ b_{ij+1} \\ c_{ij+2} \\ d_{ij+3} \\ . \\ a_{nn-3} \\ b_{nn-2} \\ c_{nn-1} \\ d_{nn} \end{bmatrix} \qquad B=\begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ . \\ y_{ij} \\ y_{i\,j+1} \\ y_{ij+2} \\ y_{ij+3} \\ . \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where, $y_{ij}=f(t_{ik})$, $i=1,2,\ldots,i,\ldots,k,\ldots n$,
hence, the existence of unique solution depends on l_{ij},
so, to solve this system of linear equation, we use a matlap programme,

Example:
The following table of values for function $f(x)=y$ *is given*

x_i	2	2.2	2.4	2.6
$f(x_i)$	−1.664587	−2.84835	−4.24739	−5.79257

Now, we construct the
1 − natural cubic spline,
2 − legendre − spline interpolation,

here, it is noted that $y=x^2\cos(x)$,
Firstly: natural cubic spline:

$$A_s=\begin{bmatrix} 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2.2 & 4.84 & 10.64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2.2 & 4.84 & 10.64 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2.4 & 5.76 & 13.8240 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2.4 & 5.74 & 13.824 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2.6 & 6.76 & 17.576 \\ 0 & 1 & 4.4 & 14.52 & 0 & -1 & -4.4 & -14.52 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2.8 & 17.28 & 0 & -1 & -2.8 & 17.28 \\ 0 & 0 & 2 & 13.2 & 0 & 0 & -2 & -13.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 14.4 & 0 & 0 & -2 & -14.4 \\ 0 & 0 & 2 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 15.6 \end{bmatrix}$$

$$X_s = \begin{bmatrix} a_{11} \\ b_{12} \\ c_{13} \\ d_{14} \\ a_{21} \\ b_{22} \\ c_{23} \\ d_{24} \\ a_{31} \\ b_{32} \\ c_{33} \\ d_{34} \end{bmatrix} \quad ; \quad B_s = \begin{bmatrix} -1.664587 \\ -2.84835 \\ -4.24739 \\ -5.79257 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

to determine *the coefficients* a_{ij}
we use matlap programme to find the inverse of A_s

so ,

$$A_s^{-1} = \begin{bmatrix}
-187.622 & 188.622 & 199.196 & -199.196 & -0.574 & 0.574 & -39.724 & 0.115 & -2.694 & 1.382 & 8.867 & 1.255 \\
292.933 & -292.933 & -298.794 & 298.794 & 0.861 & -0.861 & 59.587 & -0.172 & 4.041 & -2.072 & -12.100 & -1.883 \\
-148.967 & 148.967 & 149.397 & -149.397 & -0.431 & 0.431 & -29.793 & 0.086 & -2.021 & 1.036 & 5.500 & 0.941 \\
24.828 & -24.828 & -24.900 & 24.900 & 0.072 & -0.072 & 4.966 & -0.014 & 0.337 & -0.173 & -0.833 & -0.157 \\
204.048 & -204.048 & -194.528 & 195.528 & 2.480 & -2.480 & 40.810 & -0.496 & -4.081 & -5.968 & 0.000 & -5.423 \\
-241.564 & 241.564 & 239.871 & -239.871 & -3.307 & 3.307 & -48.313 & 0.661 & 4.831 & 7.958 & 0.000 & 7.230 \\
93.987 & -93.987 & -95.451 & 95.451 & 1.464 & -1.464 & 18.797 & -0.293 & -1.880 & -3.523 & 0.000 & -3.201 \\
-11.983 & 11.983 & 12.199 & -12.199 & -0.215 & 0.215 & -2.397 & 0.043 & 0.240 & 0.518 & 0.000 & 0.471 \\
183.007 & -183.007 & -180.963 & 180.963 & 10.955 & -9.955 & 36.601 & 0.409 & -3.660 & -6.952 & 0.000 & -4.282 \\
-244.044 & 244.044 & 241.317 & -241.317 & -2.273 & 2.273 & -48.809 & -0.545 & 4.881 & 9.271 & 0.000 & 7.371 \\
100.187 & -100.187 & -99.067 & 99.067 & -1.119 & 1.119 & 20.037 & 0.224 & -2.004 & -3.806 & 0.000 & -3.552 \\
-12.844 & 12.844 & 12.701 & -12.701 & 0.144 & -0.144 & -2.569 & -0.029 & 0.257 & 0.488 & 0.000 & 0.520
\end{bmatrix}$$

