On Construction of framelet packets on local fields with positive characteristic

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Abstract
In the present paper, using the prime member of local field with positive characteristic, some concept of multiresolution analysis (MRA) and wavelets were extended to such fields. The separation lemma was proved and applying this lemma, the wavelet packets related to multiresolution analysis was built. Also it is shown that these wavelet packets translations produces one unique orthogonal basis for desired local fields. The separation lemma for frames was proved and then, it was used to construct framelets.

Keywords:
wavelet, multiresolusion analysis, local field, wavelet packets, wavelet frame packets, positive characteristic

1. Introduction

Wavelets are of the family of dilation and includes the transformation of a function called mother wavelet (Daubechies, 1992). Wavelets were introduced in early 1980 and attracted many people of mathematical community and other fields in which wavelet can play useful role and the results of the interest were several books and high volume of research articles about them (Grochenig, 2000)

Wavelets have different application in several fields, for example, (Jiang, 2004) wavelet transform is used as a tool for analyzing frequency and time (Zheng, 1990) and wavelet analysis is used as mathematical microscope which is closely related to harmonic analysis and approximation theory. In this study, it was used for studying multiresolution analysis on local field and building the corresponding wavelet (Zheng, 1997)

Dahlke (Dahlke, 1994) introduced wavelets and multiresolution. Han et al (Stavropoulos, 1998) have developed these concepts in Hilbert spaces and Lemarie (Lemarie, 1989) examined them on Lie groups.

2. Method

Local field : Definition: Suppose that $\mathbb{k}$ is a field and topological space. In this case, $\mathbb{k}$ is called locally compact field or a local field, if $\mathbb{k}^+$ and $\mathbb{k}^*$ are locally compact abelian groups. $\mathbb{k}^+$ and $\mathbb{k}^*$ are collective and multiplicative groups of field $\mathbb{k}$, respectively.

2.1 Definition

commutative ring R is called integer domain if $a.b = 0$ in R , results in $a = 0$ or $.b = 0$.

2.2 Definition

It is said that a field $\mathbb{F}$ is with characteristic $p \neq 0$, if for every $x \in \mathbb{F}$, $px = 0$ and no positive integer smaller than p has such characteristic. In this case char($\mathbb{F}$)=p

Remark : If the field $\mathbb{F}$ has no the characteristic of p for every positive integer of p , it is called a field characterized by 0.

2.3 Example

$\mathbb{Q}$ and $\mathbb{R}$ and $\mathbb{C}$ are the fields characterized by zero, $\mathbb{Z}_3$ is a field characterized by 3.

2.4 Theorem

A field is characterized by zero or prime number.

2.5 Corollary

If $\mathbb{F}$ is a finite field of $\text{card}\mathbb{F} \leq \infty$, then it is characterized by prime number.

2.6 Theorem

[7] If $\mathbb{F}$ is a finite field, then $\text{card}\mathbb{F} - p^n$ where $n \in \mathbb{N}$ and $.p = \text{char}\mathbb{F}$.

2.7 Example

If p is prime number, then $\mathbb{Z}_p = \frac{\mathbb{Z}}{\langle p \rangle}$ is a field.

2.8 Theorem

For every prime number p and every positive integer n, there is a unique field with p^n members.

2.9 Definition

If p is prime number and $n \geq 1$ is integer, then a unique field with p^n members is called Galois field of order p^n and represented by $\mathbb{GF}(p^n)$ or $\mathbb{F}_{p^n}$.

Remark: Given the result 1 and theorem 2 we have $\text{char}\mathbb{GF}(p^n) = p$

2.10 Definition

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Suppose that $\mathbb{F}$ is a field. In this case, $\mathbb{F}((t))$ is the set of all formal infinite sums of $c_{n_0}t^{n_0} + c_{n_0+1}t^{n_0+1} + \cdots$, with the variable t where $n_0 \in \mathbb{Z}$ and $.(n \in \{n_0, n_0+1, \dots\})\ c_n \in \mathbb{F}$. If $g(t) = c_{n_0}t^{n_0} + c_{n_0+1}t^{n_0+1} + \cdots$, $f(t) = b_{n_0}t^{n_0} + b_{n_0+1}t^{n_0+1} + \cdots$, belong to $\mathbb{F}((t))$, then, it will be easily shown that $\mathbb{F}((t))$ is a field and called formal Laurent series field. It is clear that field k with the discrete topology is a local field, so from here, k is a local field with non-discrete topology.

