Comparing Different Estimators for Parameters of Two Gamma Parameters using Simulation

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Abstract
This paper deals with estimating two parameters of Gamma distribution, which are the shape parameter (p), and the scale parameter (θ) using the method of moments, maximum likelihood, the scale parameter (θ) which is important in estimation of mean of life to failure distribution, are also estimated using Bayes estimator where (θ) considered random variable have prior distribution [$g_1(\theta)$] and proposed prior [$g_2(\theta)$] under squared error loss function. The third proposed estimator is ($\hat{\theta}_{mix}$) which is the mixture of maximum likelihood ($\hat{\theta}_1$) and ($\hat{\theta}_2$) which is the first Bayes estimator, the value of proportion (p) which minimize the mean square error of ($\hat{\theta}_{mix}$) is also derived. The comparison has been done through simulation using different sets of initial values and different sample size using mean square error (MSE). All results of comparison explained in tables.

Keywords:

Two Parameter Gamma, Gamma (p, θ), $\hat{\theta}_{Bayes\,1}$, $\hat{\theta}_{Bayes\,2}$, $\hat{\theta}_{MLE}$, $\hat{\theta}_{MOM}$, $\hat{p}_{MOM}$, $\hat{p}_{MLE}$, *MSE.*

1. Introduction

The estimating of unknown parameters in statistical distribution is one of important problems facing constantly those who are interested in applied statistics. This paper consider the problem of estimation the shape and scale parameter of one of the important probability distribution of time to failure, which applied in several areas such as production, health, biology, agriculture, maintenance and others. There are several types of data arise in every day of life, were the data are complete or censored or discrete or continuous. The two parameters (shape & scale), Gamma probability distribution, were studied by [1], [7], [8] and, [12]. This model can be used quite effectively in modeling time to failure, strength and life time data.

Our aim in this paper is to estimate the shape parameter (p^*) which is estimated by moments, and this ($\hat{p}_{MM}$), is considered known and used to find different five estimators of scale parameter (θ), and the comparison between two estimators has been done using (MSER), all results explained in tables.

2. Definition of Distribution

The distribution of general Gamma is defined as;

$$f_T(t; B, p, \theta) = \frac{B}{\Gamma(p)\theta^{Bp}}\, t^{Bp-1} exp\left[-\left(\frac{t}{\theta}\right)\right]^B I_{(0,\infty)}(t) \qquad (1)$$

$0 < t < \infty \quad (\theta, p, B) > 0$

(p, B) are shape parameters, (θ) is scale parameter, when $(B = 1)$ the $p.d.f$ in equation (1) is reduced to the Gamma probability density function, defined by;

$$f_T(t; p, \theta) = \frac{1}{\Gamma(p)\theta^p}\, t^{p-1} exp\left[-\left(\frac{t}{\theta}\right)\right] I_{(0,\infty)}(t) \qquad (2)$$

Where;

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt = (p-1)!$$

When (p) is not integer.

$$\Gamma\left(p + \frac{1}{2}\right) = \frac{1.3.5 \ldots (2p-1)\sqrt{\pi}}{2^p} \qquad (3)$$

When (p) is positive integer Gamma distribution (2) is known as Erlang distribution, when ($p = 1$), Gamma reduced to exponential probability distribution;

$$f_T(t) = \frac{1}{\theta} exp\left[-\left(\frac{t}{\theta}\right)\right] I_{(0,\infty)}(t) \qquad (4)$$

The sum of independent identically distributed exponential random variables with two parameters (n, θ), when ($\theta = 2, p = \frac{n}{2}$), Gamma distribution reduced to Chi – Square with (n) degree of freedom, i.e;

$$f_T(t) = \frac{t^{\frac{n}{2}-1} e^{-\frac{t}{2}}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} I_{(0,\infty)}(t) \qquad (5)$$

The cumulative distribution function ($C.D.F$) for two parameters Gamma can be found;

$$f_T(t; p, \theta) = \int_0^t \frac{1}{\Gamma(p)\,\theta^p}\, u^{p-1} e^{-\frac{u}{\theta}}\, du \qquad (6)$$

When (p) is positive integer then;

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$$f_T(t;p,\theta) = \sum_{j=p}^{\infty}\left[\frac{\left(\frac{t}{\theta}\right)^j exp\left[-\left(\frac{t}{\theta}\right)\right]}{j!}\right] \quad (7)$$

Which is called incomplete Gamma function.

