Determining The Pattern for 1- fault Tolerant Hamiltonian Cycle From Generalized Petersen Graph P(n,k)

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Abstract
Given a Generalized Petersen Graph P(n,k) a question can be asked by looking at its specific pattern whose 2n vertices and 3n edges, and its special degree on each vertices, which is 3. The degree of each vertices which is only one more than the vertices in Hamiltonian cycle makes a question: is that possible to find a relationship between Generalized Petersen Graph P(n,k) and Hamiltonian cycle, especially with 1-fault tolerant Hamiltonian where the fault occurs because either one vertex or one edge not included in the cycle. In this paper we will discuss about Generalized Petersen Graph P(n,1), if n odd, $3 \leq n \leq 13$; and P(n,2), if n = 1(mod 6), or 3(mod 6) for $7 \leq n \leq 19$.

Keywords
generalized Petersen P(n,k) graph, Hamiltonian circuit, 1-fault-tolerant Hamiltonian

1. Introduction

Graph theory has proven to be useful as a tool for enhancing the rapid improvement in science and technology, for example in network design. An interconnection computer network connects the processors of the parallel computers, where the vertices represent the processors and the edges represent the links. On designing a network, one has to determine a suitable network depending on requirement and their properties, and one of the major requirements in designing the topology network is the Hamiltonian properties.

In some distributed operation system, the Token ring approach is used. An interconnection network requires the presence of Hamiltonian cycles in the structure to meet this approach. Fault-tolerance is also desirable feature in massive parallel system that relatively high probability of failure. A number of fault-tolerant design for specific architectures have been proposed and is represented by a graph [1].

Generalized Petersen Graph is a graph consisting of an inner star polygon (n,k) graph and an outer regular polygon where the corresponding vertices in the inner and outer polygons connected with edges and form a specific pattern. Generalized Petersen graphs is denoted by P(n,k) ([2], [3] for $n \geq 3$ and $1 \leq k \leq \lfloor (n-1)/2 \rfloor$, whose 2n vertices and 3n edges, where every vertex has degree 3.These types of graphs were introduced by [4] and [5]. The following are some examples of P(n,k):

In Section 2 we briefly discuss about 1-fault tolerant Hamiltonian graph. In Section 3 we discuss about lattice diagram, in Section 4 the lattice diagram of Generalized Petersen Graphs P (n,1) if n odd, $3 \leq n \leq 13$; and P(n,2), if n=1(mod 6) or 3(mod 6) for $7 \leq n \leq 19$ will be discussed

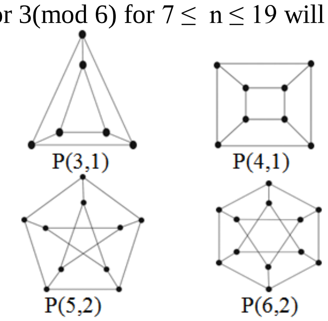

Fig. 1. Some examples of Petersen graphs

The reason for only discuss about those specific graphs because we want to determine the pattern of the lattice diagram since [1] already proposed the Theorem: P(n,1) is 1-fault tolerant Hamiltonian if and only if n odd and P(n,2) is 1-fault tolerant Hamiltonian if and only if n=1(mod 6) or 3(mod 6); and finally in Section 5 we give the conclusion.

2. 1-Fault Tolerant Hamiltonian Graph

Given graph G(V,E), a path in G that includes every vertex in G is called as Hamiltonian path of G, and a cycle in G that includes every vertex in G is called as Hamiltonian cycle of G. If G contains a Hamiltonian cycle, then G is called as Hamiltonian graph [6].

In a network (representing by a graph), when faults occur then that network is not reliable anymore. Thus, to solve it, we need to remove the nodes (vertices) or the links (edges). Indeed, if only one vertex or one edge fault then we only have to remove one vertex or one edge where the fault occurs.