2.11 Definition

Suppose that G is a group. Then the map $f: G \to G$ is called homomorphism map if f is surjective and injective homomorphism.

2.12 Definition

Suppose that G is a locally compact group with the Haar measure μ. Then, homomorphism (continuous) f on G is considered, if E is a Borel subset of G, then $f(E)$ is a Borel subset of G and $\mu o f$ is a Haar measure of G, given the uniqueness of Haar measure, there is $c > 0$ that $\mu o f = c\mu$, in this case c is called module of f and represented by $\mathrm{mod}_G(f)$. Given the definition: $\mu(f(E)) = \mathrm{mod}_G(f)\mu(E)$ For every subset of E from G.

3. Results

Multiresolution analysis for $L^2(\mathrm{k})$

From now on, by K we mean a discrete local field with positive character char(K)=p>0.

3.1 Definition

Let K be a local field with positive character p. Then we define the following subsets of K:

$D = \{x \in K : |x| \le 1\}, \qquad B = \{x \in K : |x| < 1\}$

Where D is the integer domain and B is called the prime ideal of K.

3.2 Theorem

Finite field GF(q) ($q=p^c$)where p is prime number and $c \in \mathbb{N}$, is a vector space with the dimension c on finite field GF(q).

3.3 Proposition

The quotient space $\frac{D}{B}$ is isomorphic with the finite dimensional field GF(q).

Given the above proposition, $q = p^c)\ \frac{D}{B} \cong \mathrm{GF}(q) := L$ where p is prime number and $c \in \mathbb{N}$. Standard homomorphism $\pi : D \longrightarrow L$ from D on L is considered, given the previous theorem, L is a vector space with the dimension c on finite field GF (p). Now, the set $\{1 = \epsilon_0, \epsilon_1, \dots, \epsilon_{c-1}\} \subseteq D^* \subset D$ is selected where $\{\pi(\epsilon_k)\}_{k=0}^{c-1}$ is a basis for L on finite field GF(p).

3.4 Definition

Suppose that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and ρ are the prime members of local field k. in this case, for $n \in \mathbb{N}_0$ and $0 \le n < q$ ($q=p^c$ where p is prime number and $c \in \mathbb{N}$), we have:

$$0 \le a_k < p, \quad k = 0, 1, \dots, c-1,$$
$$n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1},$$

We define:

$$u(n) = (a_0 + a_1\epsilon_1 + \ldots + a_{c-1}\epsilon_{c-1})\rho^{-1}, \qquad 0 \le n < q,$$

For $n \ge 0$, we have:

$$n = b_0 + b_1 q + \cdots + b_s q^s, \qquad 0 \le b_k < q, \quad k = 0, 1, \dots, s,$$
$$u(n) = u(b_0) + u(b_1)\rho^{-1} + \ldots + u(b_s)\rho^{-s}.$$

3.5 Proposition

Suppose that $\{u(n) : n \in \mathbb{N}_0\}$ is a set defined above. In this case,

1. $\{u(n)\rho\}_{n=0}^{q-1}$ is a set of all equivalence classes of B in D, therefore

$$:\{u(n)\rho\}_{n=0}^{q-1} \cong \frac{D}{B} \cong \mathrm{GF}(q) \cong span\{\epsilon_j\}_{j=0}^{c-1}$$

2. $\{u(n)\}_{n=0}^{q-1}$ is a set of all equivalence classes of D in B^{-1}.