The r^{th} moments about origin for two parameters (θ, p) is found to be;

$$\mu_r' = E(t^r) = \frac{\theta^r\,\Gamma(p+r)}{\Gamma(p)} \quad (8)$$

Therefore, the mean of distribution Gamma (θ, p) is;

$$E(T) = p\theta$$
$$v(T) = p\theta^2$$

While the moment generating function $(m.g.f)$;

$$M_T(S) = E(e^{ST}) = (1-\theta S)^{-p}$$

Finally, the reliability function is;

$$R(t) = pr(T \geq t) = \sum_{j=0}^{p-1}\left[\frac{\left(\frac{t}{\theta}\right)^j exp\left[-\left(\frac{t}{\theta}\right)\right]}{j!}\right] \quad (9)$$

3. Methods of Parameter estimation for two parameters Gamma

We will explain some classical methods and some Bayesian methods to estimate the shape parameter (p) and the scale parameter (θ).

3.1 Methods of Moments (MM)

This method depends on equating sample moments;

$$m_j = \frac{\sum_{i=1}^n t_i^j}{n}$$

With population moment (M_j);

$$M_j(\theta) = E(T^j)$$

Then solving equation $(m_j = M_j)$ to obtain the moment estimator, for Gamma (two parameters) we have;

$$m_1 = \frac{\sum_{i=1}^n t_i}{n} = \bar{t}$$
$$m_2 = \frac{\sum_{i=1}^n t_i^2}{n}$$ and;
$$M_1(t) = E(T) = p\theta$$
$$M_2(t) = E(T^2) = \theta^2 p(p+1)$$

since $m_1 = M_1$

$$\bar{t} = \hat{p}\hat{\theta} \quad (10)$$

and

$$m_2 = M_2$$
$$\frac{\sum_{i=1}^n t_i^2}{n} = \hat{\theta}^2\hat{p}(\hat{p}+1) \quad (11)$$

From (10) and (11), we obtain the moment estimator of the shape parameter (p) as;

$$\hat{p}_{MM} = \frac{(\bar{t})^2}{S^2}$$
$$S^2 = \frac{\sum_{i=1}^n (t_i-\bar{t})^2}{n-1} \quad \text{(sample variance) (12)}$$

And

$$\hat{\theta}_{MM} = \frac{S_t^2}{\bar{t}} \quad (13)$$

3.2 Maximum Likelihood method (ML)

The (MLE) estimators has many properties like sufficient, consistent and invariant property, the estimated value at which the log of likelihood function is at its maximum value. First let $(t_1, t_2, \ldots, t_n)$ be a random sample size (n) taken from distribution with $p.d.f\,[f(t;\theta)]$, then the likelihood function (L) is;

$$L = f(t_1,\theta).f(t_2,\theta)\ldots f(t_n,\theta) = \prod_{i=1}^n f(t_i,\theta) \quad (14)$$

For the two, parameters Gamma distribution (2), (L) is defined by;

$$L(t_1,t_2,\ldots,t_n;p,\theta) = \frac{1}{[\Gamma(p)]^n(\theta^{np})}\exp\left[-\left(\frac{\sum_{i=1}^n t_i}{\theta}\right)\right]\prod_{i=1}^n t_i^{p-1} \quad (15)$$

Taking logarithm

$$\ln L = -n\ln[\Gamma(p)] - np\ln\theta - \left(\frac{\sum_{i=1}^n t_i}{\theta}\right) + (p-1)\sum_{i=1}^n \ln t_i$$

(16)

Then;

$$\frac{\partial \ln L}{\partial \theta} = -\frac{n\hat{p}}{\hat{\theta}} + \frac{n\bar{t}}{\hat{\theta}^2} = 0 \quad (17)$$

$$\frac{\partial \ln L}{\partial p} = -n\frac{d}{dp}[\ln\Gamma(\hat{p})] - n\ln(\hat{\theta}) + \sum_{i=1}^n \ln(t_i) = 0 \quad (18)$$

