Ordered Fuzzy Highly Disconnectedness Space

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Abstract:
In this paper, we give and study a new class of fuzzy topological space called ordered fuzzy highly disconnected space. The upper fuzzy ordered and lower fuzzy ordered space have been defined. On this basis, some interesting properties of these spaces have been obtained.

Keywords:
Ordered fuzzy highly disconnected spaces, upper fuzzy ordered space, lower fuzzy ordered space, ordered fuzzy continuous mapping.

1. Introduction

Ever since the introduction of fuzzy sets by L.A.Zadeh [1], the fuzzy concept has invaded almost all branches of Mathematics. Fuzzy sets have applications in many fields such as information [2] and control [3]. C.L.Chang [4] introduced and developed the theory of fuzzy topological spaces and since then various notion in classical topology have been extended to fuzzy topological space. The concept of ordered fuzzy topological spaces was introduced and developed by A.K.Katsaras [5]. In this paper we introduce and develop the study of ordered fuzzy highly disconnectedness spaces.

2. Notations and Preliminaries

Definition2.1. Let α be any fuzzy set in the ordered fuzzy topological space $(X,U,\leq)$,Then defined

$I(\alpha)=\wedge\{\beta:\beta$ is fuzzy increasing closed and $\beta\geq\alpha\}$

$D(\alpha)=\wedge\{\beta:\beta$ is fuzzy decreasing closed and $\beta\geq\alpha\}$

$I_1(\alpha)=\vee\{\beta:\beta$ is fuzzy increasing open and $\beta\leq\alpha\}$

$D_1(\alpha)=\vee\{\beta:\beta$ is fuzzy decreasing open and $\beta\leq\alpha\}$.

Definition2.2. Let $f:(X,U)\to(Y,V)$ be a mapping form a fuzzy topological space X to another fuzzy topological space Y. f is called

(i) a fuzzy continuous mapping if $f^{-1}(\alpha)\in U$ for each $\alpha\in V$ or equivalently $f^{-1}(\beta)$ is fuzzy closed set of X for each closed set β of Y.

(ii) a fuzzy open/closed mapping if $f(\alpha)$ is open/closed set of Y for each open/closed set α of X. [6]

Definition2.3. For each $\tau\in R$, let $\eta_\tau,\eta^\tau:R(I)\to I$ be given by $\eta_\tau(\alpha)=1-\alpha(\tau)$ and $\eta^\tau(\alpha)=\alpha(\tau)$. Define

$\eta_*=\{\eta_\tau/\tau\in R\}\bigcup\{0.1\}$ and $\eta^*=\{\eta^\tau/\tau\in R\}\bigcup\{0.1\}$

Then η_* and η^* are called I-topologicies on $R(I)$.[7]

Notation. LVC(X) [UVC(X)] denotes the lattice of all lower [upper] semi continuous functions from X to $R(I)$, that is continuous with respect to η_* [η^*].

3. Ordered fuzzy highly disconnectedness space and it's properties

Definition3.1. Let $(X,U,\leq)$ be an ordered fuzzy topological space. Let α be any fuzzy open increasing /decreasing set in $(X,U,\leq)$. If $I(\alpha)/D(\alpha)$ is fuzzy open increasing /decreasing in $(X,U,\leq)$, then $(X,U,\leq)$ is said to be upper/lower fuzzy highly disconnected. $(X,U,\leq)$ is said to be ordered fuzzy highly disconnected if it is both upper and lower fuzzy highly disconnected.

Theorem3.1. For an ordered fuzzy topological space $(X,U,\leq)$ the following four equivalent.

(i) $(X,U,\leq)$ is upper fuzzy highly disconnected.

(ii) For each fuzzy closed decreasing set α,
$D_1(\alpha)$ is fuzzy closed decreasing.

(iii) For each fuzzy open increasing set α, have
$I(\alpha)+D[1-I(\alpha)]=1$.

(iv) For each pair of fuzzy open increasing set α, open decreasing set β in X with $I(\alpha)+\beta=1$, have $I(\alpha)+D(\beta)=1$.

