Probabilistic Metric Spaces and Some Contraction Mappings

Parvin Azhdari
Department of Statistics, Tehran North Branch, Islamic Azad University, Tehran, Iran.

Abstract:
There exist many types of contraction mappings in the case of single valued and multi-valued that researchers are interested in proving fixed point theorems. Now, I consider two types of contraction: \((\varphi - k) - B\) contraction and \(bn\)-contraction. In this paper, after presenting the definition of \((\varphi - k) - B\) contraction, I prove a lemma about continuity of \((\varphi - k) - B\) contraction and by using it I prove a unique single valued fixed point theorem for \((\varphi - k) - B\) contraction with \(t\)-norm which is \(q\)-convergent in probabilistic metric space. Then multi-valued \(bn\)-contraction definition is illustrated. I obtain first multi-valued theorem with new assumptions. Finally, I prove another fixed point theorem for multi-valued case by the definition of a large class of mappings called weakly demicompact mappings.

Keywords:
Probabilistic metric space, \(bn\)-contraction, \((\varphi - k) - B\) contraction, fixed point.

1. Introduction

The notion of a probabilistic metric space was introduced by Menger [1]. There exist many types of fixed point theorems in the field of contraction mappings [2]. One such theorem was formulated by Sehgal and Bharucha-Reid [3] who introduced the \(B\)-contraction mapping in probabilistic metric spaces. Hicks [4] established \(C\)-contractions in probabilistic metric spaces while Radu generalized \(C\)-contraction to multi-valued \(\psi\)-contraction [5]. Pap et. al generalized the definition of a large class of mappings called weakly demicompact mappings.

2 Preliminary Notes

We recall some concepts from probabilistic metric space, convergence and contraction. For more details, we refer the reader to [13, 14].

Let \(D^+\) be the set of all distribution of functions \(F\) such that \(F(0) = 0\) (\(F\) is a non-decreasing, left continuous mapping from \(R\) into \([0, 1]\) such that \(\lim_{x \to \infty} F(x) = 1\)).

The ordered pair \((S, F)\) is said to be a probabilistic metric space if \(S\) is a nonempty set and \(F : S \times S \to D^+ (F(p, q)\) written by \(F_{pq}\) for every \((p, q) \in S \times S\) satisfies the following conditions:

1) \(F_{uv}(x) = 1\) for every \(x \geq 0\) \(\Rightarrow u = v\) (\(u, v \in S\)),
2) \(F_{uv} = F_{vu}\) for all \(u, v \in S\),
3) \(F_{uv}(x + y) = 1\) \(\forall u, v, w \in S\), and all \(x, y \in R^+\),

\(A\) Menger space is a triple \((S, F, T)\) where \((S, F)\) is a probabilistic metric space, \(T\) is a triangular norm (abbreviated \(t\)-norm) and the following inequality holds

\[ F_{uv}(x + y) \geq T(F_{uw}(x), F_{vw}(y)) \]

for all \(u, v, w \in S\), and all \(x, y \in R^+\).

Recall that the mapping \(T: [0, 1] \times [0, 1] \to [0, 1]\) is called a triangular norm (\(a\)-norm) if the following conditions are satisfied:

\[ T(a, 1) = a \] for every \(a \in [0, 1]\); \(T(a, b) = T(b, a)\) for all \(a, b \in [0, 1]\) and \(a \geq b, c \geq d\)

\[ T(a, c) \geq T(b, d) \] for all \(a, b, c, d \in [0, 1]\); \(T(T(a, b), c) = T(a, T(b, c))\)

\(a, b, c \in [0, 1]\).

Basic examples are \(t\)-norms \(TL\) (\(Lukasiewicz\) \(t\)-norm), \(TP\) and \(TM\), defined by \(TL(a, b) = \max\{a + b - 1, 0\}\), \(TP(a, b) = ab\) and \(TM(a, b) = \min\{a, b\}\).

If \(T\) is a \(t\)-norm and \((x_1, x_2, \ldots, x_n) \in [0, 1]^n\) \(n \in N^*\) in which \(N^* = N \cup \{+\infty\}\), one can define recurrently \(x_i = T(T_{i-1} x_1, x_i)\) for all \(n \geq 2\). One can also extend \(T\) to a countable infinitary operation by defining \(T_{i=1}^n x_i\) for any sequence \((x_i)_{i \in \mathbb{N}}\) as \(\lim_{n \to \infty} T_{i=1}^n x_i\).