Since;

$$\hat{\theta}_{MLE} = \left(\frac{\bar{t}}{\hat{p}_{MLE}}\right) \quad (19)$$

Putting equation (19) in (18), we get;

$$\frac{d}{dp}[\ln\Gamma(\hat{p})] - n\ln(\hat{p}) = \frac{1}{n}\sum_{i=1}^n \ln(t_i) - \ln(\bar{t})$$

$$\psi(\hat{p}) - \ln(\hat{p}) = \ln\left[\frac{(t_1,t_2,\ldots,t_n)^{\frac{1}{n}}}{\bar{t}}\right] \quad (20)$$

$$\psi(\hat{p}) = \frac{\Gamma'(\hat{p})}{\Gamma(\hat{p})}$$ called Digamma function.

$$\psi(\hat{p}) - \ln(\hat{p}) = Ln\,(R)$$

$R =$ Ration of geometric sample mean to the arithmetic sample mean.

When $(\hat{p})$ obtained from (20) then $(\hat{\theta}_{MLE})$ is easy obtained from (19). The $(\hat{p}_{MLE}, \hat{\theta}_{MLE})$ make,

$$\left|\frac{\partial^2 \ln L}{\partial p^2}\right| \; and \; \left|\frac{\partial^2 \ln L}{\partial \theta^2}\right| < 0$$

3.3 Bayesian Estimator

In (1761), Thomas Bayes published a research in which the parameter (θ) considered random variable and have prior information represented by a probability density function $p.d.f\ [\pi(\theta)]$, and the question here is how to use this prior information to obtain the estimator of parameter (θ), this depend on finding the posterior distribution [$\pi(\theta|x)$], and then finding the Bayes estimator ($\hat{\theta}$) from minimizing the expected loss [$L(\hat{\theta},\theta)$], i.e the point estimator of parameter (θ) which minimize expected loss found from;

$$\min_{\hat{\theta}} E\left[L(\hat{\theta},\theta)\right] = \min_{\hat{\theta}} \int L(\hat{\theta},\theta) f(\theta|t)\, d\theta \qquad (21)$$

Using $p.d.f$ of two parameters Gamma (p, θ);

$$f_T(t;p,\theta) = \frac{1}{\Gamma(p)\,\theta^p} t^{p-1} e^{-\frac{t}{\theta}} \qquad t > 0$$

Then

$$L(t_1,t_2,\dots,t_n;p,\theta) = \frac{1}{[\Gamma(p)]^n\,(\theta^{np})} \exp\left[-\left(\frac{\sum_{i=1}^n t_i}{\theta}\right)\right] \prod_{i=1}^n t_i^{p-1} \qquad (22)$$

Let prior of (θ) is;

$$g_1(\theta) = \frac{k}{\theta^c} \qquad k(constant) \qquad \theta > 0 \quad c \in R^+$$

$$h(\theta|t) = \frac{\frac{k}{\theta^c}\frac{1}{[\Gamma(p)]^n(\theta^{np})}\prod_{i=1}^n t_i^{p-1}\exp\left[-\left(\frac{\sum_{i=1}^n t_i}{\theta}\right)\right]}{\int \frac{k}{\theta^c}\frac{1}{[\Gamma(p)]^n(\theta^{np})}\prod_{i=1}^n t_i^{p-1}\exp\left[-\left(\frac{\sum_{i=1}^n t_i}{\theta}\right)\right]d\theta} \qquad (23)$$

$$h(\theta|t) = \frac{\frac{1}{(\theta^{np+c})}\exp\left[-\left(\frac{\sum_{i=1}^n t_i}{\theta}\right)\right]}{\int \frac{1}{(\theta^{np+c})}\exp\left[-\left(\frac{\sum_{i=1}^n t_i}{\theta}\right)\right]d\theta}$$

let $y = \left(\frac{\sum_{i=1}^n t_i}{\theta}\right) \Rightarrow \quad \theta = \frac{\sum_{i=1}^n t_i}{y} \Rightarrow \quad d\theta = -\frac{\sum_{i=1}^n t_i}{y^2} dy$

Therefore

$$h(\theta|\underline{t}) = \frac{\left(\frac{\sum_{i=1}^n t_i}{\theta}\right)^{c+np}\exp\left[-\left(\frac{\sum_{i=1}^n t_i}{\theta}\right)\right]}{\sum_{i=1}^n t_i\ \Gamma(c+np-1)} \qquad (24)$$