Proof. (i)$\Rightarrow$(ii) Let α be any fuzzy closed decreasing set. Then $1-\alpha$ is fuzzy open increasing set and so by assumption (i), $I(1-\alpha)$ is fuzzy open increasing. As $D_1(\alpha)=1-I(1-\alpha)$, That is $D_1(\alpha)$ is fuzzy closed decreasing.

(ii)$\Rightarrow$(iii) Let α be any fuzzy open increasing set. Then $1-I(\alpha)=D_1(1-\alpha)$ Consider $I(\alpha)+D[1-I(\alpha)]=I(\alpha)+D[D_1(1-\alpha)]$. As α be any fuzzy open increasing ,$1-\alpha$ is fuzzy closed decreasing and by assumption (ii), $D_1(1-\alpha)$ is fuzzy closed decreasing. Therefore $D[D_1(1-\alpha)]=D_1(1-\alpha)$.

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Now $I(\alpha)+D[D_1(1-\alpha)]=I(\alpha)+D_1(1-\alpha)=I(\alpha)+1-I(\alpha)=1$. That is $I(\alpha)+D[1-I(\alpha)]=1$.

(iii)$\Rightarrow$(iv) Let α is any fuzzy open increasing set, β is any fuzzy open decreasing set in X with $I(\alpha)+\beta=1$, so $\beta=1-I(\alpha)$, $D(\beta)=D[1-I(\alpha)]\cdots(1)$. By assumption (iii) $I(\alpha)+D[1-I(\alpha)]=1\cdots(2)$. From (1)and (2) $I(\alpha)+D(\beta)=1$

(iv) $\Rightarrow$ (i) Let α be any fuzzy open increasing set, put $\beta=1-I(\alpha)$. Then β is fuzzy open decreasing and from the construction of β it follows that $I(\alpha)+\beta=1$. By assumption (iv) have $I(\alpha)+D(\beta)=1$ and so $I(\alpha)=1-D(\beta)$ is fuzzy open increasing. Therefore $(X,U,\leq)$ is upper fuzzy highly disconnected.

Theorem3.2. The image $(Y,V,\leq)$ of a upper fuzzy highly disconnected space $(X,U,\leq)$ under fuzzy continuous, fuzzy open mapping and order preserving be also upper fuzzy highly disconnected.

Proof. Let $f:(X,U,\leq)\to(Y,V,\leq)$ is fuzzy continuous and fuzzy open mapping. Let β is fuzzy open increasing in Y. Since f is fuzzy continuous and order preserving, $f^{-1}(\beta)$ be fuzzy open increasing in X. As X is upper fuzzy highly disconnected, $I[f^{-1}(\beta)]$ is fuzzy open increasing in X. Since f is fuzzy continuous, $f^{-1}[I(\beta)]$ $=I[f^{-1}(\beta)]$. Hence $I(\beta)=f\{f^{-1}[I(\beta)]\}=f\{I[f^{-1}(\beta)]\}$ $\leq I\{f[f^{-1}(\beta)]\}=I(\beta)$, then $f\{I[f^{-1}(\beta)]\}=I(\beta)$. Since f is fuzzy open and order preserving, $I[f^{-1}(\beta)]$ is fuzzy open increasing in X, so $f\{I[f^{-1}(\beta)]\}=I(\beta)$ is fuzzy open increasing in Y. This proves that Y be upper fuzzy highly disconnected

Definition3.2. An ordered fuzzy topological space $(X,U,\leq)$ is upper fuzzy ordered if for every fuzzy closed increasing set α and fuzzy open increasing set β such that $\alpha\leq\beta$,there exists a fuzzy open increasing set δ such that $\alpha\leq\delta\leq I(\delta)\leq\beta$. Similarly we can define lower fuzzy ordered space. A ordered fuzzy topological space which is both upper and lower fuzzy ordered is called fuzzy ordered.