If \(q \in (0, 1)\) is given, we say that the \(t\)-norm \(T\) is \(q\)-convergent if \(\lim_{n \to \infty} T_{i=1}^n (1 - q_i) = 1\).

We remark that if \(T\) is \(q\)-convergent, then, \(\forall \lambda \in (0, 1) \exists \exists s(\lambda) \in N \forall n \in N T_{i=1}^n (1 - q_i^{s(\lambda)}) > 1 - \lambda\). Also note that if the \(t\)-norm \(T\) is \(q\)-convergent, then \(\sup_{0 \leq t < 1} T(t, t) = 1\).
Definition 2.1: Let \((S, F, T)\) be a Menger space. If \(\sup_{\varepsilon \leq t < 1} T(t, t) = 1\), then the family \(\{U_s\}_{s \in S}\) where,
\[
U_s = \{(x, y) \in S \times S \mid F_{xy}(\varepsilon) > 1 - \varepsilon\}
\]
is a base for a metrizable uniformity on \(S\), called the F-uniformity \([2,13,14]\). The F-uniformity naturally determines a metrizable topology on \(S\), called the strong topology or F-topology \([15]\), a subset \(O\) of \(S\) is F-open if for every \(p \in O\) there exists \(t > 0\) such that \(N_p = \{q \in S \mid F_{pq}(t) > 1 - t\} \subset O\).

Definition 2.2: Let \(\varphi : (0, 1) \rightarrow (0, 1)\) be a mapping. We say that the \(t\)-norm \(T\) is \(\varphi\)-convergent if
\[
\forall \delta \in (0,1) \forall \lambda \in (0,1) \exists s = s(\delta, \lambda) \in N \forall n \geq 0 \frac{T^\varphi(s)}{1 - \varphi^s(\delta)} > 1 - \lambda
\]

Definition 2.3: A sequence \((x_n)_{n \in N}\) is called \(F\)-convergent sequence \(x \in S\) if for all \(\varepsilon > 0\) and \(\lambda \in (0, 1)\) there exists \(n_0 = n_0(\varepsilon, \lambda) \in N\) such that \(\forall n \geq n_0 \frac{F_{xn}(\varepsilon)}{1 - \lambda} > 1 - \lambda\). A probabilistic metric space \((S, F, T)\) is called sequentially complete if every \(F\)-sequence is \(F\)-convergent. In the following, \(2^S\) denotes the class of all nonempty subsets of the set \(S\) and \(C(S)\) is the class of all nonempty closed (in the F-topology) subsets of \(S\).

Definition 2.4: A sequence \((x_n)_{n \in N}\) is called a Cauchy sequence \(x \in S\) if for all \(\varepsilon > 0\) and \(\lambda \in (0, 1)\) there exists \(n_0 = n_0(\varepsilon, \lambda) \in N\) such that \(\forall n \geq n_0 \forall m \in N \frac{F_{xn, x}(\varepsilon)}{1 - \lambda} > 1 - \lambda\). We also have
\[
\forall \varepsilon > 0 \quad x_n \rightarrow x \Leftrightarrow F_{xn, x}(\varepsilon) \rightarrow 1.
\]

A probabilistic metric space \((S, F, T)\) is called sequentially complete if every Cauchy sequence is \(F\)-convergent. In the following, \(2^S\) denotes the class of all nonempty subsets of the set \(S\) and \(C(S)\) is the class of all nonempty closed (in the F-topology) subsets of \(S\). Now, we need a number of results about contraction mappings. We review them at below.

