On Construction of framelet packets on local fields with positive characteristic

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Abstract
In the present paper, using the prime member of local field with positive characteristic, some concept of multiresolution analysis (MRA) and wavelets were extended to such fields. The separation lemma was proved and applying this lemma, the wavelet packets related to multiresolution analysis was built. Also it is shown that these wavelet packets translations produces one unique orthogonal basis for desired local fields. The separation lemma for frames was proved and then, it was used to construct framelets.

Keywords:
wavelet, multiresolution analysis, local field, wavelet packets, wavelet frame packets, positive characteristic

1. Introduction
Wavelets are of the family of dilation and includes the transformation of a function called mother wavelet (Daubechies, 1992). Wavelets were introduced in early 1980 and attracted many people of mathematical community and other fields in which wavelet can play useful role and the results of the interest were several books and high volume of research articles about them (Grochenig, 2000)
Wavelets have different application in several fields, for example, (Jiang, 2004) wavelet transform is used as a tool for analyzing frequency and time (Zheng, 1990) and wavelet analysis is used as mathematical microscope which is closely related to harmonic analysis and approximation theory. In this study, it was used for studying multiresolution analysis on local field and building the corresponding wavelet (Zheng, 1997)
Dahlke (Dahlke, 1994) introduced wavelets and multiresolution. Han et al (Stavropoulos, 1998) have developed these concepts in Hilbert spaces and Lemarie (Lemarie, 1989) examined them on Lie groups.

2. Method
Local field : Definition: Suppose that $\mathbb{K}$ is a field and topological space. In this case, $\mathbb{K}$ is called locally compact field or a local field, if $\mathbb{K}^+$ and $\mathbb{K}^-$ are locally compact abelian groups. $\mathbb{K}^+$ and $\mathbb{K}^-$ are collective and multiplicative groups of field $\mathbb{K}$, respectively.

2.1 Definition
A commutative ring $\mathbb{R}$ is called integer domain if $\alpha \cdot \beta = 0$ in $\mathbb{R}$, results in $\alpha = 0$ or $\beta = 0$.

2.2 Definition
It is said that a field $\mathbb{F}$ is with characteristic $p \neq 0$, if for every $\alpha \in \mathbb{F}$, $\alpha^p = 0$ and no positive integer smaller than $P$ has such characteristic. In this case $\text{char}(\mathbb{F}) = p$

Remark : If the field $\mathbb{F}$ has no the characteristic of $F$ for every positive integer of $P$, it is called a field characterized by 0.

2.3 Example
$\mathbb{Q}$ and $\mathbb{R}$ and $\mathbb{C}$ are the fields characterized by zero, $\mathbb{Z}$ is a field characterized by 3.

2.4 Theorem
A field is characterized by zero or prime number.

2.5 Corollary
If $\mathbb{F}$ is a finite field of $\text{card} \mathbb{F} \leq \infty$, then it is characterized by prime number.

2.6 Theorem
If $\mathbb{F}$ is a finite field, then $\text{char} \mathbb{F} = p$ where $n \in \mathbb{N}$ and $\cdot p = \text{char} \mathbb{F}$

2.7 Example
If $\mathbb{F}$ is prime number, then $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ is a field.

2.8 Theorem
For every prime number $p$ and every positive integer n, there is a unique field with $p^n$ members.

2.9 Definition
If $\mathbb{P}$ is prime number and $n \geq 1$ is integer, then a unique field with $p^n$ members is called Galois field of order $p^n$ and represented by $\mathbb{GF}(p^n)$ or $\mathbb{F}_{p^n}$.

