Beyond The Sampling Theorem- Compressive Sensing and its Applications

AISHA SULTHANA
IV B.TECH IN C.S.E GOKULA KRISHNA COLLEGE OF ENGINEERING SULLURPETA, SPSR NELLORE (Dt), A.P.

R.BALAMURALI
ASSOCIATE PROFESSOR & HEAD IN CSE GOKULA KRISHNA COLLEGE OF ENGINEERING SULLURPETA, SPSR NELLORE (Dt), A.P.

Abstract
The compressive sensing (or compressive sampling, CS) theorem states that a sparse signal can be perfectly reconstructed even though it is sampled at a rate lower than the Nyquist rate. It has gained an increasing interest due to its promising results in various applications. There are two popular reconstruction methods for CS: basis pursuit (BP) and matching pursuit (MP).Introductory papers on CS often concentrated either on mathematical fundamentals or reconstruction algorithms for CS. Newcomers in this field are required to study a number of papers to fully understand the idea of CS. This paper aims to provide both the basic idea of CS and how to implement BP and MP, so that newcomers no longer need to survey multiple papers to understand CS and can readily apply CS for their works.

Keywords:
Nyquist rate, basis pursuit, matching pursuit.

1. Introduction

Mind Map

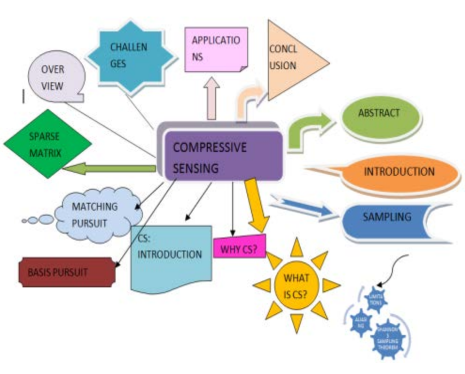


Fig.1. Mind Map for Compressive Sensing

Fig.1. represents a mind map which gives the overview of the paper. The various categories are represented in different shapes and colours for easy identification. The sub branches are listed under the main branches.

1.1 What Compressed Sensing is?

Compressed sensing (CS) addresses the situation in which there is an unknown mathematical object of interest that lies in a large ambient space, but because of prior information, the information content of the signal is less than the dimension of the space. The other key aspect of compressed sensing is the idea of incoherent measurements. Each measurement of a signal should give some global information; when all the measurements are combined, the reconstruction algorithm is non-linear.

However, compressed sensing does not give "something for nothing". In the case of noisy measurements, it is always helpful to take more measurements since this reduces the effect of the noise, and so under-sampling will always perform worse than exact- or over-sampling.

1.2 What Compressed Sensing is not?

Because compressed sensing deals with sparsity and compressibility, it is related to many other fields, such as sparse approximation, classic problems in image and signal processing such as denoising and deconvolution, dictionary learning, computational harmonic analysis etc. The architecture proposed is a pure compressed sensing architecture, because fundamental to its operation is the fact that measurements are incoherent.

Perhaps one of the biggest impact of CS is that it has spurred research in related fields, with the idea of exploiting prior knowledge. Yet the impact on hardware devices is much more limited even though compressed sensing theory is about 7 years old and is quite well understood , there are very few pure compressed sensing applications.

Manuscript received January 5, 2014
Manuscript revised January 20, 2014

1.3 How can one compress an image?

It is quite typical for an image to have a large featureless component – for instance, in a landscape; up to half of the picture might be taken up by a monochromatic sky background. Suppose for instance that we locate a large square, say 100×100pixels, which are all exactly the same colour – e.g. all white.

Without compression, this square would take 10,000 bytes to store (using 8-bit grayscale); however, instead, one can simply record the dimensions and location of the square, and note a single colour with which to paint the entire square; this will require only four or five bytes in all to record, leading to a massive space saving.