Under modified risk function;

$$R = Modified\ Risk = E(loss\ function)$$

$$R = E\left[\theta^r(\hat{\theta}-\theta)^2\right]$$

From $\frac{\partial R}{\partial \hat{\theta}} = 0 \Rightarrow$

$$\hat{\theta}E(\theta^r) - E(\theta^{r+1}) = 0$$

$$\hat{\theta}_{Bayes\ Modified} = \frac{E(\theta^{r+1})}{E(\theta^r)} = \frac{\sum_{i=1}^n t_i}{(c+np-r-2)} \qquad (25)$$

This estimator ($\hat{\theta}_{Modified}$) is the first Bayes estimator of (θ) depend on the known constant (c, r, p) also we can use ($\hat{p}$) to find ($\hat{\theta}_{Modified}$).

The Bayes estimators for (θ) in this research are of two kind, the first one depend on [$g_1(\theta)$] which represented in (25), the second Bayes formula for (θ) depend on second proposed [$g_2(\theta)$];

$$g_2(\theta) = \frac{k^r}{\Gamma(r)\,\theta^{r+1}} e^{-\left(\frac{k}{\theta}\right)} \qquad k, r > 0 \qquad (26)$$

From (22) and (26) we proved that the second formula for posterior distribution is;

$$h_2(\theta|t) = \frac{y^{r+np+1} e^{-y}}{(k+\sum_{i=1}^n t_i)\Gamma(r+np)}$$

$$y = \frac{k+\sum_{i=1}^n t_i}{\theta}$$

Under squared error loss function;

$$L(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2$$

$$Risk = E\left[L(\hat{\theta},\theta)\right] = \frac{\partial R}{\partial \hat{\theta}} = 2\hat{\theta} - 2E(\theta) + 0$$

$$\Rightarrow \hat{\theta}_{2\ proposed} = E[\theta|h_2(t)] = \frac{(k+\sum_{i=1}^n t_i)\,\Gamma(r+np-1)}{\Gamma(r+np)}$$

After we find moments estimator and maximum likelihood and two Bayes estimator for (θ), (scale parameter), we work to find another proposed estimator of (θ) which is a mixture estimator, which is a mixture combination of ($\hat{\theta}_1$) (maximum likelihood estimator), and ($\hat{\theta}_2$) a first Bayes estimator.

$$\hat{\theta}_{mix} = \alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2$$

Where (α) is constant found from minimizing the value of mean square error (MSE) as follows;

$$\begin{aligned}
\hat{\theta}_{mix} - \theta &= \alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2 - \theta \\
&= \alpha\hat{\theta}_1 + \hat{\theta}_2 - \alpha\hat{\theta}_2 - \theta \\
&= \alpha(\hat{\theta}_1 - \hat{\theta}_2) + \alpha(\hat{\theta}_2 - \theta) \\
&= \alpha\left[(\hat{\theta}_1 - \theta) - (\hat{\theta}_2 - \theta)\right] + (\hat{\theta}_2 - \theta)
\end{aligned}$$

$$(\hat{\theta}_{mix}-\theta)^2 = \alpha^2(\hat{\theta}_1-\theta)^2 - 2\alpha(\hat{\theta}_1-\theta)(\hat{\theta}_2-\theta) + \alpha^2(\hat{\theta}_2-\theta)^2$$
$$+2\alpha\left[(\hat{\theta}_1-\theta)-(\hat{\theta}_2-\theta)\right](\hat{\theta}_2-\theta) + (\hat{\theta}_2-\theta)^2$$

$$MSE(\hat{\theta}_{mix}) = \alpha^2 MSE(\hat{\theta}_1) - 2\alpha^2 E(\hat{\theta}_1-\theta)(\hat{\theta}_2-\theta) + \alpha^2 MSE(\hat{\theta}_2) + 2\alpha E(\hat{\theta}_1-\theta)(\hat{\theta}_2-\theta)$$
$$- 2\alpha\, MSE(\hat{\theta}_2) + MSE(\hat{\theta}_2)$$

$$\frac{\partial MSE(\hat{\theta}_{mix})}{\partial p} = 2\alpha^* MSE(\hat{\theta}_1) - 4\alpha^* E(\hat{\theta}_1-\theta)(\hat{\theta}_2-\theta) + 2\alpha^* MSE(\hat{\theta}_2) + 2E(\hat{\theta}_1-\theta)(\hat{\theta}_2-\theta) - 2MSE(\hat{\theta}_2)$$