3.6 Proposition

For every $n, m \in \mathbb{N}$, the equation $u(n+m) = u(n) + u(m)$ is not established, but if $0 \le s < q^k,\ r, k \ge 0$, then $u(rq^k + s) = u(r)\rho^{-k} + u(s)$.

3.7 Proposition

If $k, l \in \mathbb{N}_0$ and χ is a characteristic defined on K, then

$$\chi(\epsilon_\mu \rho^{-j}) = \begin{cases} \exp(\frac{2\pi i}{p}) & \mu = 0, j = 1 \\ 1 & \mu = 1, \dots, c-1 \ or j \neq 1 \end{cases},$$

3.8 Wavelet frame packets (Framelets)

Definition: Suppose that H is a separable Hilbert space. The sequence $\{x_k : k \in \mathbb{Z}\}$ in the Hilbert space H is a frame for H, if there are the constants c1 and c2, such that for every $x \in H$, the following equation holds:

$$C_1\|x\|^2 \le \sum_{k \in \mathbb{Z}} |\langle x, x_k\rangle|^2 \le C_2\|x\|^2.$$

The largest c1 and the smallest c2 which are satisfied in above equation, are called the bounds of frame.

Now we are going to establish some part of classical frame theory, for L2 (K) where K is a local field of positive characteristic.

Suppose that $\Phi=\{\varphi_1,\varphi_2,\ldots,\varphi_N\}\subset L^2(\mathbb{K})$ where the following system

$\{\varphi_l(.-u(k)):1\leq l\leq N, k\in\mathbb{N}_0\},$

is a frame for

$S(\Phi)=\overline{span}\{\varphi_l(.-u(k)):1\leq l\leq N, k\in\mathbb{N}_0\},$

Suppose that $\{\psi_1,\psi_2,\ldots,\psi_N\}$ is a subset of $S(\Phi)$, there is a question:

When can it be said that $\{\psi_l(.-u(k)):1\leq l\leq N, k\in\mathbb{N}_0\}$ is also a frame for $S(\Phi)$?

If $\psi_j\in S(\Phi)$, then, there is a sequence $\{r_{jlk}:k\in\mathbb{N}_0\}$ in $l^2(\mathbb{N}_0)$ where $\psi_j(x)=\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0} r_{jlk}\varphi_l(x-u(k)),$

By applying Fourier transform, we have:

$$\hat{\psi}_j(\xi)=\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0} r_{jlk}\overline{\chi_k(\xi)}\hat{\varphi}_l(\xi)$$
$$=\sum_{l=1}^{N} R_{jl}(\xi)\hat{\varphi}_l(\xi),$$

$R_{jl}(\xi)$ و $R_{jl}(\xi)=\sum_{k\in\mathbb{N}_0} r_{jlk}\overline{\chi_k(\xi)}$ is an integer periodic function, because for every $n\in\mathbb{N}_0$, we have:

$$R_{jl}(\xi+u(n))=\sum_{k\in\mathbb{N}_0} r_{jlk}\overline{\chi_k(\xi+u(n))}$$
$$=\sum_{k\in\mathbb{N}_0} r_{jlk}\overline{\chi_k(\xi)\chi_k(u(n))}=\sum_{k\in\mathbb{N}_0} r_{jlk}\overline{\chi_k(\xi)}=R_{jl}(\xi).$$

So our discussion about ψ_{jk} to be frame or not, reduced to these periodic functions.

3.7 Proposition

If $k,l\in\mathbb{N}_0$ and χ is a characteristic on K, i.e., $\chi(\rho^{-1})=\exp(\frac{2\pi}{p})\neq 1$, then $\chi_k(u(l))=1$. If k=0 or k=1, then it is clear that $\chi_k(u(l))=\chi(u(k)u(l))=\chi(0)=1,$

4. Discussion

Separation lemma for wavelet frame packets

The main idea of separation lemma is building wavelet frame packets and following theorem.