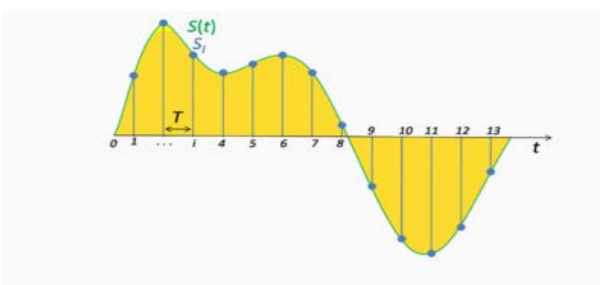
2. SAMPLING

Many signals originate continuous-time signals, e.g. conventional music or voice. By sampling a continuous-time signal at isolated, equally-spaced points in time, we obtain a sequence of numbers.

s[n]=s(n Ts)

$n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

T_s is the sampling period.



In signal processing, sampling is the reduction of a continuous signal to a discrete signal. A common example is the conversion of a sound wave (a continuous signal) to a sequence of samples (a discrete-time signal).

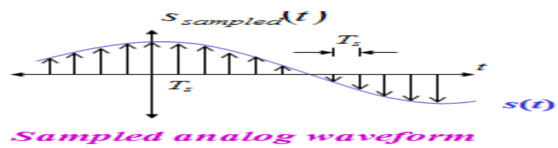
A sample refers to a value or set of values at a point in time and/or space. A sampler is a subsystem or operation that extracts samples from a continuous signal. A theoretical ideal sampler produces samples equivalent to the instantaneous value of the continuous signal at the desired points.

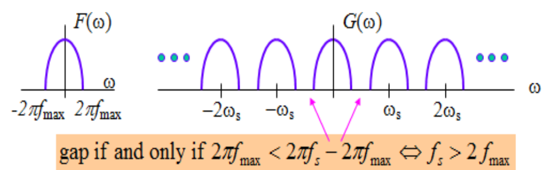
Sampling can be done for functions varying in space, time, or any other dimension, and similar results are obtained in two or more dimensions. For functions that vary with time, let *s*(*t*) be a continuous function (or "signal") to be sampled, and let sampling be performed by measuring the value of the continuous function every *T* seconds, which is called the sampling interval. Thus, the sampled function is given by the sequence**:**

s (nT), for integer values of *n*.

The sampling frequency or sampling rate $\mathbf{f_s}$ is defined as the number of samples obtained in one second (samples per second), thus $\mathbf{f_s = 1/T}$. Reconstructing a continuous function from samples is done by interpolation algorithms.

$$s_{sampled}(t) = s(t)\underbrace{\sum_{n=-\infty}^{\infty}\delta(t-nT_s)}_{impulse\ train}$$

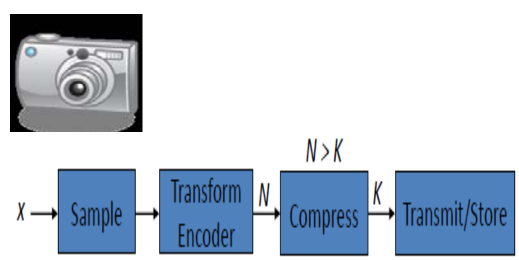


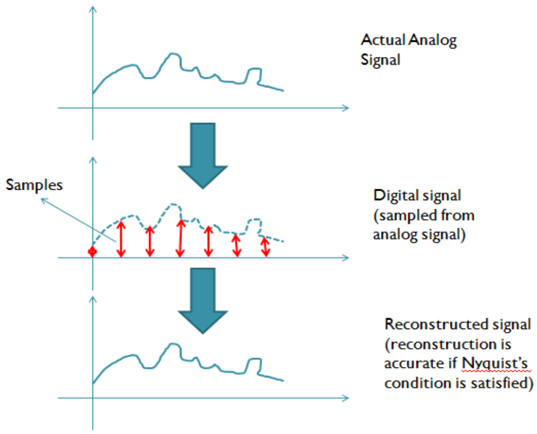


3. Shannon's Sampling Theorem

A band-limited signal with maximum frequency B can be accurately reconstructed from its uniformly spaced digital samples if the rate of sampling exceeds 2B (called Nyquist rate).Independently discovered by Shannon, Whitaker, Kotelnikov and Nyquist.