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Theorem3.3. If X is upper fuzzy highly disconnected and upper fuzzy ordered space. Let $Y\subset X$ be such that γ_Y is fuzzy closed increasing and $P\subset Y$ be such that γ_P is fuzzy closed increasing and open increasing. Then there exists a fuzzy closed increasing set μ in X such that $\mu\wedge\gamma_Y=\gamma_P$.

Proof. As γ_P is fuzzy open increasing in Y, there exists a fuzzy open increasing set α in X such that $\alpha\wedge\gamma_Y=\gamma_P$. Since γ_Y is fuzzy closed increasing in X, γ_P is also fuzzy closed increasing in X. Now α is a fuzzy open increasing set in upper fuzzy ordered space X such that $\gamma_P<\gamma_Y$. Therefore there exists a fuzzy open increasing set δ in X such that $\gamma_P<\delta<I(\delta)<\alpha$. Let $\mu=I(\delta)$, as X is upper fuzzy highly disconnected, $I(\delta)$ is fuzzy open increasing in X. Thus $\mu\wedge\gamma_Y=I(\delta)\wedge\gamma_Y\leq\alpha\wedge\gamma_Y=\gamma_P$;

Other $\gamma_P<\mu$ and $\gamma_P\leq\gamma_Y$, have $\gamma_P\leq\mu\wedge\gamma_Y$, thus $\mu\wedge\gamma_Y=\gamma_P$.

Theorem3.4. Let $(X,U,\leq)$ is ordered fuzzy topological space and $(Y,V,\leq)$ is upper fuzzy highly disconnected space. Let $f:(X,U,\leq)\to(Y,V,\leq)$ is fuzzy continuous, order preserving and fuzzy open mapping. Then $(X,U,\leq)$ is an upper fuzzy highly disconnected space.

Proof. Let α is a fuzzy open increasing set in $(X,U,\leq)$. As f is order preserving and fuzzy open mapping, $f(\alpha)$ is a fuzzy open increasing set in $(Y,V,\leq)$.Since Y is upper fuzzy highly disconnected space, $I[f(\alpha)]$ is fuzzy open increasing. As f is fuzzy continuous, order preserving and fuzzy open mapping, $f^{-1}\{I[f(\alpha)]\}=I(\alpha)$ and $f^{-1}\{I[f(\alpha)]\}$ is fuzzy open increasing. That is $I(\alpha)$ is fuzzy open increasing. Therefore $(X,U,\leq)$ is an upper fuzzy highly disconnected space.

Theorem3.5. Let $(X,U,\leq)$ is a ordered fuzzy topological space. Then $(X,U,\leq)$ be upper fuzzy highly disconnected iff for all fuzzy decreasing open set α and decreasing closed set β such that $\alpha\leq\beta$, then $D(\alpha)\leq D_1(\beta)$.

Proof. Let $(X,U,\leq)$ be upper fuzzy highly disconnected, α be any fuzzy open decreasing set, β is any fuzzy closed decreasing set and $\alpha\leq\beta$. Then by (ii) Theorem3.1, $D_1(\beta)$ is fuzzy closed decreasing. As α is fuzzy open decreasing and $\alpha\leq\beta$, thus $D(\alpha)\leq D_1(\beta)$. Conversely, let β is any fuzzy closed decreasing set, then $D_1(\beta)$ is fuzzy open decreasing. So $D_1(\beta)\leq\beta$. Therefore by assumption have $D[D_1(\beta)]\leq D_1(\beta)$. This implies that $D_1(\beta)$ is fuzzy closed decreasing. Hence by (ii) Theorem3.1, has $(X,U,\leq)$ be upper fuzzy highly disconnected.

Remark3.1. If $(X,U,\leq)$ be an upper fuzzy highly disconnected space. Let $(\alpha_i,1-\beta_i), i\in N$ be a collection such that fuzzy open decreasing sets, β_j be fuzzy closed decreasing sets and let $\alpha, 1-\beta$ be fuzzy open decreasing and fuzzy open increasing respectively. If $\alpha_i \le \alpha, \beta \le \beta_j$ for all $i, j \in N$, then there exists fuzzy open decreasing set μ such that $D(\alpha_i) \le \mu \le D_1(\beta_j)$ for all $i, j \in N$.