Theorem: Suppose that $\{\varphi_l:1\leq l\leq N\}\subset L^2(\mathbb{K})$ where $\{\varphi_l(.-u(k)):1\leq l\leq N, k\in\mathbb{N}_0\},$ is a frame for $V_0=\overline{span}\{\varphi_l(.-u(k)):1\leq l\leq N, k\in\mathbb{N}_0\}$ with the bounds of c1 and c2, if $\{\psi_l^n:n\geq 0, 1\leq l\leq N\}$ is wavelet frame packets and $V_j=\{f\in L^2(\mathbb{K}):f(\rho^j.)\in V_0\},$ then for every $j\geq 0$ the set $\{\psi_l^n(.-u(k)):0\leq n\leq q^j-1, 1\leq l\leq N, k\in\mathbb{N}_0\},$ is of the functions of a frame for V_j with the bounds of λjc1 and λjc2.

To clarify this theorem and to prove it, following items are used.

Definition: Suppose that $\{\varphi_j:1\leq j\leq N\}$ is a set of functions in $L^2(\mathbb{K})$ where $\{\varphi_j(.-u(k)):1\leq j\leq N, k\in\mathbb{N}_0\},$ is a frame for $V=\overline{span}\{\varphi_j(.-u(k)):1\leq j\leq N, k\in\mathbb{N}_0\},$ and there is a sequence $\{h_{ljk}^r:k\in\mathbb{N}_0\}\in l^2(\mathbb{N}_0),$ for $0\leq r\leq q-1$ ، $1\leq l\leq N$, in this case, we define:

$$\psi_l^r(x)=q^{\frac{1}{2}}\sum_{j=1}^{N}\sum_{k\in\mathbb{N}_0} h_{ljk}^r\varphi_j(\rho^{-1}x-u(k)).$$

With the Fourier transform of above equation, we have:

$$\hat{\psi}_l^r(\xi)=\sum_{j=1}^{N}\sum_{k\in\mathbb{N}_0} h_{ljk}^r q^{-\frac{1}{2}}\overline{\chi_k(\rho\xi)}\hat{\varphi}_j(\rho\xi)=\sum_{j=1}^{N} h_{lj}^r\hat{\varphi}_j(\rho\xi),$$

$$h_{lj}^r(\xi)=\sum_{k\in\mathbb{N}_0} q^{-\frac{1}{2}}h_{ljk}^r\overline{\chi_k(\xi)}.$$

We also define

$H_r(\xi)=\left(h_{lj}^r(\xi)\right)_{1\leq l,j\leq N},$

and

$H(\xi)=\left(H_r(\xi+\rho u(s))\right)_{0\leq r,s\leq q-1},$

It is clear that $H(\xi)$ is a square matrix of order qN.

Lemma : ψ_l^r can be written as follows:

$$\psi_l^r(x)=\sum_{j=1}^{N}\sum_{k\in\mathbb{N}_0} h_{ljk}^r q^{\frac{1}{2}}\varphi_j(\rho^{-1}x-u(k))$$
$$=\sum_{j=1}^{N}\sum_{s=0}^{q-1}\sum_{k\in\mathbb{N}_0} h_{lj,qk+s}^r q^{\frac{1}{2}}\varphi_j(\rho^{-1}x-u(qk+s))$$
$$=\sum_{j=1}^{N}\sum_{s=0}^{q-1}\sum_{k\in\mathbb{N}_0} h_{lj,qk+s}^r q^{\frac{1}{2}}\varphi_j(\rho^{-1}x-\rho^{-1}u(k)-u(s)))$$
$$=\sum_{j=1}^{N}\sum_{s=0}^{q-1}\sum_{k\in\mathbb{N}_0} h_{lj,qk+s}^r\varphi_j^s(x-u(k)),$$

(8)

That proposition top is applied above, where

$\varphi_j^s(x)=q^{\frac{1}{2}}\varphi_j(\rho^{-1}x-u(s))),\qquad 0\leq s\leq q-1,$

With the Fourier transform of Eq. (8), we have:

$$(\psi_l^r)^\wedge(\xi)=\sum_{j=1}\sum_{s=0}\sum_{k\in\mathbb{N}_0} h_{lj,qk+s}^r\overline{\chi_k(\xi)}(\varphi_j^s)^\wedge(\xi)$$
$$=\sum_{j=1}^{N}\sum_{s=0}^{q-1} a_{lj}^{rs}(\xi)(\varphi_j^s)^\wedge(\xi),$$

where $a_{lj}^{rs}(\xi)=\sum_{k\in\mathbb{N}_0} h_{lj,qk+s}^r\overline{\chi_k(\xi)}$.