Shannon/Nyquist Sampling Theorem states that the signals must be sampled more than twice the signal bandwidth. It might end up with a huge number of samples which is needed to compress.





i. What happens if fs = 2*B*?
The solution is: Consider a sinusoid sin(2πB*t*)
Use a sampling period of T_s = 1/ fs = 1/2*B*.
Sketch: sinusoid with zeros at *t* = 0, 1/2*B*, 1/B …

ii. What happens if fs < 2*B*?
The solution is: Mixing of data may occur that may lead to "aliasing".

3.1 Limitations of Shannon's Sampling Theorem

i. The samples need to be uniformly spaced.
ii. The sampling rate needs to be very high if the original signal contains higher frequencies
iii. Does not account for several nice properties of naturally occurring signals (except for band-limitedness).

4. Aliasing

A precondition of the sampling theorem is that the signal be band limited. However, in practice, no time-limited signal can be band limited. Since signals of interest are almost always time-limited (e.g., at most spanning the lifetime of the sampling device in question), it follows that they are not band limited. Sampling an analog signal with maximum frequency B at a rate less than or equal to 2B causes an artifact called aliasing.

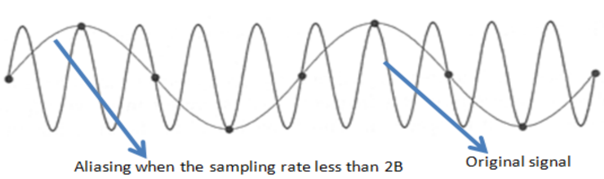


Fig: Effect of Aliasing

5. COMPRESSIVE SENSING

Signal/image compression algorithms like MPEG, JPEG, and JPEG-2000 etc. typically measure large amounts of data. The data is then converted to a transform domain where a majority of the transform coefficients turn out to have near-zero magnitude and are discarded.

Split image into small non-overlapping blocks of equal size and then apply DCT on blocks found to be sparse.

$$K \approx M \ll N$$

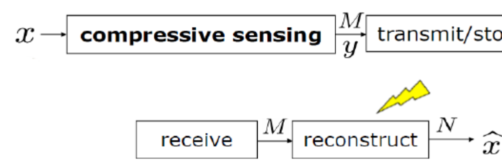


Fig. Compressive sensing based data acquisition system

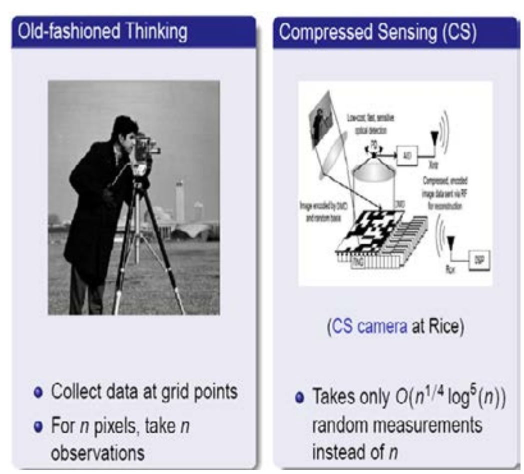
5.1 Why Compressive Sensing?

Much (as in more than 1000 times) fewer measurements need to be made. It may dramatically improve acquisition speed for MRI, range images, hyper-spectral data. It is potential to dramatically improve video-camera frame rates without sacrificing spatial resolution.

5.2 Traditional signal processing

To record images, let us think of an image as a rectangular array, e.g. a 1024 x 2048 array of pixels can take up a lot of disk space on the camera and also take a non-trivial amount of time (and energy) to transfer.

So, it is common practice to get the camera to compress the image, from an initial large size (e.g. 2MB) to a much smaller size (e.g. 200KB, which is 10% of the size). After data acquisition, consider the DCT coefficients that are zero which are discarded before quantization.



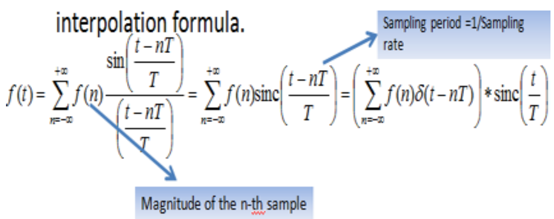
5.3 Signal processing using CS

Using Sampling theorem, the theorem is commonly called the Nyquist sampling theorem. In essence, the theorem shows that a band limited analog signal can be perfectly reconstructed from an infinite sequence of samples if the sampling rate exceeds 2*B* samples per second.