Let G(V,E) be a graph and $v \in V(G)$ and $e \in E(G)$. G is 1-vertex fault tolerant Hamiltonian graph if $G \backslash \{v\}$ is Hamiltonian $\forall$ $v \in V$, and G is 1-edge fault tolerant

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Hamiltonian graph if G\{e} is Hamiltonian $\forall$ e $\in$ E(G). G = (V,E) is 1-fault-tolerant Hamiltonian if G\{f} Hamiltonian $\forall$ f $\in$ E $\cup$ V; in other word G is 1-fault-tolerant Hamiltonian if G is 1-edge fault-tolerant Hamiltonian or 1-vertex fault-tolerant Hamiltonian [7].

3. Lattice Diagram

One model for describing a Hamiltonian cycle was proposed by [8] which is called as lattice diagram. With this model, the Generalized Petersen graphs is transformed into a labeled graph in the (x,y) plane that possesses a closed or open Eulerian trail. The lattice model of a lattice graph L consists of lattice points in (x,y) plane. Two lattice points (a1,b1) and (a2,b2) in L is called adjacent iff | a1-a2| + | b1-b2| = 1.
Let n and k are positive integers with $n \geq 2k+1$. The following rule used to determine a labeled Lattice graph L(n,k):

a. Suppose that a lattice point is labeled with an integer i with $0 \leq i \leq n-1$.

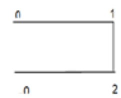
b. (a+1,b) is labeled with $i \oplus 1$, and (a, b -1) is labeled with $i \oplus k$, where $\oplus$ is addition in integer modulo n.

A lattice diagram for P(n,k) is denoted as D(n,k), is a subgraph of L(n,k) induced by vertices with label 0,1,..., n-1. The labeled graph obtained by that rule possessed close or open Eulerian Trail.
According [1], the edges in L(n,k) are interpreted as follow:

1. Horizontal edge $(i, i \oplus 1)$ in L(n,k) corresponds to an edge $(i, i \oplus 1)$ in P(n,k)
2. Vertical edge $(i, i \oplus k)$ in L(n,k) corresponds to an edge $(i', (i \oplus k)')$ in P(n,k)
3. Two edges in different directions incident to the vertex i of degree 2 in L(n,k) corresponds to an edge (i, i') in P(n, k).

4. Lattice Diagrams for Some Generalized Petersen Graphs

To represent the Generalized Petersen Graph G (V,E) using lattice diagram, we use notation proposed by [1] as following : The Generalized Petersen Graph P(n,k) is the graph with vertex set $\{i \backslash 0 \leq i \leq n-1\} \cup \{i' \backslash 0 \leq i \leq n-1\}$ and edge set $\{(i, i \oplus 1) \backslash 0 \leq i \leq n-1\} \cup \{(i, i') \backslash 0 \leq i \leq n-1\} \cup \{(i', i' \oplus 1)') \backslash 0 \leq i \leq n-1\}$.
As already stated in Section 1, we will discuss only the lattice diagram for Generalized Petersen Graph P(n,1), if n odd, $3 \leq n \leq 13$; and P(n,2), if n=1 (mod 6) or 3(mod 6) for $7 \leq n \leq 19$. In this paper we bound n as $3 \leq n \leq 13$, if n odd and $7 \leq n \leq 19$.
a. P(n, 1), n odd, and $3 \leq n \leq 13$
To determine the lattice diagram, first we draw the graph and then set the labels for every vertex according the rule already given above. To find a 1-fault tolerant Hamiltonian we keep in mind that removing one vertex or one edge still constitute Hamiltonian. In this paper we concern only removing one vertex, not removing edge. The other thing that we have to consider is the choice: is the vertex in inner polygon or in the outer?
For P(3,1).
The 1-fault tolerant Hamiltonian circuit pattern for P(3,1) - i′



The pattern for P(3,1) - i′ , $0 \leq i \leq n-1$ is $i,(i\oplus1),(i\oplus1)', (i\oplus2)', (i\oplus2),i$, where $\oplus$ notated as addition of modulo 3. From the pattern of this lattice diagram we can determine the 1-fault tolerant Hamiltonian cycle pattern for P(3,1) - i′ is 0, 1, 1′, 2′, 2, 0 as shown in the following graph:

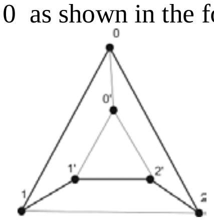


Fig 2. P(3,1)

Notice here that there exists one vertex which is not include in the 1-fault tolerant Hamiltonian circuit (bold line) with labeled.
The 1-fault tolerant Hamiltonian circuit pattern for P(3,1)-i
The pattern for P(3,1) – i, where is $i',(i\oplus)1)',(i\oplus)1), (i\oplus2), (i\oplus2)',i'$. For example, if i = 0, the 1-fault tolerant Hamiltonian cycle is the bold line of the following graph:

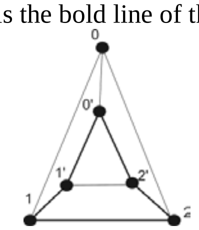


Fig. 3. P(3,1) – 0

Due to the space efficiency, we put the lattice diagram and the pattern of the 1-fault tolerant Hamiltonian graph in Table 1.

b. P(n,2), n=1 (mod 6) or 3(mod 6) for $7 \leq n \leq 19$.
To find a 1-fault tolerant Hamiltonian for P(n,2) – i and P(n,2) – i′ doing inspection of the graph is preferable to using lattice diagram.

For P(7, 2)
The 1- fault tolerant Hamiltonian pattern of P(7,2) - i′

The pattern for 1-fault tolerant Hamiltonian of P(7,2) - i′ is
i, (i⊕1),(i⊕1)′,(i⊕3)′, (i⊕5)′,(i⊕5),(i⊕4),
(i⊕3),(i⊕2),(i⊕2)′,(i⊕4)′, (i⊕6)′, (i⊕6),i

The following figure shows the 1-fault Hamiltonian circuit when i = 0′, and ⊕ notates addition modulo 7.

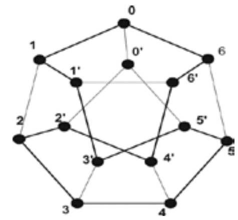


Fig. 4. P(7,2) - 0′The 1- fault tolerant Hamiltonian pattern of P(7,2) - i′

Table 1: Lattice diagram and pattern of 1-fault tolerant Hamiltonian for P(n,1), n odd, 3 ≤ n ≤13