Remark: Given the result 1 and theorem 2 we have $\text{char}(\mathbb{GF}(p^n)) = p$

2.10 Definition
Suppose that $\mathbb{F}$ is a field. In this case, $\mathbb{F}((t))$ is the set of all formal infinite sums of 
\[c_0 + c_1 t + c_2 t^2 + \cdots,\]
with the variable $t$ where $n_0 \in \mathbb{Z}$ and $n \in \{n_0, n_0 + 1, \ldots, \}$. If
\[g(t) = c_0 t^{n_0} + c_1 t^{n_0 + 1} + \cdots,\]
\[f(t) = b_0 t^{m_0} + b_1 t^{m_0 + 1} + \cdots,\]
belong to $\mathbb{F}((t))$, then it will be easily shown that $\mathbb{F}((t))$ is a field and called formal Laurent series field. It is clear that field $\mathbb{K}$ with the discrete topology is a local field, so from here, $\mathbb{K}$ is a local field with non-discrete topology.

2.11 Definition
Suppose that $G$ is a group. Then the map $\phi: G \to \mathbb{G}$ is called homomorphism map if $\phi$ is surjective and injective homomorphism.

2.12 Definition
Suppose that $G$ is a locally compact group with the Haar measure $\mu$. Then, homomorphism (continuous) $\phi: G \to \mathbb{G}$ is considered, if $E$ is a Borel subset of $G$, then $\phi(E)$ is a Borel subset of $\mathbb{G}$ and $\mu(\phi(E))$ is a Haar measure of $G$, given the uniqueness of Haar measure, there is $\mu(\phi(\psi)) = \mu(\phi(c))$. Given the definition:
\[\mu((\phi(c))(\psi)) = \mu(\phi(\psi))(\phi(\psi))\]
For every subset of $E$ from $G$.

3. Results
Multiresolution analysis for $L^1(\mathbb{R})$
From now on, by $K$ we mean a discrete local field with positive character $\operatorname{char}(K)=p>0$.

3.1 Definition
Let $K$ be a local field with positive character $p$. Then we define the following subsets of $K$:
\[D = \{x \in K : |x| \leq 1\}, \quad B = \{x \in K : |x| < 1\}\]
Where $D$ is the integer domain and $B$ is called the prime ideal of $K$.

3.2 Theorem
Finite field $\mathbb{F}(q)$ (where $p$ is prime number and $c \in \mathbb{N}$) is a vector space with the dimension $c$ on finite field $\mathbb{F}(q)$.

3.3 Proposition
The quotient space $\frac{D}{B}$ is isomorphic with the finite dimensional field $\mathbb{F}(q)$.

3.4 Proposition
Given the above proposition, $q = p^c$ where $p$ is prime number and $c \in \mathbb{N}$, Standard homomorphism $\pi: D \to \mathbb{F}(q) := L$ from $D$ on $L$ is considered, given the previous theorem, $L$ is a vector space with the dimension $c$ on finite field $\mathbb{F}(p)$. Now, the set $\{\pi^{-1}(\alpha) : \alpha \in \mathbb{B}\}$ is a basis for $L$ on finite field $\mathbb{F}(p)$.

3.4 Definition
Suppose that $N_\delta = N \cup \{0\}$ and $\rho$ are the prime members of local field $\mathbb{K}$, in this case, for $n \in N_\delta$ and $\rho$ are the prime members of local field $\mathbb{K}$, we have:
\[0 \leq n < \rho (q=p^c), \quad n = a_n + a_{n+1} \rho + \cdots + a_{n+c-1} \rho^{c-1},\]
We define:
\[u(n) = (a_n + a_{n+1} \rho + \cdots + a_{n+c-1} \rho^{c-1}) \rho^{-n}, \quad 0 \leq n < \rho.\]
For $n$ positive, we have:
\[n = a_0 + a_1 \rho + \cdots + a_{n-1} \rho^{n-1}, \quad n = a_0 + \cdots + a_n \rho^{n-1} + \cdots + \rho^{n-1}, u(n) = u(b_0) + u(b_1) \rho^{-1} + \cdots + u(b_n) \rho^{-n}.

3.5 Proposition
Suppose that $\{u(n) : n \in N_\delta\}$ is a set defined above. In this case,
\[\{u(n) : n \in N_\delta\}\]
is a set of all equivalence classes of $D$ in $B$, therefore
\[\{u(n) : n \in N_\delta\} \approx \mathbb{F}(q) \approx \mathbb{F}(q) / \mathbb{G}(\phi(\psi))\]
is a set of all equivalence classes of $D$ in $B$.