Now, the matrices $A(\xi)$ و $A^{rs}(\xi)$ are defined as follows:

$A^{rs}(\xi)=\left(a_{lj}^{rs}(\xi)\right)_{1\leq l,j\leq N},\qquad A(\xi)=\left(a^{rs}(\xi)\right)_{0\leq r,s\leq q-1}.$

then put:

$\varepsilon_{lj}^{rs}(\xi)=\delta_{lj}q^{-\frac{1}{2}}\overline{\chi(u(r)(\xi+\rho u(s)))},$

Now $E^{rs}(\xi)=\left(\varepsilon_{lj}^{rs}(\xi)\right)_{1\leq l,j\leq N},$ almost everywhere for $\xi\in D$ ،,

now we define matrix $E(\xi)$ as follows: $E(\xi)=\left(E^{rs}(\xi)\right)_{0\leq r,s\leq q-1}.$

Proposition: Show that for $1 \le l, j \le N$ ، $0 \le r, s \le q-1$, $H(\xi) = A(\rho^{-1}\xi)E(\xi)$ that $E(\xi)$ is a unitary matrix defined in above equation.

Proof: Given the definition, it is clear that (r,s)th block of matrix $A(\rho^{-1}\xi)E(\xi)$ is the matrix $\sum_{t=0}^{q-1} A^{rt}(\rho^{-1}\xi)E^{ts}(\xi)$, (l, j)th component of this block is equal to

$$\sum_{t=0}^{q-1}\sum_{m=1}^{N} a_{lm}^{rt}(\rho^{-1}\xi)\varepsilon_{mj}^{ts}(\xi) = \sum_{t=0}^{q-1}\sum_{m=1}^{N}\sum_{k\in N_0} h_{lm,qk+t}^{r}\overline{\chi_k(\rho^{-1}\xi)}\delta_{mj}q^{-\frac{1}{2}}\overline{\chi(u(t)(\xi+\rho u(s)))}$$

(Given the definition ($\varepsilon_{mj}^{ts}(\xi)$ و $a_{lm}^{rt}(\xi)$))

$$= \sum_{t=0}^{q-1}\sum_{k\in N_0} h_{lj,qk+t}^{r}\overline{\chi_k(\rho^{-1}\xi)}q^{-\frac{1}{2}}\overline{\chi(u(t)(\xi+\rho u(s)))},$$

Now, (l, j)th component of (r,s)th block of matrix $H(\xi)$ is

$$h_{lj}^{r}(\xi+\rho u(s)) = q^{-\frac{1}{2}}\sum_{k\in N_0} h_{ljk}^{r}\overline{\chi(u(k)(\xi+\rho u(s)))}$$

$$= q^{-\frac{1}{2}}\sum_{t=0}^{q-1}\sum_{k\in N_0} h_{lj,qk+t}^{r}\overline{\chi(u(qk+t)(\xi+\rho u(s)))}$$

$$= q^{-\frac{1}{2}}\sum_{t=0}^{q-1}\sum_{k\in N_0} h_{lj,qk+t}^{r}\overline{\chi(\rho^{-1}u(k)\xi+u(k)u(s)+u(t)\xi+\rho u(t)u(s))}$$

$$= q^{-\frac{1}{2}}\sum_{t=0}^{q-1}\sum_{k\in N_0} h_{lj,qk+t}^{r}\overline{\chi_k(\rho^{-1}\xi)\chi(u(t)(\xi+\rho u(s)))},$$

If the last equality holds, then, $\cdot\chi_k(u(l)) = 1$.