$$B_s = \begin{bmatrix} -1.664587 \\ -2.84835 \\ -4.24739 \\ -5.79257 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad , \qquad \textit{now, compute } A_s^{-1}B$$

$$B_s = \begin{bmatrix} -1.664587 \\ -2.84835 \\ -4.24739 \\ -5.79257 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad , \qquad \textit{now, compute } A_s^{-1}B$$

$$X_s = A^{-1}B = \begin{bmatrix} 52.8480 \\ -69.9312 \\ 32.0062 \\ -5.3344 \\ -31.0234 \\ 44.5248 \\ -20.0193 \\ 2.5483 \\ -25.4014 \\ 45.2097 \\ -21.7315 \\ 2.7861 \end{bmatrix}$$

Hence; the cubic spline interpolation ,

$$S(x) = \begin{cases} 52.8480 + -69.9312x + 32.0062x^2 + -5.3344x^3 & , \quad x \in [2 \ , \ 2.2] \\ -31.0234 + 44.5248x - 20.0193x^2 + 2.5483x^3 & , \quad x \in [2.2 \ , 2.4] \\ -25.4014 + 45.2097x + -21.7315x^2 + 2.7861x^3 & , \quad x \in [2.4 \ , \ 2.6] \end{cases}$$

Secondly , we construct the legendre – spline interpolation, auxiliary matrix can be written as

$$A_L = \begin{bmatrix}
1.0000 & 2.0000 & 5.5000 & 17.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
1.0000 & 2.2000 & 6.7600 & 23.3200 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 2.2000 & 6.7600 & 23.3200 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 2.4000 & 8.1400 & 30.9600 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 2.4000 & 8.1400 & 30.9600 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 2.6000 & 9.6400 & 40.0400 \\
0.0000 & 1.0000 & 6.6000 & 34.8000 & 0.0000 & -1.0000 & -6.6000 & -34.8000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 7.2000 & 41.7000 & 0.0000 & -1.0000 & -7.2000 & -41.7000 \\
0.0000 & 0.0000 & 3.0000 & 33.0000 & 0.0000 & 0.0000 & -3.0000 & -33.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 3.0000 & 36.0000 & 0.0000 & 0.0000 & -3.0000 & -36.0000 \\
0.0000 & 0.0000 & 3.0000 & 3.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 3.0000 & 39.0000
\end{bmatrix}$$

$$X_L = \begin{bmatrix} a_{11} \\ b_{12} \\ c_{13} \\ d_{14} \\ a_{21} \\ b_{22} \\ c_{23} \\ d_{24} \\ a_{31} \\ b_{32} \\ c_{33} \\ d_{34} \end{bmatrix} \quad ; \quad B_L = \begin{bmatrix} -1.664587 \\ -2.84835 \\ -4.24739 \\ -5.79257 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

to determine *the coefficients* a_{ij} *we use matlap programme to find the inverse of* A_L *,ie,*

$$A_L^{-1} = \begin{bmatrix}
53.043 & -52.043 & -52.554 & 52.554 & 10.511 & -10.511 & 8.409 & -2.102 & 0.491 & -0.140 & 1.035 & 0.070 \\
-32.204 & 32.204 & 34.005 & -34.005 & -6.801 & 6.801 & -5.441 & 1.360 & -0.317 & 0.091 & -1.239 & -0.045 \\
-1.075 & 1.075 & 1.344 & -1.344 & -0.269 & 0.269 & -0.215 & 0.054 & -0.013 & 0.004 & 0.367 & -0.002 \\
1.075 & -1.075 & -1.344 & 1.344 & 0.269 & -0.269 & 0.215 & -0.054 & 0.013 & -0.004 & -0.034 & 0.002 \\
519.301 & -519.301 & -1035.876 & 1036.876 & 528.575 & -528.575 & 103.860 & -105.715 & -10.040 & -7.048 & -0.346 & 3.524 \\
-580.914 & 580.914 & 1192.392 & -1192.392 & -616.478 & 616.478 & -116.183 & 123.296 & 11.231 & 8.220 & 0.387 & -4.110 \\
158.602 & -158.602 & -335.753 & 335.753 & 177.151 & -177.151 & 31.720 & -35.430 & -3.066 & -2.362 & -0.106 & 1.181 \\
-13.441 & 13.441 & 29.301 & -29.301 & -15.860 & 15.860 & -2.688 & 3.172 & 0.260 & 0.211 & 0.009 & -0.106 \\
-134.892 & 134.892 & 795.866 & -795.866 & -647.973 & 648.973 & -26.978 & 132.195 & 2.608 & -7.914 & 0.090 & -17.846 \\
140.054 & -140.054 & -826.317 & 826.317 & 681.263 & -681.263 & 28.011 & -137.253 & -2.708 & 8.216 & -0.093 & 19.442 \\
-34.946 & 34.946 & 206.183 & -206.183 & -171.237 & 171.237 & -6.989 & 34.247 & 0.676 & -2.050 & 0.023 & -5.142 \\
2.688 & -2.688 & -15.860 & 15.860 & 13.172 & -13.172 & 0.538 & -2.634 & -0.052 & 0.158 & -0.002 & 0.421
\end{bmatrix}$$