3.6 Proposition
For every $n, m \in B^1$, the equation $u(n + m) = u(n) + u(m)$ is not established, but if $n < m$, then $u(n + m) = u(n) + u(m)$.

3.7 Proposition
If $k, l \in N_\delta$ and $\chi$ is a characteristic defined on $K$, then
\[\chi(x) = \left\{ \begin{array}{ll}
\chi \in \mathbb{F}(q) & \text{if } x \in D, \\
\chi & \text{if } x \in B.
\end{array} \right.\]

3.8 Wavelet frame packets (Framelets)
Definition: Suppose that $H$ is a separable Hilbert space. The sequence $\{x_k : k \in \mathbb{Z}\}$ in the Hilbert space $H$ is a frame for $H$, if there are the constants $c1$ and $c2$, such that for every $x \in H$, the following equation holds:
\[c_1 \|x\|^2 \leq \sum_{k \in \mathbb{Z}} \|x_k, x\| \leq c_2 \|x\|^2.\]
The largest \( c_1 \) and the smallest \( c_2 \) which are satisfied in above equation, are called the bounds of frame.

Now we are going to establish some part of classical frame theory, for \( L^2(K) \) where \( K \) is a local field of positive characteristic.

Suppose that \( \Phi = \{ \varphi_1, \varphi_2, \ldots, \varphi_N \} \subset L^2(K) \) where the following system

\[
\{ \varphi_l(\cdot - u(k)) : 1 \leq l \leq N, k \in \mathbb{N}_1 \},
\]

is a frame for

\[
S(\Phi) = \text{span}\{ \varphi_l(\cdot - u(k)) : 1 \leq l \leq N, k \in \mathbb{N}_1 \}.
\]

Suppose that \( \{ \psi_1, \psi_2, \ldots, \psi_N \} \) is a subset of \( S(\Phi) \), there is a question:

When can it be said that \( \{ \psi_l(\cdot - u(k)) : 1 \leq l \leq N, k \in \mathbb{N}_1 \} \) is also a frame for \( S(\Phi) \)?

If \( \psi_j \in S(\Phi) \), then there is a sequence \( \{ r_{jk} : k \in \mathbb{N}_1 \} \) in \( l^1(\mathbb{N}_1) \), where

\[
\psi_j(x) = \sum_{k \in \mathbb{N}_1} r_{jk} \varphi_j(x - u(k)).
\]

By applying Fourier transform, we have:

\[
\hat{\psi}_j(\xi) = \sum_{k \in \mathbb{N}_1} r_{jk} \hat{\varphi}_j(\xi - u(k)),
\]

\[
R_{\varphi}(\xi) \ast \overline{R_{\psi}(\xi)} = \sum_{j \in \mathbb{N}_1} r_{jk} \varphi_j(\xi - u(k))
\]

is an integer periodic function, for every \( n \in \mathbb{N}_1 \), we have:

\[
R_{\varphi}(\xi + u(n)) = \sum_{j \in \mathbb{N}_1} r_{jk} \varphi_j(\xi + u(n))
\]

\[
- \sum_{j \in \mathbb{N}_1} r_{jk} \varphi_j(\xi + u(n)) = \sum_{j \in \mathbb{N}_1} r_{jk} \varphi_j(\xi - u(k)) = R_{\varphi}(\xi).
\]

So our discussion about \( \phi_{jk} \) to be frame or not, reduced to these periodic functions.

### 3.7 Proposition

If \( k, l \in \mathbb{N}_1 \) and \( \chi \) is a characteristic on \( K \), i.e.,

\[
\chi(x + u(k)) = \chi(x), \quad \chi(x)(x) = 0,
\]

then \( \chi_k(k) = \chi(x) \).