If $k = 0$ or $l = 0$, then, it is clear that

$$\chi_k(u(l)) = \chi(u(k)u(l)) = \chi(0) = 1,$$

, because (l, j) is selected arbitrarily. From Eq.(11) and Eq.(12), it is concluded that all components of the matrices $H(\xi)$ and $A(\rho^{-1}\xi)E(\xi)$ are equal.

Corollary : Given above proposition, we have:

$$H^*(\xi)H(\xi) = E^*(\xi)A^*(\rho^{-1}\xi)A(\rho^{-1}\xi)E(\xi),$$

Because $E(\xi)$ is a unitary matrix, so the matrices $H^*(\xi)H(\xi)$ و $A^*(\rho^{-1}\xi)A(\rho^{-1}\xi)$ are similar.

Definition 5.6 Suppose that $\Lambda(\xi)$ و $\lambda(\xi)$ are the maximum and minimum members of the set of all eigenvalues of the positive definite matrix of $H^*(\xi)H(\xi)$, in this case, put $.\Lambda = \sup_\xi \Lambda(\xi)$ و $\lambda = \inf_\xi \lambda(\xi)$.

Lemma: Almost everywhere for $\xi \in D$, we have:

$$\lambda I \le H^*(\xi)H(\xi) \le \Lambda I. \quad (13)$$

Proof. Because $H^*(\xi)H(\xi)$ is a positive definite matrix for every $\xi \in \mathbb{k}$, so, $0 < \lambda \le \Lambda < \infty$ and there is a unitary matrix like $U(\xi)$ that

$$U^*(\xi)H^*(\xi)H(\xi)U(\xi) = C(\xi),$$

where $C(\xi)$ is a diagonal matrix and on its main diagonal there are eigenvalues of a matrix $H^*(\xi)H(\xi)$, so, given λ, it is clear that $C(\xi) - \lambda I$ is a positive definite matrix, so by definition of a positive matrix, we have

$$\lambda I \le C(\xi), \quad (14)$$

And given the similarity of the matrices $C(\xi)$ و $H^*(\xi)H(\xi)$, it is concluded that Eq. (14) is equivalent to following equation:

$$\lambda I \le H^*(\xi)H(\xi),$$

Also, similarly, it can be shown that

$$H^*(\xi)H(\xi) \le \Lambda I,$$

So, almost everywhere, for $\xi \in D$

$$\lambda I \le H^*(\xi)H(\xi) \le \Lambda I. \quad (15)$$

Note: Considering the similarity of the matrices $A^*(\xi)A(\xi)$ و $H^*(\xi)H(\xi)$, it is concluded that Eq. (15) is equivalent to following equation:

$$\lambda I \le A^*(\xi)A(\xi) \le \Lambda I.,$$

almost everywhere for $.\xi \in D$

Now for every $g \in L^2(\mathbb{k})$ we have:

$$\lambda\sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in N_0}\left|\left\langle g,\varphi_l^s(\cdot-u(k))\right\rangle\right|^2 \le \sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in N_0}|\langle g,\psi_l^s(\cdot-u(k))\rangle|^2$$

$$\le \Lambda\sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in N_0}\left|\left\langle g,\varphi_l^s(\cdot-u(k))\right\rangle\right|^2, \quad (16)$$

Where φ_l^s is defined in (10). Given Eq. (10), we have:

$$\sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in N_0}\left|\left\langle g,\varphi_l^s(\cdot-u(k))\right\rangle\right|^2 = \sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in N_0}|\langle g,q^{\frac{1}{2}}\varphi_l(\rho^{-1}x-\rho^{-1}u(k)-u(s))|^2$$

$$= \sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in N_0}|\langle g,q^{\frac{1}{2}}\varphi_l(\rho^{-1}x-u(qk+s))|^2$$

$$= \sum_{l=1}^{N}\sum_{k\in N_0}|\langle g,q^{\frac{1}{2}}\varphi_l(\rho^{-1}\cdot-u(k))\rangle|^2,$$