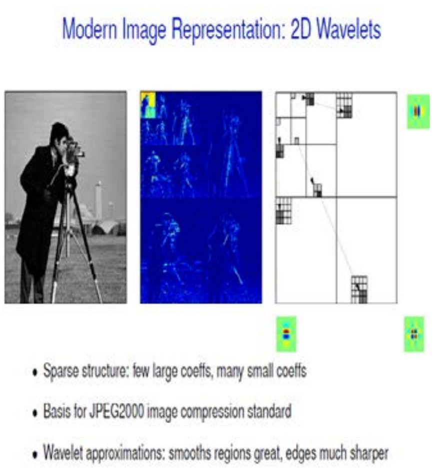
The theorem also leads to a formula for reconstruction of the original signal. The field of compressed sensing provides a stricter sampling condition when the underlying signal is known to be sparse.

5.4 How to reconstruct x(t) from x(n)?

The solution to this is "Interpolation". The optimal reconstruction for band-limited signals from their digital samples proceeds using the sinc interpolant.



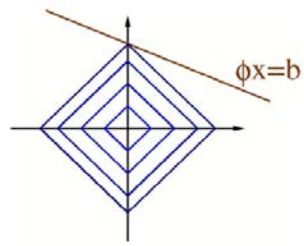
6. WAVELET COEFFICIENTS



7. BASIS PURSUIT

Chen and Donoho have suggested a method of decomposition based on true global optimization which is atleast theoretically feasible, due to recent advances in linear programming. Among the many possible solutions to $\Phi\alpha=s$ or $\Phi x=b$, they pick one whose coefficients have minimum l^1 norm.

$$\min \|\alpha\|_1 \text{ subject to } \Phi\alpha = s.$$



8. MATCHING PURSUIT

Matching pursuit involves finding the "best matching" projections of multidimensional data onto an over-complete dictionary. Mallat and Zhang have proposed this greedy algorithm that builds up a sequence of sparse approximations.

9. SPARSE MATRIX

A sparse matrix is a matrix populated primarily with zeros (Stoer & Bulirsch 2002, p. 619) as elements of the table. The term itself was coined by Harry M. Markowitz. If the majority of elements differ from zero, then it is common to refer to the matrix as a dense matrix.

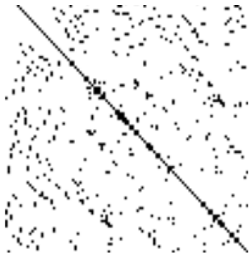
9.1 Example of sparse matrix

[11 22 0 0 0 0 0]
[0 33 44 0 0 0 0]
[0 0 55 66 77 0 0]
[0 0 0 0 0 88 0]
[0 0 0 0 0 0 99]

The above sparse matrix contains only 9 nonzero elements of the 35, with 26 of those elements as zero.

Conceptually, sparsity corresponds to systems which are loosely coupled. Consider a line of balls connected by springs from one to the next; this is a sparse system. By contrast, if the same line of balls had springs connecting each ball to all other balls, the system would be represented by a **dense matrix**. The concept of sparsity is useful in combinatorics and application areas such as network theory, which have a low density of significant data or connections.

Huge sparse matrices often appear in science or engineering when solving partial differential equations.



10. APPLICATIONS

10.1 Analog to Digital Conversion

This is a fundamental aspect of Wireless Communications.

E.g. CDMA in which voice message of 4096 hertz standard frequency that spreads over radio spectrum can span thousands of hertz.

Here if the signal is still sparse, so detect or recover signal more rapidly than Shannon's theorem.

10.2 Image restoration and image inpainting:

Image restoration is the operation of taking a corrupted/ noisy image and estimating the clean original image. Image inpainting is the process of recover missing pixels of given image.