If \( k = 0 \) or \( k = 1 \), then

\[
\chi_k(k) = \chi(x) = \chi(x) = 1.
\]

It is clear that

\[
\chi_k(\xi) = \chi(x) = \chi(x) = 1.
\]

### 4. Discussion

**Separation lemma for wavelet frame packets**

The main idea of separation lemma is building wavelet frame packets and following theorem.

**Theorem:** Suppose that \( \{ \varphi_l : 1 \leq l \leq N \} \subset L^2(K) \) where \( \{ \varphi_l(\cdot - u(k)) : 1 \leq l \leq N, k \in \mathbb{N}_1 \} \) is a frame for

\[
V = \text{span}\{ \varphi_l(\cdot - u(k)) : 1 \leq l \leq N, k \in \mathbb{N}_1 \}
\]

with the bounds of \( c_1 \) and \( c_2 \), if \( \{ \psi_l : n \geq \lambda, 1 \leq l \leq N \} \) is wavelet frame packets and

\[
V_j = \{ f \in L^2(K) : f(\mu^j) \in V \}
\]

then for every \( j \geq \lambda \), the set \( \{ \psi_l(k - u(k)) : n \geq \lambda, 1 \leq l \leq N, k \in \mathbb{N}_1 \} \) is of the functions of a frame for \( V_j \) with the bounds of \( \lambda_jc_1 \) and \( \lambda_jc_2 \).

To clarify this theorem and to prove it, following items are used.

**Definition:** Suppose that \( \{ \varphi_j : 1 \leq j \leq N \} \) is a set of functions in \( L^2(K) \) where \( \{ \varphi_j(\cdot - u(k)) : 1 \leq j \leq N, k \in \mathbb{N}_1 \} \) is a frame for

\[
V = \text{span}\{ \varphi_j(\cdot - u(k)) : 1 \leq j \leq N, k \in \mathbb{N}_1 \}
\]

and there is a sequence

\[
\{ b_{jk} : k \in \mathbb{N}_1 \} \in l^1(\mathbb{N}_1),
\]

for \( a \leq r \leq q - 1 \), \( 1 \leq l \leq N \), in this case, we define:

\[
s_j(\xi) = a^l \sum_{k \in \mathbb{N}_1} b_{jk} \varphi_j(\xi - u(k)).
\]

With the Fourier transform of above equation, we have:

\[
\hat{s}_j(\xi) = a^l \sum_{k \in \mathbb{N}_1} b_{jk} \chi_k(\xi - u(k))
\]

\[
\hat{b}_{jk} = a^{-1} b_{jk} \chi_k(\xi - u(k)).
\]

**Lemma:** \( \hat{s}_j(\xi) \) can be written as follows:

\[
\hat{s}_j(\xi) = a^l \sum_{k \in \mathbb{N}_1} b_{jk} \psi_j(\xi - u(k)),
\]

\[
\hat{b}_{jk} = a^{-1} b_{jk} \chi_k(\xi - u(k)).
\]

It is clear that \( \hat{s}_j(\xi) \) is a square matrix of order \( q \).

**Lemma:** \( \hat{s}_j(\xi) \) can be written as follows:

\[
\hat{s}_j(\xi) = a^l \sum_{k \in \mathbb{N}_1} b_{jk} \psi_j(\xi - u(k)),
\]

\[
\hat{b}_{jk} = a^{-1} b_{jk} \chi_k(\xi - u(k)).
\]

**Theorem:** Suppose that \( \{ \varphi_j : 1 \leq j \leq N \} \) is a set of functions in \( L^2(K) \) where \( \{ \varphi_j(\cdot - u(k)) : 1 \leq j \leq N, k \in \mathbb{N}_1 \} \) is a frame for

\[
V = \text{span}\{ \varphi_j(\cdot - u(k)) : 1 \leq j \leq N, k \in \mathbb{N}_1 \}
\]

and there is a sequence

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\]