By the above property $D(\alpha_i) \le D(\alpha) \wedge D_1(\beta) \le D_1(\beta_j)$, $i, j \in N$. Setting $\mu = D(\alpha) \wedge D_1(\beta)$, μ satisfies our required condition.

Theorem3.6. If $(X, U, \le)$ be an upper fuzzy highly disconnected space. Let $\{\alpha_q\}_{q\in Q}$ and $\{\beta_q\}_{q\in Q}$ be monotone increasing collections of fuzzy open decreasing sets and fuzzy closed decreasing sets of X and suppose that $\alpha_{q_1} \le \beta_{q_2}$ whenever $q_1 < q_2$ (Q is the set of rational numbers). Then there exists a monotone increasing collection $\{\mu_q\}_{q\in Q}$ of fuzzy open decreasing sets of X such that $D(\alpha_{q_1}) \le \mu_{q_2}$ and $\mu_{q_1} \le D_1(\beta_{q_2})$ whenever $q_1 < q_2$.

Proof. Let us arrange into sequence $\{q_n\}$ of rational numbers without repetitions. For every $n \ge 2$, we shall define inductively a collection $\{\mu_{q_i}; 1 \le i \le n\}$ such that

$$\left.\begin{array}{ll} D\,(\alpha_q) \le \mu_{q_i} & if\ \ q < q_i \\ \mu_{q_i} \le D_1(\beta_q) & if\ \ q_i < q \end{array}\right\} S_n, \text{ for all } i < n.$$

By Theorem3.5 the family $\{D(\alpha_q)\}$ and $\{D_1(\beta_q)\}$ satisfying $D(\alpha_{q_1}) \le D_1(\beta_{q_2})$ if $q_1 < q_2$. By Remark3.1 there exists fuzzy open decreasing set δ_1 such that $D(\alpha_1) \le \delta_1 \le D_1(\beta_{q_2})$. Seting $\mu_{q_1} = \delta_1$ get S_2 . Assume that fuzzy sets μ_{qi} are already defined for $i < n$ and satisfy S_n . Define $\Phi = \vee\{\mu_{qi} : i < n, q_i < q_n\} \vee \alpha_{q_n}$ and $\Psi = \wedge\{\mu_{q_j} : j < n, q_j > q_n\} \wedge \beta_{q_n}$. Then we have that $D(\mu_{q_i}) \le D(\Phi) \le D_1(\mu_{q_j})$ and $D(\mu_{q_i}) \le D_1(\Psi) \le D_1(\mu_{q_j})$ whenever $q_i < q_n < q_j\, (i, j < n)$ as well as $\alpha_q \le D(\Phi) \le \beta_{q'}$ and $\alpha_q \le D_1(\Psi) \le \beta_{q'}$ whenever $q < q_n < q'$. This shows that the countable collection $\{\mu_{q_i} : i < n, q_i < q_n\} \bigcup \{\alpha_q : q < q_n\}$ and $\{\mu_{q_j} : j < n, q_j > q_n\} \bigcup \{\beta_q : q > q_n\}$ together with Φ and Ψ fulfill all conditions of Remark3.1. Hence there exists a fuzzy open decreasing set δ_n such that $D(\delta_n) \le \beta_q$ if $q_n < q$, $\alpha_q \le D_1(\delta_n)$ if $q < q_n$, $D(\mu_{q_i}) \le D_1(\delta_n)$ if $q_i < q_n$, $D(\delta_n) \le D_1(\mu_{qj})$ if $q_n < q_j$, where $1 \le i, j \le n-1$. Now setting $\mu_{qn} = \delta_n$ we obtain the fuzzy sets $\mu_{q1}, \mu_{q2}, \cdots, \mu_{qn}$ that satisfy S_{n+1} . Therefore the collection $\{\mu_{qi} : i = 1, 2\cdots\}$ has required property. This completes the proof.