10.3 Magnetic Resonance Imaging (MRI):

MRI is a medical imaging technique used in radiology to visualize detailed internal structures. In MRI, samples are collected directly in Fourier frequency domain (k-space) of object. The scan time in MRI is proportional to the number of Fourier coefficients. Using compressive sensing technique, we can reduce the number of samples and scan time. Real MR images are known to be sparse in discrete cosine transform (DCT) and wavelet transform.

Some other applications include

i. Analogue to digital Conversion
ii. Single-pixel imaging
iii. Data compression
iv. Astronomical signal
v. Geophysical data analysis
vi. Compressive radar imaging.
vii. Data Acquisition
viii. Data Compression
ix. Image and Video Compression

10.4 Progress Chart

i. February28,2010–Research Reading on compressive sensing and information gathering.
ii. March20,2010–complete research reading and jpeg simulation part.
iii. April10,2010–completecompressive sensing coding.
iv. April20,2010–Final touch and documentation report.

11. CONCLUSION

Compressive sensing established itself by now as a new sampling theory which exhibits fundamental and intriguing connections with several mathematical fields, such as probability, geometry of Banach spaces, harmonic analysis, theory of computability and information-based complexity. The link to convex optimization and the development of very efficient and robust numerical methods make compressive sensing a concept useful for a broad spectrum of natural science and engineering applications, in particular, in signal and image processing and acquisition.

Moreover, new challenges are now emerging in numerical analysis and simulation where high- dimensional problems (e.g., stochastic partial differential equations in finance and electron structure calculations in chemistry and biochemistry) became the frontier. In this context, besides other forms of efficient approximation, such as sparse grid and tensor product methods, compressive sensing is a promising concept which is likely to cope with the "curse of dimensionality".

In particular, further systematic developments of adaptivity in the presence of different scales, randomized algorithms, an increasing role for combinatorial aspects of the underlying algorithms, are examples of possible future developments, which are inspired by the successful history of compressive sensing.

REFERENCES

[1] David L. Donoho, "Compressed sensing", IEEE Trans. on Information Theory, Vol. 52, No. 4, pp. 1289 - 1306, April 2006
[2] Richard G. Baraniuk, "Compressive Sensing", IEEE Signal Processing Magazine, July 2007.
[3] Andreas F. Molisch, "Ultra wide band communications - an overview," in Proceedings of General Assembly of the International Union of Radio Science (USRIGA), 2008.
[4] Z. Wang, G. R. Arce, B. M. Sadler, J. L. Paredes, and X. Ma,"Compressed Detection for Pilot Assisted Ultra-Wideband Impulse Radio", in Proceedings of IEEE ICUWB, 2007.
[5] Jose L. Paredes, Gonzalo R. Arce, and Zhongmin Wang, "Ultra-wideband Compressed Sensing: channel estimation", IEEE Journal of Selected Topics in Signal Processing, Vol. 1, No. 3, October 2007.

[6] Emmanuel Candèsand Justin Romberg, Practical signal recovery from random projections. (Preprint, Jan. 2005)
[7] Emmanuel Candès, Justin Romberg, and Terence Tao, Stable signal recovery from incomplete and inaccurate measurements. (Communications on Pure and Applied Mathematics, 59(8), pp. 1207-1223, August 2006)

Aisha Sulthana is pursuing her B.Tech degree in C.S.E from Gokula Krishna College of engineering. She is from Sullurpeta, SPSR Nellore (Dt). She has an aggregate of 80.2% in B.Tech. She has completed her intermediate in Sri Satyam Jr.College with 94.4% and SSC in Sri Satyam Children’s Academy with 89%.She has participated in various National Level Symposiums in Paper Presentations, Technical C-Quiz and also organized a National level Symposium and various events in college.

R.Balamurali has completed his B.E in Computer Science and Engineering from Manonmaniam Sundaranar University in the year 2002 and M.Tech in Information Technology from Sathyabama University in the year 2008. He is pursuing his PhD from SRM University, Kattankulathur. He is presently working as a Associate Professor and Head of CSE Department in Gokula Krishna College of Engineering. He is having an overall teaching experience of 9 years. His research interest includes Wireless Sensor Networks, Compressive Sensing and Computer Architecture.s