So, Eq.(16) can be written as follows:

$$\lambda\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{1}{2}}\varphi_l(\rho^{-1}.-u(k))\rangle|^2 \le \sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,\psi_l^s(.-u(k))\rangle|^2 \le \Lambda\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{1}{2}}\varphi_l(\rho^{-1}.-u(k))\rangle|^2, \quad (17)$$

Above contents can be summarized in following theorem known as the separation lemma for frames:

Theorem: (separation lemma for frames) Suppose that $(0 \le s \le q-1 \;,\; 1 \le l \le N)$ ψ_l^s , φ_l and Λ , λ are defined as above, then for every $g \in L^2(\mathbb{k})$, we have:

$$\lambda\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{1}{2}}\varphi_l(\rho^{-1}.-u(k))\rangle|^2 \le \sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,\psi_l^s(.-u(k))\rangle|^2 \le \Lambda\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{1}{2}}\varphi_l(\rho^{-1}.-u(k))\rangle|^2.$$

Now, we use the lemma of separation for function $\{\psi_l^s : 1 \le l \le N\}$ where $0 \le s \le q-1$, so, we have:

$$\lambda\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{1}{2}}\psi_l^s(\rho^{-1}.-u(k))\rangle|^2 \le \sum_{r=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,\psi_l^{s,r}(.-u(k))\rangle|^2 \le \Lambda\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{1}{2}}\psi_l^s(\rho^{-1}.-u(k))\rangle|^2, \quad (17)$$

Where $(0 \le r \le q-1)$ $\psi_l^{s,r}$ is defined as follows:

$$\psi_l^{s,r}(x) = \sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0} h_{ljk}^r q^{\frac{1}{2}}\psi_j^s(\rho^{-1}x-u(k)), \qquad 0 \le s \le q-1, \quad 1 \le l \le N, \quad (18)$$

And $\{h_{ljk}^s : k \in \mathbb{N}_0\} \in l^2(\mathbb{N}_0)$.

With Eq. (17) on $0 \le s \le q-1$, we have:

$$\lambda\sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{1}{2}}\psi_l^s(\rho^{-1}.-u(k))\rangle|^2 \le \sum_{s=0}^{q-1}\sum_{r=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,\psi_l^{s,r}(.-u(k))\rangle|^2 \le \Lambda\sum_{s=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{1}{2}}\psi_l^s(\rho^{-1}.-u(k))\rangle|^2 \quad (19)$$

Given Eq. (16), we have:

$$\lambda^2\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{2}{2}}\varphi_l(\rho^{-2}.-u(k))\rangle|^2 \le \sum_{s=0}^{q-1}\sum_{r=0}^{q-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,\psi_l^{s,r}(.-u(k))\rangle|^2 \le \Lambda^2\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{2}{2}}\varphi_l(\rho^{-2}.-u(k))\rangle|^2. \quad (20)$$

Now we define wavelet frame packets with the similar method of wavelet packets.

Definition: Considering the functions $\varphi_1,\varphi_2,\ldots,\varphi_N$, and use the lemma of separation for space

$$\overline{span}\{q^{\frac{1}{2}}\varphi_l(\rho^{-1}.-u(k)) : 1 \le l \le N, k \in \mathbb{N}_0\},$$

we obtain the functions $\{\psi_l^s : 1 \le l \le N, 0 \le s \le q-1\}$, see Eq.(16). Now, for any integer, we recursively define $1 \le l \le N$, ψ_l^n , $(n \ge 0)$ n as follows:

Definition: Suppose that there are functions ψ_l^r for $r \in \mathbb{N}_0$ and $1 \le l \le N$. Then for $0 \le s \le q-1$ and $1 \le l \le N$, we define:

$$\psi_l^{s+qr} = \sum_{j=0}^{N}\sum_{k\in\mathbb{N}_0} h_{ljk}^s q^{\frac{1}{2}}\psi_j^r(\rho^{-1}.-u(k)). \quad (21)$$

The set $\{\psi_l^n : n \ge 0, 1 \le l \le N\}$ of functions defined above is called wavelet frame packets(Framelet).