Now, compute $A_L^{-1}B$ *so*

$$A_L^{-1}B = \begin{bmatrix} 2.6582 \\ -1.0563 \\ 0.1922 \\ -0.1922 \\ -22.0065 \\ 27.9701 \\ -8.2546 \\ 0.5757 \\ -53.2606 \\ 62.4143 \\ -17.5014 \\ 1.3463 \end{bmatrix} .$$

so , Legender – spline interpolation can be written as

$$L(x) = \begin{cases} 2.6582 - 1.0563x + \frac{0.1922}{2}(3x^2 - 1) - \frac{0.1922}{2}(5x^3 - 3x) & , \ x \in [2 \ , \ 2.2] \\ -22.0065 + 27.9701x - \frac{8.2546}{2}(3x^2 - 1) + \frac{0.5757}{2}(5x^3 - 3x) & , \ x \in [2.2 \ , \ 2.4] \\ -53.2606 + 62.41431x - \frac{17.5014}{2}(3x^2 - 1) + \frac{1.3463}{2}(5x^3 - 3x) & , \ x \in [2.2 \ , \ 2.4] \end{cases}$$

Here : in the following table some comparison values between Legender – spline , Cubic spline interpolation and exact solution . see the figures (1, 2, 3) .

x_i	exact value $f(x_i)$	legendre – spline $L(x_i)$	cubic spline $S(x_i)$	$E\lvert f(x_i)-L(x_i)\rvert$	$E\lvert f(x_i)-S(x_i)\rvert$
2.123000	−2.36428	−2.366557511	−2.402791235	0.002276993	0.038510717
2.125140	−2.37727	−2.37949743	−2.415985616	0.002225264	0.038713451
2.121200	−2.35337	−2.355693127	−2.391707131	0.002319191	0.038333195
2.199900	−2.8477	−2.847702411	−2.890667291	7.22859E − 06	0.042972109
2.224000	−3.00596	−3.005419055	−2.987173168	0.000538727	0.018784614
2.300100	−3.5253	−3.524059351	−3.51400317	0.00124179	0.011297971
2.333400	−3.76127	−3.760159547	−3.753803619	0.001112135	0.007468063
2.232900	−3.06518	−3.06448567	−3.047111778	0.000697462	0.018071354
2.399990	−4.24731	−4.247315434	−4.247273468	1.9336E − 06	4.00321E − 05
2.400010	−4.24746	−4.247464567	−3.556623177	2.4634E − 06	0.690838926
2.421100	−4.40497	−4.405592706	−3.788438933	0.000618816	0.616534957
2.499900	−5.00637	−5.009107099	−4.665091666	0.002734106	0.002734106
2.500000	−5.00715	−5.009883044	−4.6662125	0.002735447	0.340935097
2.540000	−5.31893	−5.321665416	−5.11570839	0.002738708	0.203218319
2.578990	−5.62604	−5.627451561	−5.555407452	0.001411633	0.070632476
2.599900	−5.79177	−5.791783947	−5.791497201	1.00293E − 05	0.000276717

Table (a)

Error analysis and order of convergence :

In this example we show the convergence (numarical) of the Method is good ,in addition ,the Legendre – spline method give more accuracy (Table a)than cubic spline , figures 1,2,3 .

Here ,the Legendre – spline method produce function values $L(x)$ as approximation to $y(t_i)$,
The unknown values may be replaced by $L(x)$,
also all commands we used in appendix a ,

The challenge ,is Legendre – spline method a good numerical approximation solution method of an initial value problem of ordinary differential equations. Fractional differential equation, partial Fractional differential equation,?
What is the stability and convergence of the solution?
What is a bout the perturbation of solution ?
these qustions need more and more studing and researching to answer it ,and this is the our gools.

5. Conclusion

This research will add new numerical method (Legendre – spline interpolation method) to approximate the solution of Fractional differential equations, and also concern with the problem that tries to approximate ate the solution of fractional differential equations through and how.

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