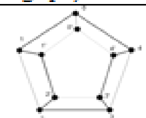
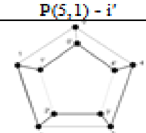
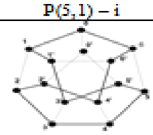
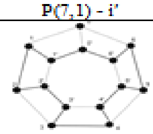
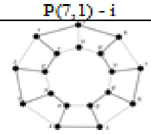
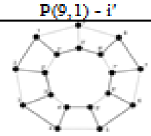
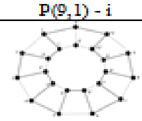
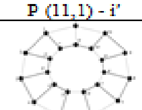
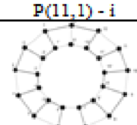
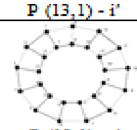
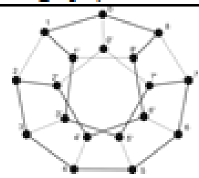
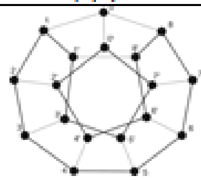
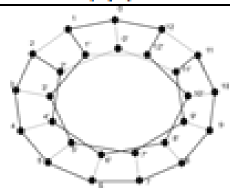
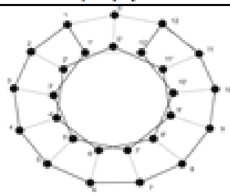
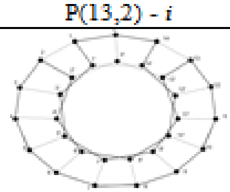
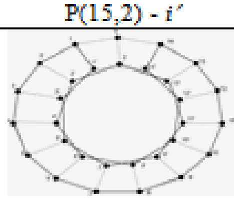
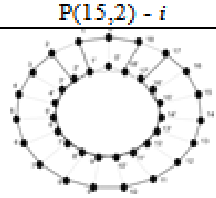
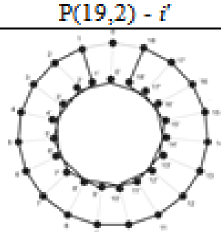
n	The graph, when i = 0	Pattern of 1-fault tolerant Hamiltonian, with ⊕ is addition modulo n
5	 P(5,1) - i′	i, (i⊕1),(i⊕1)′,(i⊕2)′,(i⊕2),(i⊕3),(i⊕3)′,(i⊕4)′,(i⊕4),i
5	 P(5,1) – i	i′, (i⊕1)′,(i⊕1),(i⊕2),(i⊕2)′,(i⊕3)′,(i⊕3),(i⊕4),(i⊕4)′,i′
7	 P(7,1) - i′	i, (i⊕1),(i⊕1)′,(i⊕2)′,(i⊕2),(i⊕3),(i⊕3)′,(i⊕4)′,(i⊕4), (i⊕5),(i⊕5)′,(i⊕6)′,(i⊕6),i
7	 P(7,1) - i	i′, (i⊕1)′,(i⊕1),(i⊕2),(i⊕2)′,(i⊕3)′,(i⊕3),(i⊕4),(i⊕4)′, (i⊕5)′,(i⊕5),(i⊕6),(i⊕6)′,i′
9	 P(9,1) - i′	i, (i⊕1),(i⊕1)′,(i⊕2)′,(i⊕2),(i⊕3),(i⊕3)′,(i⊕4)′,(i⊕4), (i⊕5),(i⊕5)′,(i⊕6)′,(i⊕6), (i⊕7), (i⊕7)′, (i⊕8)′, (i⊕8),i
9	 P(9,1) - i	i′, (i⊕1)′,(i⊕1),(i⊕2),(i⊕2)′,(i⊕3)′,(i⊕3),(i⊕4),(i⊕4)′,(i⊕5)′,(i⊕5),(i⊕6),(i⊕6)′, (i⊕7)′, (i⊕7), (i⊕8), (i⊕8)′,i′
11	 P (11,1) - i′	i, (i⊕1),(i⊕1)′,(i⊕2)′,(i⊕2),(i⊕3),(i⊕3)′,(i⊕4)′,(i⊕4),(i⊕5),(i⊕5)′,(i⊕6)′,(i⊕6), (i⊕7), (i⊕7)′, (i⊕8)′, (i⊕8), (i⊕9), (i⊕9)′, (i⊕10)′, (i⊕10),i
11	 P(11,1) - i	i′, (i⊕1)′,(i⊕1),(i⊕2),(i⊕2)′,(i⊕3)′,(i⊕3),(i⊕4),(i⊕4)′, (i⊕5)′,(i⊕5),(i⊕6),(i⊕6)′, (i⊕7)′, (i⊕7), (i⊕8), (i⊕8)′,(i⊕9)′, (i⊕9), (i⊕10), (i⊕10)′,i′
13	 P (13,1) - i′	i, (i⊕1),(i⊕1)′,(i⊕2)′,(i⊕2),(i⊕3),(i⊕3)′,(i⊕4)′,(i⊕4),(i⊕5),(i⊕5)′,(i⊕6)′,(i⊕6), (i⊕7), (i⊕7)′, (i⊕8)′, (i⊕8),(i⊕9), (i⊕9)′, (i⊕10)′, (i⊕10), (i⊕11), (i⊕11)′, (i⊕12)′,(i⊕11),i
13	 P (13,1) – i	i′, (i⊕1)′,(i⊕1),(i⊕2),(i⊕2)′,(i⊕3)′,(i⊕3),(i⊕4),(i⊕4)′,(i⊕5)′,(i⊕5),(i⊕6),(i⊕6)′, (i⊕7)′, (i⊕7), (i⊕8), (i⊕8)′,(i⊕9)′, (i⊕9), (i⊕10), (i⊕10)′, (i⊕11)′, (i⊕11), (i⊕12),(i⊕12)′, i′