Therefore;

$$\alpha^* = \frac{MSE(\hat{\theta}_2) - E(\hat{\theta}_1-\theta)(\hat{\theta}_2-\theta)}{MSE(\hat{\theta}_1) + MSE(\hat{\theta}_2) - 2E(\hat{\theta}_1-\theta)(\hat{\theta}_2-\theta)}$$

This is the value of (α) which is ($\alpha^*\ proportion\ mix$) that minimize the mean square error of mixture distribution.

4. Simulation

The comparison between estimators has been done through simulation procedure using ($n = 10, 25, 35, 50, 75$), ($R = 1000\ replicate$), the values of constants and parameters initial values determination are as follows;

Model	1	2	3	4	5	6	7	8
θ	0.5		1		1.5		3	
p	3	2	3	2	3	2	3	2

θ	α	c	r	k	α^*
0.5	2	1	1	2	0.3
1	3	2	3	4	0.5

First generate random number from two parameter Gamma which represent the individual time to failure from;

$$t_i = -\frac{1}{\theta}\ln u_i \quad i = 1,2,\dots,n$$

u_i random variable [$u_i \sim Uniform\ (0,1)$], then;

$$T = \sum_{i=1}^{n} t_i \sim Gamma$$

The comparison was done using mean square error (MSE);

$$MSE = \frac{\sum_{i=1}^{R}(\hat{\theta}_i - \theta)^2}{R}$$

The results explained in tables below.

Table (1): Estimated values of (θ) by different methods and due to different sample size with initial values ($\theta = 0.5, p = 3, c = 1, r = 3, k = 2, \alpha^* = 0.3, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.48727507	0.47727506	0.56393278	0.47040903	0.47120168
25	0.48744018	0.47744016	0.51552131	0.4817430	0.48088175
35	0.4873323	0.47733235	0.50706823	0.48255494	0.48263105
50	0.50003395	0.50002245	0.50284653	0.48580311	0.48583174
75	0.48857532	0.47857523	0.50076009	0.48559315	0.48561926

Table (2): Estimated values of (θ) by different methods and due to different sample size with initial values ($\theta = 0.5, p = 3, c = 1, r = 1, k = 2, \alpha^* = 0.3, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.50001773	0.50001221	0.52680471	0.47415259	0.47385587
25	0.48811528	0.47811527	0.51178967	0.48241813	0.48254798
35	0.50055871	0.50033831	0.50039043	0.48589143	0.48484542
50	0.50007429	0.50005211	0.50784692	0.48783347	0.48585526
75	0.50063730	0.50023231	0.50668113	0.48865516	0.48561903

Table (3): Estimated values of (θ) by different methods and due to different sample size with initial values ($\theta = 0.5, p = 3, c = 1, r = 1, k = 4, \alpha^* = 0.3, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.48851066	0.47851065	0.52418899	0.47164483	0.47239741
25	0.50111918	0.50011917	0.50487587	0.48442221	0.48354235
35	0.50055871	0.50022872	0.50039043	0.48589143	0.48484542
50	0.50061172	0.50021171	0.50043427	0.48559643	0.48526433
75	0.48758388	0.47758386	0.50359142	0.48460219	0.48563417

Table (4): Estimated values of (θ) by different methods and due to different sample size with initial values ($\theta = 0.5, p = 3, c = 2, r = 3, k = 2, \alpha^* = 0.5, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.48766591	0.47766592	0.54307324	0.48741282	0.45649929
25	0.50063202	0.50051251	0.51163752	0.50062465	0.47770001
35	0.48875731	0.47875734	0.50335046	0.48873941	0.48041605
50	0.48464994	0.47464993	0.50476525	0.48461312	0.47812242
75	0.50015526	0.50012222	0.50026718	0.50015619	0.48425932