**Lemma:** \( \hat{s}_j(\xi) \) can be written as follows:

\[
\hat{s}_j(\xi) = a^l \sum_{k \in \mathbb{N}_1} b_{jk} \psi_j(\xi - u(k)),
\]

\[
\hat{b}_{jk} = a^{-1} b_{jk} \chi_k(\xi - u(k)).
\]

Now, the matrices \( A(\xi) \) and \( A^*(\xi) \) are defined as follows:

then put:

\[
A(\xi) = \left( a_{jk}(\xi) \right)_{1 \leq j,k \leq N},
\]

\[
A^*(\xi) = \left( a_{jk}(\xi) \right)_{1 \leq j,k \leq N}.
\]

Now, the matrices \( E(\xi) \) almost everywhere for \( \xi \in D \), now we define matrix \( E(\xi) \) as follows:
Proposition: Show that for \(1 \leq l, j \leq N\), \(r, s \leq q - 1\),
\[
H(\xi) = A(\rho^{-1} \xi)E(\xi)
\]
that \(E(\xi)\) is a unitary matrix defined in above equation.
Proof: Given the definition, it is clear that \((r, s)\) th block of matrix \(A(\rho^{-1} \xi)E(\xi)\) is the matrix
\[
\sum_{i=r}^{s} A(\rho^{-1} \xi)E(\xi).
\]
(Given the definition \(A(\rho^{-1} \xi)E(\xi)\))
Now, \((l, j)\) th component of \((r, s)\) th block of matrix \(H(\xi)\) is
\[
\sum_{k=1}^{N} A(\rho^{-1} \xi)E(\xi).
\]
If the last equality holds, then, \(\chi_k(u(l)) = 1\).
If \(k = 0\) or \(l = 0\), then, it is clear that
\[
\chi_k(u(l)) = \chi(u(k)u(l)) = \chi(0) = 1
\]
because \((l, j)\) is selected arbitrarily. From Eq.(11) and Eq.(12), it is concluded that all components of the matrices \(H(\xi)\) and \(A(\rho^{-1} \xi)E(\xi)\) are equal.
Corollary: Given above proposition, we have:
\[
H^*(\xi)H(\xi) = E^*(\xi)A^*(\rho^{-1} \xi)A(\rho^{-1} \xi)E(\xi),
\]
Because \(E(\xi)\) is a unitary matrix, so the matrices \(H^*(\xi)H(\xi) \triangleright A(\rho^{-1} \xi)A(\rho^{-1} \xi)E(\xi)\) are similar.
Definition 5.6 Suppose that \(\Lambda(\xi) \triangleright \lambda(\xi)\) are the maximum and minimum members of the set of all eigenvalues of the positive definite matrix of \(H^*(\xi)H(\xi)\), in this case, put
\[
\Lambda = \sup_{\xi} \Lambda(\xi) \triangleright \lambda = \inf_{\xi} \lambda(\xi).
\]
Lemma: Almost everywhere for \(\xi \in D\), we have:
\[
\lambda I \leq H^*(\xi)H(\xi) \leq \Lambda I.
\]
Proof. Because \(H^*(\xi)H(\xi)\) is a positive definite matrix for every \(\xi \in D\), so, \(\rho < \lambda \leq \Lambda < \infty\) and there is a unitary matrix \(U(\xi)\) that
\[
U^*(\xi)H^*(\xi)H(\xi)U(\xi) = C(\xi),
\]
where \(C(\xi)\) is a diagonal matrix and on its main diagonal there are eigenvalues of a matrix \(H^*(\xi)H(\xi)\), so, given \(\lambda\), it is clear that \(C(\xi) - \lambda I\) is a positive definite matrix, so by definition of a positive matrix, we have
\[
\lambda \leq C(\xi),
\]
And given the similarity of the matrices \(C(\xi) \triangleright H^*(\xi)H(\xi)\), it is concluded that Eq. (14) is equivalent to following equation:
\[
\lambda I \leq H^*(\xi)H(\xi),
\]
Also, similarly, it can be shown
\[
H^*(\xi)H(\xi) \leq \Lambda I,
\]
that
\[
\xi \in D,
\]
So, almost everywhere for \(\xi \in D\)
\[
\lambda I \leq H^*(\xi)H(\xi) \leq \Lambda I.
\]
Note: Considering the similarity of the matrices \(A(\xi)A(\xi) \triangleright H^*(\xi)H(\xi)\), it is concluded that Eq. (15) is equivalent to following equation:
\[
\lambda \leq A(\xi)A(\xi) \leq \Lambda,\]
almost everywhere for \(\xi \in D\).
Now for every \(\rho \in D^+(\infty)\) we have:
\[
\sum_{l=1}^{N} \sum_{k=1}^{N} |(\rho, \varphi^{(l, - u(k))})| \leq \sum_{l=1}^{N} \sum_{k=1}^{N} |(\rho, \varphi^{(l, - u(k))})| \leq \sum_{l=1}^{N} \sum_{k=1}^{N} |(\rho, \varphi^{(l, - u(k))})|
\]
(16)
Where \(\varphi^{(l, - u(k))}\) is defined in (10). Given Eq. (10), we have:
\[
\sum_{l=1}^{N} \sum_{k=1}^{N} |(\rho, \varphi^{(l, - u(k))})| - \sum_{l=1}^{N} \sum_{k=1}^{N} |(\rho, \varphi^{(l, - u(k))})| - \sum_{l=1}^{N} \sum_{k=1}^{N} |(\rho, \varphi^{(l, - u(k))})|,
\]
So, Eq.(16) can be written as follows:
Above contents can be summarized in following theorem known as the separation lemma for frames:

**Theorem:** (separation lemma for frames) Suppose that \( \varphi \) and \( \lambda \) are defined as above, then for every \( g \in L^2(k) \), we have:

\[
\sum_{i=1}^{n} \left| \left( \int_{k} \varphi_{j,k}^*[g \varphi_{j,k}(\rho^{-1}, -u(k))] \right)^2 \right| \leq \sum_{i=1}^{n} \sum_{k \in K_n} \left| (g \varphi_{j,k}(\rho^{-1}, -u(k)))^2 \right|
\]

(17)

Now, we use the lemma of separation for function \( \psi_i : \ \forall \leq i \leq N \) where \( \lambda \leq s \leq q - 1 \), so, we have:

\[
\sum_{i=1}^{n} \left| \left( \int_{k} \psi_i^*[g \psi_i(\rho^{-1}, -u(k))] \right)^2 \right| \leq \sum_{i=1}^{n} \sum_{k \in K_n} \left| (g \psi_i(\rho^{-1}, -u(k)))^2 \right|
\]

(18)

And \( \{h_{j,k}^i : k \in K_n\} \in L^2(N_n) \).

With Eq. (17) on \( \lambda \leq \leq s \leq q - 1 \), we have:

\[
\sum_{i=1}^{n} \left| \left( \int_{k} \varphi_{j,k}^*[g \varphi_{j,k}(\rho^{-1}, -u(k))] \right)^2 \right| \leq \sum_{i=1}^{n} \sum_{k \in K_n} \left| (g \varphi_{j,k}(\rho^{-1}, -u(k)))^2 \right|
\]

(19)

Given Eq. (16), we have:

\[
\sum_{i=1}^{n} \left| \left( \int_{k} \varphi_{j,k}^*[g \varphi_{j,k}(\rho^{-1}, -u(k))] \right)^2 \right| \leq \sum_{i=1}^{n} \sum_{k \in K_n} \left| (g \varphi_{j,k}(\rho^{-1}, -u(k)))^2 \right|
\]

(20)

Now we define wavelet frame packets with the similar method of wavelets.

\[
\varphi_1, \varphi_2, \ldots, \varphi_N
\]

Definition: Considering the functions \( \varphi \) and use the lemma of separation for space \( \sum_{i=1}^{n} \left| \left( \int_{k} \varphi_{j,k}^*[g \varphi_{j,k}(\rho^{-1}, -u(k))] \right)^2 \right| \), we obtain the functions \( \{\psi_i : \ \forall \leq i \leq N, \ \lambda \leq s \leq q - 1 \} \), see Eq.(16). Now, for any integer, we recursively define \( \forall \leq i \leq N, \ \psi_i^* : (n \geq \lambda) \) as follows:

Definition: Suppose that there are functions \( \psi_i^* \) for \( \forall \leq i \leq N \) and \( \lambda \leq \leq q - 1 \). Then for \( \lambda \leq s \leq q - 1 \) and \( \lambda \leq \leq q - 1 \), we define:

\[
\psi_i^{\lambda,s} = \sum_{j=1}^{\lambda} \sum_{k \in K_n} h_{j,k}^i \varphi_{j,k}(\rho^{-1}, -u(k)).
\]

(21)

The set \( \{\psi_i^* : n \geq \lambda, \ \forall \leq i \leq N\} \) of functions defined above is called wavelet frame packets (Framelet).

Lemma: For \( \psi_i^* \) and \( \psi_i^* \) defined above, we have following inequality:

\[
\sum_{i=1}^{n} \left| \left( \int_{k} \varphi_{j,k}^*[g \varphi_{j,k}(\rho^{-1}, -u(k))] \right)^2 \right| \leq \sum_{i=1}^{n} \sum_{k \in K_n} \left| (g \varphi_{j,k}(\rho^{-1}, -u(k)))^2 \right|
\]

(22)

With the use of induction on \( j \in N \) and given Eq.(16), following inequity can be easily solved:

\[
\sum_{i=1}^{n} \left| \left( \int_{k} \varphi_{j,k}^*[g \varphi_{j,k}(\rho^{-1}, -u(k))] \right)^2 \right| \leq \sum_{i=1}^{n} \sum_{k \in K_n} \left| (g \varphi_{j,k}(\rho^{-1}, -u(k)))^2 \right|
\]

(2)

3)

**Theorem:** \( \{\psi_i^* : n \geq \lambda, \ \forall \leq i \leq N\} \) is a frame for \( V_j \) with the bounds of \( \lambda \) and \( \lambda \).

Proof: since \( \varphi_i(-u(k)) : \ \forall \leq i \leq N, k \in N \) is a frame for \( V_j \) with the frame bounds \( C_1 \) and \( C_2 \), then for every \( g \in V \), by multiplying above equation by \( \varphi_i^* \) and putting \( x = \rho^{-1}x \), we have:

\[
C_1 \|g\| \leq \sum_{i=1}^{n} \left| \left( \int_{k} \varphi_{j,k}^*[g \varphi_{j,k}(\rho^{-1}, -u(k))] \right)^2 \right| \leq C_2 \|g\|, \quad g \in V,
\]

where \( g_{i,x} = \rho^i \varphi_i(x) \) and given the proposition \( \|g\| - \|g\| \), therefore

\[
C_1 \|g\| \leq \sum_{i=1}^{n} \left| \left( \int_{k} \varphi_{j,k}^*[g \varphi_{j,k}(\rho^{-1}, -u(k))] \right)^2 \right| \leq C_2 \|g\|.
\]

(24)

According to above equation and \( \{\varphi_i(-u(k)) : \ \forall \leq i \leq N, k \in N \} \subseteq V_j \), it is concluded that \( \{\varphi_i(-u(k)) : \ \forall \leq i \leq N, k \in N \} \) is a frame for \( V_j \) with the bounds of \( C_1 \) and \( C_2 \), now, according to Eq.(23) and Eq.(24), we have:

\[
\sum_{i=1}^{n} \left| \left( \int_{k} \varphi_{j,k}^*[g \varphi_{j,k}(\rho^{-1}, -u(k))] \right)^2 \right| \leq C_2 \|g\|.
\]
For every $g \in V_J$, so, $\{ \psi^n_l : n \geq \sigma, 1 \leq l \leq N \}$ is a frame for $V_J$ with the bounds of $\lambda^G \alpha$ and $\lambda^G \gamma$.

References