Table 2: Lattice diagram and pattern of 1-fault tolerant Hamiltonian for P(n,2), n=1 (mod 6) or 3(mod 6) for 7 ≤ n ≤ 19

n	The graph, when i = 0	Pattern of 1-fault tolerant Hamiltonian
9	 P(9,2) - i'	i, (i⊕1),(i⊕1)',(i⊕3)',(i⊕5)',(i⊕7)',(i⊕7),(i⊕6),(i⊕5),(i⊕4),(i⊕3),(i⊕2), (i⊕2)', (i⊕4)', (i⊕6)', (i⊕8)', (i⊕8),i
9	 P(9,2) - *i*	i', (i⊕2)',(i⊕4)',(i⊕6)',(i⊕8)',(i⊕8),(i⊕7),(i⊕6),(i⊕5),(i⊕4),(i⊕3),(i⊕2), (i⊕1), (i⊕1)', (i⊕3)', (i⊕5)', (i⊕7)',i'
13	 P(13,2) - *i'*	i, (i⊕1),(i⊕1)',(i⊕3)',(i⊕5)',(i⊕7)', (i⊕9)', (i⊕11)', (i⊕11),(i⊕6),(i⊕10),(i⊕9),(i⊕8),(i⊕7),(i⊕6), (i⊕5), (i⊕4), (i⊕3), (i⊕2), (i⊕2)',(i⊕4)', (i⊕6)', (i⊕8)', (i⊕10)', (i⊕12)', (i⊕12),i
13	 P(13,2) - *i*	i', (i⊕2)',(i⊕4)',(i⊕6)',(i⊕8)',(i⊕10)',(i⊕12)', (i⊕12),(i⊕11),(i⊕10),(i⊕9), (i⊕8),(i⊕7),(i⊕6),(i⊕5), (i⊕4), (i⊕3), (i⊕2), (i⊕1), (i⊕1)',(i⊕3)',(i⊕5)',(i⊕7)',)',(i⊕9)', (i⊕11)', i'
15	 P(15,2) - *i'*	i, (i⊕1),(i⊕1)',(i⊕3)',(i⊕5)',(i⊕7)', (i⊕9)', (i⊕11)', (i⊕13)',(i⊕13), (i⊕12),(i⊕11),(i⊕10),(i⊕9),(i⊕8), (i⊕7), (i⊕6), (i⊕5), (i⊕4), (i⊕3), (i⊕2), (i⊕2)', (i⊕4)', (i⊕6)', (i⊕8)', (i⊕10)', (i⊕12)', (i⊕14)', (i⊕14),i
15	 P(15,2) - *i*	i', (i⊕2)',(i⊕4)',(i⊕6)',(i⊕8)',(i⊕10)',(i⊕12)', (i⊕14)', (i⊕14), (i⊕13), (i⊕12), (i⊕11), (i⊕10),(i⊕9), (i⊕8),(i⊕7),(i⊕6),(i⊕5), (i⊕4), (i⊕3), (i⊕2), (i⊕1), (i⊕1)',(i⊕3)',(i⊕5)',(i⊕7)',)',(i⊕9)', (i⊕11)', (i⊕13)',i'
19	 P(19,2) - *i'*	i, (i⊕1),(i⊕1)',(i⊕3)',(i⊕5)',(i⊕7)', (i⊕9)', (i⊕11)', (i⊕13)', (i⊕15)', (i⊕17)' (i⊕17), (i⊕16),(i⊕15),(i⊕14),(i⊕13), (i⊕12),(i⊕11),(i⊕10), (i⊕9),(i⊕8), (i⊕7), (i⊕6), (i⊕5), (i⊕4), (i⊕3), (i⊕2), (i⊕2)', (i⊕4)', (i⊕6)', (i⊕8)', (i⊕10)', (i⊕12)', (i⊕14)', (i⊕16)', (i⊕18)', (i⊕18),i
19	 P(19,2) - *i*	i', (i⊕2)',(i⊕4)',(i⊕6)',(i⊕8)',(i⊕10), (i⊕12)', (i⊕14)', (i⊕16)', (i⊕18)', (i⊕18), (i⊕17), (i⊕16), (i⊕15), (i⊕14),(i⊕13),(i⊕12), (i⊕11), (i⊕10), (i⊕9),(i⊕8),(i⊕7),(i⊕6), (i⊕5), (i⊕4), (i⊕3), (i⊕2), (i⊕1), (i⊕1)',(i⊕3)', (i⊕5)',(i⊕7)',(i⊕9)', (i⊕11)', (i⊕13)',(i⊕15)', (i⊕17)', i'