Definition3.3. Let $(X, U, \le)$ and $(Y, V, \le)$ be ordered fuzzy topological space. A mapping $f : (X, U, \le) \to (Y, V, \le)$ is called increasing/decreasing fuzzy continuous if $f^{-1}(\alpha)$ is fuzzy open (closed) increasing/decreasing set of X for every fuzzy open (closed) set α of Y. If f is both increasing and decreasing then it is called ordered fuzzy continuous.

Theorem3.7. Let $(X, U, \le)$ is a ordered fuzzy topological space. Then the following statements are equivalent.

(i) $(X, U, \le)$ is upper fuzzy highly disconnected.

(ii) If $g \in \text{LVC(X)}$, $h \in \text{UVC(X)}$ and $g \le h$, then there exists an increasing fuzzy continuous function $f : (X, U, \le) \to R(I)$, such that $g \le f \le h$.

(iii) If $1-\alpha, \beta \in U$ and $\beta \le \alpha$ then there exists an increasing fuzzy continuous function $f : (X, U, \le) \to R(I)$ such that $\beta \le R_{0f} \le \alpha$.

Proof. (i) $\Rightarrow$ (ii) define $H_\tau = \eta_\tau h$ and $G_\tau = (1-\eta^\tau)g$, $\tau \in Q$. Thus get two monotone increasing families of respectively open decreasing and closed decreasing sets of X. Moreover $H_\tau \le G_s$ if $\tau < s$. By Theorem3.6 there exists a monotone increasing family $\{F_\tau\}_{\tau\in Q}$ of fuzzy open decreasing sets of X such that $D(H_\tau) \le F_s$ and $F_\tau \le D_1(G_s)$ whenever $\tau < s$. Define $V_t = \wedge_{\tau<t}(1-F_\tau)$ for all $t \in R$, we have $\vee_{t\in R} V_t = \vee_{t\in R} \wedge_{\tau<t} (1-F_\tau) \ge \vee_{t\in R} \wedge_{\tau<t} (1-G_\tau)$ $= \vee_{t\in R} \wedge_{\tau<t} g^{-1}(1-\eta_\tau) = \vee_{t\in R} g^{-1}(1-\eta_t) = g^{-1}[\vee_{t\in R}(1-\eta_t)] = 1$. Let a function $f : (X, U, \le) \to R(I)$ satisfying the required properties $f(x)(t) = V_t(x)$ for all $x \in X$ and $t \in R$. To prove f is a increasing fuzzy continuous function. We observe that $\vee_{s>t} V_s = \vee_{s>t} I_1(V_s)$, $\wedge_{s<t} V_s = \wedge_{s<t} I(V_s)$. Then $f^{-1}(\eta^t) = \vee_{s>t} V_s = \vee_{s>t} I_1(V_s)$ is fuzzy open increasing. $f^{-1}(1-\eta_t) = \wedge_{s<t} V_s = \wedge_{s<t} I(V_s)$ is fuzzy closed increasing, so that f is increasing fuzzy continuous. To conclude the proof it remains to show that $g \le f \le h$, that is $g^{-1}(1-\eta_t)$ $\le f^{-1}(1-\eta_t) \le h^{-1}(1-\eta_t)$ and $g^{-1}(\eta^t) \le f^{-1}(\eta^t) \le h^{-1}(\eta^t)$ for each $t \in R$. We have $g^{-1}(1-\eta_t) = \wedge_{s<t} g^{-1}(1-\eta_s)$ $= \wedge_{s<t} \wedge_{r<s} g^{-1}(1-\eta_r) \le \wedge_{s<t} \wedge_{r<s} (1-F_r) = \wedge_{s<t} V_s = f^{-1}(1-\eta_t)$ and $f^{-1}(1-\eta_t) = \wedge_{s<t} \wedge_{r<s} (1-F_r) \le \wedge_{s<t} \wedge_{r<s} (1-H_r)$ $= \wedge_{s<t} \wedge_{r<s} h^{-1}(\eta^r) = \wedge_{s<t} h^{-1}(1-\eta_s) = f^{-1}(1-\eta_t)$. Similarly we obtain $g^{-1}(\eta^t) = \vee_{s>t} g^{-1}(\eta^s) = \vee_{s>t} \vee_{r>s} g^{-1}(1-\eta_r) \le \vee_{s>t} \wedge_{r<s} (1-F_r)$ $= \vee_{s>t} V_s = f^{-1}(\eta^t)$ and $f^{-1}(\eta^t) = \vee_{s>t} V_s = \vee_{s>t} \wedge_{r<s} (1-F_r)$ $\le \vee_{s>t} \vee_{r>s} (1-H_r) = \vee_{s>t} \vee_{r>s} h^{-1}(\eta^r) = \vee_{s>t} h^{-1}(\eta^s) = h^{-1}(\eta^t)$. Thus (i) $\Rightarrow$ (ii) is proved.