Lemma: For φ_l and ψ_l^n defined above, we have following inequality:

$$\lambda^j\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{j}{2}}\varphi_l(\rho^{-j}.-u(k))\rangle|^2 \le \sum_{n=0}^{q^j-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,\psi_l^n(.-u(k))\rangle|^2 \le \Lambda^j\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{j}{2}}\varphi_l(\rho^{-j}.-u(k))\rangle|^2.$$

Proof. By comparing the Eq.(21) and Eq.(18), we have:

$$\{\psi_l^{s,r} : 0 \le r,s \le q-1\} = \{\psi_l^{s+qr} : 0 \le r,s \le q-1\} = \{\psi_l^n : 0 \le n \le q^2-1\}.$$

So Eq.(20) can be written as follows:

$$\lambda^2\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{2}{2}}\varphi_l(\rho^{-2}.-u(k))\rangle|^2 \le \sum_{n=0}^{q^2-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,\psi_l^n(.-u(k))\rangle|^2 \le \Lambda^2\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{2}{2}}\varphi_l(\rho^{-2}.-u(k))\rangle|^2. \quad (22)$$

With the use of induction on $j \in \mathbb{N}$ and given Eq.(16), following inequity can be easily solved:

$$\lambda^j\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{j}{2}}\varphi_l(\rho^{-j}.-u(k))\rangle|^2 \le \sum_{n=0}^{q^j-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,\psi_l^n(.-u(k))\rangle|^2 \le \Lambda^j\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{j}{2}}\varphi_l(\rho^{-j}.-u(k))\rangle|^2. \quad (23)$$

Theorem: $\{\psi_l^n : n \ge 0, 1 \le l \le N\}$ is a frame for V_j with the bounds of $\lambda^j C_1$ and $\Lambda^j C_2$.

Proof: since $\{\varphi_l(.-u(k)) : 1 \le l \le N, k \in \mathbb{N}_0\}$ is a frame for V_0 with the frame bounds C1 and C2, then for every $C_1\|g\|^2 \le \sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,\varphi_l(.-u(k))\rangle|^2 \le C_2\|g\|^2$, $g \in V_0$, by multiplying above equation by q^j and putting $x = \rho^{-j}x$, we have:

$$C_1\|g_{j,0}\|^2 \le \sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{j}{2}}\varphi_l(\rho^{-j}x-u(k))\rangle|^2 \le C_2\|g_{j,0}\|^2,$$

where $g_{j,0}(x) = q^{\frac{j}{2}}g(\rho^{-1}x)$ and given the proposition $\|g_{j,0}\|^2 = \|g\|^2$, therefore

$$C_1\|g\|^2 \le \sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,q^{\frac{j}{2}}\varphi_l(\rho^{-j}x-u(k))\rangle|^2 \le C_2\|g\|^2, \quad (24)$$

According to above equation and $\{q^{\frac{j}{2}}\varphi_l(\rho^{-j}.-u(k)) : 1 \le l \le N, k \in \mathbb{N}_0\} \subseteq V_j$, it is concluded that $\{q^{\frac{j}{2}}\varphi_l(\rho^{-j}.-u(k)) : 1 \le l \le N, k \in \mathbb{N}_0\}$ is a frame for V_j with the bounds of C1 and C2, now, according to Eq.(23) and Eq.(24), we have:

$$\lambda^j C_1\|g\|^2 \le \sum_{n=0}^{q^j-1}\sum_{l=1}^{N}\sum_{k\in\mathbb{N}_0}|\langle g,\psi_l^n(.-u(k))\rangle|^2 \le \Lambda^j C_2\|g\|^2,$$

For every $g \in V_j$, so, $\{\psi_l^n : n \geq 0, 1 \leq l \leq N\}$ is a frame for V_j with the bounds of $\lambda^j C_1$ and $\Lambda^j C_2$.

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