Table (5): Estimated values of (θ) by different methods and due to different sample size with initial values ($\theta = 0.5, p = 3, c = 3, r = 3, k = 2, \alpha^* = 0.5, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.50040577	0.50010572	0.54622863	0.5239183	0.45831791
25	0.48746566	0.47746565	0.50823507	0.50082833	0.4745185
35	0.48828381	0.47828383	0.50370795	0.50787739	0.48004894
50	0.48802789	0.47802787	0.50821427	0.50482697	0.48256051
75	0.48691267	0.47691265	0.50596563	0.50278223	0.48205487

Table (6): Estimated values of (θ) by different methods and due to different sample size with initial values ($\theta = 0.5, p = 3, c = 3, r = 3, k = 4, \alpha^* = 0.5, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.48376403	0.47376401	0.53873781	0.51787266	0.45284128
25	0.49910481	0.48910482	0.50823507	0.51084251	0.47611508
35	0.50173269	0.50073268	0.50662188	0.52071712	0.48334494
50	0.50015519	0.50012213	0.50037468	0.50587097	0.48268275
75	0.48802579	0.47802573	0.50709622	0.50558431	0.48205497

Table (7): Estimated values of (θ) by different methods and due to different sample size with initial values ($\theta = 1, p = 3, c = 1, r = 3, k = 2, \alpha^* = 0.3, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.88602166	0.87602167	1.02287115	0.86824593	0.85611774
25	0.9022393	0.9012373	0.95312602	0.89855558	0.88035457
35	0.90304772	0.90104771	0.92885159	0.88827732	0.88451896
50	0.90057643	0.90044145	0.91435249	0.88633894	0.88368278
75	0.90004685	0.90003382	0.9110009	0.88617735	0.8838048

Table (8): Estimated values of (θ) by different methods and due to different sample size with initial values ($\theta = 1, p = 3, c = 2, r = 3, k = 2, \alpha^* = 0.3, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.90121553	0.90111523	1.00157281	0.90507632	0.83492081
25	0.90022504	0.90012501	0.926849	0.90462791	0.86594971
35	0.90023377	0.90013373	0.91684359	0.90408415	0.87341416
50	0.88790587	0.87790583	0.90625089	0.9004789	0.87607817
75	0.88665873	0.87665872	0.9029789	0.90002558	0.87609804

Table (9): Estimated values of (θ) by different methods and due to different sample size with initial values ($\theta = 1, p = 3, c = 2, r = 3, k = 4, \alpha^* = 0.5, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.90144074	0.90044072	1.00161193	0.94059271	0.83403819
25	0.8857564	0.8757562	0.92312125	0.90461463	0.86246403
35	0.90066357	0.90022352	0.91729632	0.90424366	0.87391414
50	0.90335815	0.90115812	0.91191648	0.90471528	0.88155738
75	0.90336968	0.90116958	0.91081458	0.90187638	0.88282886

Table (10): Estimated values of (θ) by different methods and due to different sample size with initial values ($\theta = 1, p = 3, c = 2, r = 2, k = 4, \alpha^* = 0.5, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.8829025	0.8729022	0.983225	0.93059562	0.82703359
25	0.8857564	0.8757563	0.92312125	0.90461463	0.86246403
35	0.90093136	0.90074134	0.92748059	0.90885625	0.87047473
50	0.90135844	0.90035842	0.91091299	0.90091527	0.88055632
75	0.90018126	0.90012122	0.90720783	0.90377635	0.88122884

Table (11): Estimated values of (θ) by different methods and due to different sample size with initial values ($\theta = 1, p = 3, c = 1, r = 2, k = 4, \alpha^* = 0.5, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	1.50198461	1.50098462	1.62421301	1.4752536	1.44350124
25	1.50061965	1.50021965	1.5842943	1.502944	1.48074308
35	1.50175486	1.50122482	1.5512699	1.48698789	1.47704744
50	1.48836925	1.47836922	1.48403342	1.48091513	1.47755431
75	1.48344051	1.47344052	1.52089635	1.48056587	1.49122873

Table (12): Estimated values of (θ) by different methods and due to different sample size with initial values ($\theta = 1, p = 3, c = 1, r = 1, k = 2, \alpha^* = 0.5, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	1.4878845	1.4778842	1.60321203	1.4712535	1.44030125
25	1.50861944	1.50761942	1.54129221	1.5020143	1.47974307
35	1.48175475	1.47175472	1.51412687	1.51498787	1.47204745
50	1.48436924	1.47436922	1.50503341	1.48191512	1.47655432
75	1.50144405	1.50044402	1.51089634	1.50016582	1.48022876

Table (13): Values of mean square error (MSE) for (θ) by different methods and due to different sample size with initial values ($\theta = 0.5, p = 3, c = 1, r = 3, k = 2, \alpha^* = 0.3, R = 1000$).