Definition 2.5 \([16]\): Let \(F\) be a probabilistic distance on \(S\) and \(M \in \mathbb{R}^k\). A mapping \(f : S \rightarrow 2^S\) is called continuous if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that
\[
F_{ux}(\varepsilon) > 1 - \delta \Rightarrow \forall x \in u \exists y \in v \forall t \in F_{uv}(\varepsilon) > 1 - \varepsilon.
\]

Theorem 2.1 \([16]\): Let \((S, F, T)\) be a complete Menger space such that \(\sup_{\varepsilon \leq t < 1} T(t, t) = 1\) and \(f : S \rightarrow C(S)\) is a \(t\)-norm continuous mapping. If there exists a sequence \((t_n)_{n \in N}\) \( \subset (0,1) \) with \(\sum_{n=1}^{\infty} t_n < \infty\) and a sequence \((x_n)_{n \in N}\) \( \subset S\) with the properties:
\[
x_{n+1} \in f x_n \text{ for all } n \text{ and } \lim_{n \rightarrow \infty} T^\varphi_{i=1} g_{n+i-1} = 1,
\]
where \(g_n = F_{xn, x_{n+1}}(t_n)\), then \(f\) has a fixed point. One of important concept in this paper is \((\varphi-k)\)-B contraction. Mihet introduced this concept in \([6]\). Now, we define comparison functions from the class \(\varphi\) of all mapping \(\varphi : (0, 1) \rightarrow (0, 1)\) with the properties:
1) \(\varphi\) is an increasing bijection;
2) \(\varphi(\lambda) < \lambda\) \( \forall \lambda \in (0,1)\).

Since every such a comparison mapping is continuous, it is easy to see that if \(\varphi \in \Phi\) then
\[
\lim_{n \rightarrow \infty} \varphi^n(\lambda) = 0 \text{ for every } \lambda \in (0, 1).
\]

Definition 2.6 \([6]\): Let \((S, F)\) be a probabilistic metric space, \(\varphi \in \Phi\) and \(k \in (0, 1)\) be given. A mapping \(f : S \rightarrow S\) is called a \((\varphi - k)\)-B contraction on \(S\) if the following condition holds:
\[
x, y \in S, \varepsilon \in (0,1) \lambda \in (0,1) F_{xy}(\varepsilon) > 1 - \lambda \Rightarrow 1 - \lambda \Rightarrow F_{f(x), f(y)}((k \varepsilon)) > 1 - \varphi(\lambda)
\]

Another contraction that we use is \(b_n\)-contraction.

Definition 2.7 \([10]\): Let \((X, F)\) be a probabilistic metric space and \((b_n)_{n \in N}\) increasing sequence from \((0, 1)\) such that \(\lim_{n \rightarrow \infty} b_n = 1\).

A mapping \(f : X \rightarrow X\) is strict \(b_n\)-contraction if for every \(n \in N\), there exists \(q_0 \in (0, 1)\) and for all \(x_1, x_2 \in X, t > 0\)
\[
F_{x_1, x_2}(t) > b_n \Rightarrow F_{f(x_1), f(x_2)}(q_n t) > b_n.
\]

3 Main Results

This section consists of two parts. The first one is related to single valued fixed point, while the second one is about multi-valued theorems. In the first part, the definition of a continuous mapping is recalled and by using it, a Lemma and then a Theorem with new assumptions is proven.

Definition 3.1: Let \(F\) be a probabilistic distance on \(S\). A mapping \(f : S \rightarrow S\) is called continuous if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that
\[
F_{uf}(\delta) > 1 - \delta \Rightarrow F_{uf, v}(\varepsilon) > 1 - \varepsilon.
\]

The definition of the \((\varphi-k)\)-B contraction was mentioned in Section 2. We proved two fixed point Theorems about \((\varphi-k)\)-B contractions in \([17]\). These Theorems are recalled below.

Theorem 3.1: Let \((S, F, T)\) be a complete Menger space, \(T\) be a \(t\)-norm such that \(\sup_{\varepsilon \leq t < 1} T(t, t) = 1\) and \(f : S \rightarrow S\) a \((\varphi-k)\)-B contraction. If \(\lim_{t \rightarrow \infty} F_{x_0, x_0}(t) = 1\) for some \(x_0 \in S\) and \(m \in N\), then there exists a unique fixed point \(x\) of the mapping \(f\) so that
\[
x = \lim_{n \rightarrow \infty} f^n(x_0).
\]

Theorem 3.2: Let \((S, F, T)\) be a complete Menger space, \(T\) be a \(t\)-norm such that \(\sup_{\varepsilon \leq t < 1} T(t, t) = 1\) and \(f : S \rightarrow S\) a \((\varphi-k)\)-B contraction and suppose that for some \(p \in S\) and \(j > 0\)
If t-norm T is \( \varphi \)-convergent, then there exists a unique fixed point \( z \) of mapping \( f \) and
\[
\sup_{x_{>0}} x f \left( 1 - F_{p,f}(x) \right) < \infty
\]

Before starting the new Theorem, we need the following lemma and its proof.