The pattern for 1-fault tolerant Hamiltonian of P(7,2) - i′ is : i′, (i⊕2)′,(i⊕4)′,(i⊕6)′,(i⊕6),(i⊕5),(i⊕4), (i⊕3), (i⊕2),(i⊕1),(i⊕1)′, (i⊕3)′, (i⊕5)′,i′

The following figure shows the 1-fault Hamiltonian circuit when i = 0, and ⊕ notates addition modulo 7.

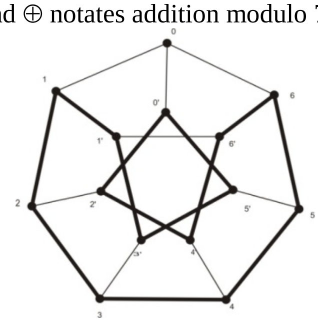


Fig. 5. P (7,2) - 0

For P(n,2), n=1 (mod 6) or 3(mod 6) for 7 ≤ n ≤ 19, the 1-fault tolerant Hamiltonian makes a specific pattern : if the initial point is in the outer polygon (the removing vertex is in the inner polygon), then there is two adjacent edges connected with that point, and the adjacent vertices with the initial point will be connected with the inner polygon in a specific pattern so that half of the vertices in the inner polygon have been connected. The process will continue to connect the other vertices in the outer polygon, then connects again with the vertices in the inner polygon and finally connect with the vertex in the outer polygon which adjacent with the initial vertex. With this pattern we will see that there exist exactly two edges in the outer polygon which not include in the 1-fault tolerant Hamiltonian, where the two edges are not adjacent each other. This pattern also similar if the removing vertex is in the outer polygon, where the resulting 1-fault tolerant Hamiltonian also does not include two edges in the outer polygon, but those two edges are adjacent.

5. Conclusion

From the discussion above we can conclude that for P(n,1), 3 ≤ n ≤ 13, n odd, the lattice diagram that forms 1-fault tolerant Hamiltonian graph make a specific pattern either we remove one vertex in the inner polygon or outer polygon. The pattern is that the point lattice that includes in 1-fault tolerant Hamiltonian periodically changes from outer to inner polygon when two points already in that polygon, except the initial points. Therefore the edges in the 1-fault tolerant Hamiltonian graph interchangeably in the outer and inner polygon, except in the polygon where the initial point is in, where there exist exactly two edges adjacent in that polygon. For P(n,2), n=1 (mod 6) or 3(mod 6) for 7 ≤ n ≤ 19, if the removing vertex in the inner polygon, there exists exactly two edges in the outer polygon which are not include in the 1-fault tolerant Hamiltonian and the two edges are not adjacent each other; while if the removing vertex is in the outer polygon, the resulting 1-fault tolerant Hamiltonian also does not include two edges in the outer polygon, but those two edges are adjacent.

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