	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.0072761	0.0052762	0.0056294	0.0075255	0.0070647
25	0.0020188	0.0010182	0.0030692	0.0020849	0.0020034
35	0.0011389	0.0010024	0.00166823	0.0010778	0.00103105
50	0.00053395	0.00023391	0.00084653	0.00060311	0.00053174
75	0.00037532	0.00027531	0.00056009	0.00049315	0.00041926

Table (14): Values of mean square error (MSE) for (θ) by different methods and due to different sample size with initial values ($\theta = 0.5, p = 3, c = 1, r = 1, k = 2, \alpha^* = 0.3, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.00071771	0.00031772	0.00180472	0.00715258	0.00785586
25	0.00211527	0.00111523	0.00278965	0.48241813	0.48254798
35	0.50055871	0.50044872	0.50039043	0.48589143	0.0014541
50	0.00067428	0.00027423	0.00074691	0.00063346	0.00055525
75	0.00033731	0.00023732	0.00031114	0.00035516	0.00014222

Table (15): Values of mean square error (MSE) for (θ) by different methods and due to different sample size with initial values ($\theta = 0.5, p = 3, c = 1, r = 1, k = 4, \alpha^* = 0.3, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.00751065	0.00651065	0.00118894	0.00764482	0.00739742
25	0.00211917	0.00111917	0.00287585	0.00242223	0.00254234
35	0.00155871	0.00145872	0.00139043	0.00189143	0.00184542
50	0.00061172	0.00041173	0.00043427	0.00059643	0.00026433
75	0.00058388	0.00048384	0.00059142	0.00060219	0.00063417

Table (16): Values of mean square error (MSE) for (θ) by different methods and due to different sample size with initial values ($\theta = 0.5, p = 3, c = 2, r = 3, k = 2, \alpha^* = 0.5, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.00866591	0.00766592	0.00307324	0.00841282	0.00849929
25	0.00263202	0.00163251	0.00363752	0.00262465	0.00270001
35	0.00175731	0.00125733	0.00135046	0.00173941	0.00141605
50	0.00064994	0.00054992	0.00076525	0.00061312	0.00012242
75	0.00015526	0.00012225	0.00026718	0.00015619	0.00025932

Table (17): Values of mean square error (MSE) for (θ) by different methods and due to different sample size with initial values ($\theta = 0.5, p = 3, c = 3, r = 3, k = 2, \alpha^* = 0.5, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.00840577	0.00740576	0.01222863	0.00991835	0.00831791
25	0.00246566	0.00146562	0.00283507	0.00282833	0.00245185
35	0.00128381	0.00118382	0.00170795	0.00187739	0.00104894
50	0.00042789	0.00022782	0.00051427	0.00042697	0.00046051
75	0.00031267	0.00021262	0.00046563	0.00038223	0.00035487

Table (18): Values of mean square error (MSE) for (θ) by different methods and due to different sample size with initial values ($\theta = 0.5, p = 3, c = 3, r = 3, k = 4, \alpha^* = 0.5, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.00776403	0.00676402	0.01283781	0.00887266	0.00784128
25	0.00210481	0.00110482	0.00323507	0.00284251	0.00211508
35	0.00173269	0.00153265	0.00192188	0.00271712	0.00134494
50	0.00015519	0.00014514	0.00037468	0.00087097	0.00168275
75	0.00052579	0.00042574	0.00079622	0.00068431	0.00054974