(ii) $\Leftrightarrow$ (iii) Suppose $1-\alpha$, $\beta \in U$ and $\beta \leq \alpha$. Then $\gamma_\beta \leq \gamma_\alpha$ and $\gamma_\beta \in LVC(X)$, $\gamma_\alpha \in UVC(X)$. Hence by (ii) there exists increasing fuzzy continuous function $f:(X,U,\leq) \rightarrow R(I)$ such that $\gamma_\beta \leq f \leq \gamma_\alpha$. Clearly $f(x) \in R(I)$ for all $x \in X$ and $\beta = (1-\eta_1)\gamma_\beta \leq (1-\eta_1) f \leq \eta^1 f \leq \eta^1 \gamma_\alpha = \alpha$.

(iii) $\Rightarrow$ (i) This follows Theorem3.5 and the fact that $(1-\eta_1) f$ and $\eta^1 f$ are fuzzy closed decreasing and fuzzy open decreasing set respectively. So the result.

Theorem3.8. Let $(X,U,\leq)$ is a upper fuzzy highly disconnected space and $K \subset X$ be such that $\beta_K \in U$. Let $f:(K,U/K) \rightarrow R(I)$ be increasing fuzzy continuous. Then f has an increasing fuzzy continuous extension over X.

Proof. Let $g,h: X \rightarrow R(I)$ be such that $g = f = h$ on K , and $g(x) = (0)$, $h(x) = (1)$. Have $\eta^t g = \beta_t \wedge \gamma_K$ where $\beta_t \in U$ is such that $\beta_t / K = \eta^t f$ and $\eta_t h = \alpha_t \wedge \gamma_K$ where $\alpha_t \in U$ is such that $\alpha_t / K = \eta_t f$. Thus $g \in LVC(X)$, $h \in UVC(X)$ and $g \leq h$. By Theorem3.7 there is an increasing fuzzy continuous function $F:(K,U,\leq) \rightarrow R(I)$ such that $g \leq F \leq h$ and $F = f$ on K .

References

[1] L.A.Zadeh, Fuzzy sets, Information and Control, 8(1965)338-353.

[2] P.Smets, The degree of belief in a fuzzy event, Information Sciences, 25(1981)1-19.

[3] M.Sugeno, An introductory survey of fuzzy control, Information Sciences, 36(1985)59-83.

[4] C.L.Cheng, Fuzzy topological space, J.Math. Anal. Appl., 24(1968) 182 - 190 .

[5] A.K.Katsaras, Ordered fuzzy topological space, J.Math. Anal. Appl., 84 (1981)44-58.

[6] K.K.Azad, On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly continuity, J.Math. Anal. Appl., 82(1982)14-32.

[7] G.Thangaraj and G.Balasubramanian, On fuzzy basically disconnected spacs, The Journal of Fuzzy Mathematics, 9(2001)103-110.

[8] H. Wang, Image and inverse image of refinement of fuzzy topogenous Order, International Journal of Computer Science and Network Security, 15(1) (2015) 31-35.

[9] H. Wang, The refinement of fuzzy topogenous structure and It's properties. Fuzzy Systems and Math. 25(4) (2011), 83-86