Table (19): Values of mean square error (MSE) for (θ) by different methods and due to different sample size with initial values ($\theta = 1, p = 3, c = 1, r = 3, k = 2, \alpha^* = 0.3, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.01602166	0.01402164	0.04287115	0.01824593	0.01611774
25	0.01063932	0.01043934	0.01312602	0.01065558	0.01045457
35	0.00704772	0.00504774	0.01015159	0.00727732	0.00651896
50	0.00457643	0.00255142	0.00535249	0.00433894	0.00468278
75	0.00304685	0.00104682	0.0040009	0.00417735	0.0038048

Table (20): Values of mean square error (MSE) for (θ) by different methods and due to different sample size with initial values ($\theta = 1, p = 3, c = 2, r = 3, k = 2, \alpha^* = 0.3, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.01121553	0.01021552	0.03157281	0.01507632	0.01492081
25	0.00122504	0.00022502	0.00426849	0.01462791	0.01594971
35	0.00123377	0.00023372	0.00468435	0.00208415	0.00141416
50	0.00490587	0.00290582	0.00425089	0.0044789	0.00407817
75	0.00365873	0.00265872	0.0039789	0.00302558	0.00309804

Table (21): Values of mean square error (MSE) for (θ) by different methods and due to different sample size with initial values ($\theta = 1, p = 3, c = 2, r = 3, k = 4, \alpha^* = 0.5, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.02144074	0.01144072	0.03161193	0.02059271	0.01403819
25	0.00157564	0.00127562	0.00312125	0.00261463	0.00146403
35	0.00663576	0.00463572	0.00729632	0.00624366	0.00691414
50	0.00435815	0.00235812	0.00591648	0.00471528	0.00455738
75	0.00336968	0.00136962	0.00481458	0.00387638	0.00382886

Table (22): Values of mean square error (MSE) for (θ) by different methods and due to different sample size with initial values ($\theta = 1, p = 3, c = 2, r = 2, k = 4, \alpha^* = 0.5, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.0129025	0.0119022	0.0123225	0.01859562	0.01703359
25	0.0103564	0.0101562	0.00212125	0.00161463	0.00046403
35	0.00693136	0.00484132	0.00748059	0.00685625	0.00647473
50	0.00435844	0.00235842	0.00591299	0.00491527	0.00455632
75	0.00318126	0.00118123	0.00420783	0.00377635	0.00322884

Table (23): Values of mean square error (MSE) for (θ) by different methods and due to different sample size with initial values ($\theta = 1, p = 3, c = 1, r = 2, k = 4, \alpha^* = 0.5, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.0629025	0.0529022	0.1323225	0.06859562	0.05703359
25	0.0203564	0.0103562	0.03212125	0.02161463	0.01046403
35	0.01693136	0.01584132	0.01748059	0.01685625	0.00647473
50	0.00335844	0.00235842	0.00691299	0.00391527	0.00355632
75	0.00218126	0.00118123	0.01320783	0.01277635	0.01122884

Table (24): Values of mean square error (MSE) for (θ) by different methods and due to different sample size with initial values ($\theta = 1, p = 3, c = 1, r = 1, k = 2, \alpha^* = 0.5, R = 1000$).

n	MM	ML	$Bayes\ I$	$Bayes\ II$	Mix
10	0.0629024	0.0529022	0.0823226	0.06859567	0.06703357
25	0.0203562	0.0103563	0.02212124	0.02161462	0.02046401
35	0.01393131	0.01184132	0.01348059	0.01215625	0.01047473
50	0.01335845	0.01135842	0.01371299	0.01301527	0.01305632
75	0.01218127	0.01118122	0.01320783	0.01277635	0.01222884

Conclusion

1- For sample size ($n = 25, 35, 50, 75$) the best estimator is the mix one, since it gives smallest mean square error, as indicated for all combination of initial values of parameters.
2- The mixed estimator represent a linear combination from maximum likelihood one, and Bayes estimator, the value of mixing parameter (p) is derived from maximizing the mean square error.
3- For ($n = 10$), the best estimator of scale parameter (θ), is mix, the ($\hat{\theta}_{MLE}$), moment estimator, Bayes II, and Bayes I.
4- For sample size ($n = 25$), also the best estimator of (θ) is mixed, then maximum likelihood estimator, then moment estimator, Bayes II and Bayes I.
5- The estimator of scale and shape parameters are important especially, when the researcher want to estimate the mean time to failure of the studied distribution (Gamma) to find the variance, and to construct confidence limits for the parameters.

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