**Lemma 3.1:** Every \( (\varphi - k) - B \) contraction is continuous.

**Proof:** Let \( \varepsilon > 0 \) then there exists \( \delta \in (0, 1) \) such that \( \delta < \varepsilon \).
If \( F_{u,v}(\delta) > 1 - \delta \) then, since \( f \) is a \( (\varphi - k) - B \) contraction we have
\[
F_{fu,fv}(\delta) > 1 - \varphi(\delta),
\]
where we obtain that
\[
F_{fu,fv}(\delta) > 1 - \varphi(\delta) > 1 - \delta > 1 - \varepsilon. \quad \text{Therefore; is continuous.}
\]

**Theorem 3.3:** Let \((S, F, T)\) be a complete Menger space such that \( \sup \leq t < 1 \) \( T(t, t) = 1 \) and let \( f : S \rightarrow S \) be a \( (\varphi - k) - B \) contraction. If \( T \) is \( \varphi \)-convergent, i.e.,
\[
\forall \delta > 0 \lim_{n \rightarrow \infty} T_{i=1}^n(1 - \varphi(\delta)) = 1 \quad (1)
\]
and \( F_{p,f} \in D^+ \) for every \( p \in S \), then there exists a unique fixed point \( x \) of the mapping \( f \)
and \( x = \lim_{n \rightarrow \infty} f^n(p) \) for every \( p \in S \).

**Proof:** Let \( p \in S \) and \( \delta > 0 \) be such that \( F_{p,f}(\delta) > 0 \). Since
\[
F_{p,f}(\delta) > 1 - \lambda;
\]
we have:
\[
F_{p,f}(k \delta) > 1 - \varphi(\lambda_1)
\]
and generally for every \( n \in N \)
\[
F_{p,f}(k^n \delta) > 1 - \varphi^n(\lambda_1) \quad (2)
\]
We will prove that \( (f^n(p))_{n \in N} \) is a Cauchy sequence, i.e., for all \( \varepsilon > 0 \) and \( \lambda \in (0, 1) \) there
exists \( n_0(\varepsilon, \lambda) \in N \) such that \( F_{p,f}(k^n \delta) > 1 - \lambda \) for all \( n \geq n_0 \).
Let \( \varepsilon > 0 \) and \( \lambda \in (0, 1) \) be given. Since the series \( \sum_{i=1}^{\infty} k^i \delta \) converges, there exists \( n_0 = n_0(\varepsilon) \) such that
\[
\sum_{i=1}^{\infty} k^i \delta < \varepsilon.
\]
Then for every \( n \geq n_0 \)
\[
F_{p,f}(k^n \delta) \geq F_{p,f}(k^n \delta) \geq F_{p,f}(k^n \delta) \geq T(...(T(F_{p,f}(k^i \delta), F_{p,f}(k^{i+1} \delta)))...), F_{p,f}(k^{n-1} \delta))
\]
\[
\geq T(...(T(...(T(1 - \varphi(\lambda_1)^1), 1 - \varphi(\lambda_1)^2), 1 - \varphi(\lambda_1)^3), ..., 1 - \varphi(\lambda_1))})
\]
Let \( n_1 = n_1(\lambda) \in N \) be such that \( T_{i=n_1}^n(1 - \varphi(\lambda_1)^i) > 1 - \lambda \).
Since relation (1) holds, the number \( n_1 \) exists. Using relation (2) we obtain for all \( n \geq \max \{n_0, n_1\} \) and \( m \in M \),
\[
F_{f^m(p,f^n+mp)}(\varepsilon) \geq T_{i=1}^{n+m-1}(1 - \varphi^i(\lambda_1))
\]
\[
\geq 1 - \lambda.
\]
By assumption the Menger space \((S, F, T)\) is complete, so the sequence \( (f^m(p))_{n \in N} \) is convergent to a value like \( x \).

It remains to prove the uniqueness of the fixed point \( x \). Suppose that \( y = fy, y \neq x \). If
\( \varepsilon > 0 \) be such that \( F_{x,y}(\varepsilon) > 0 \) and \( F_{x,y}(\varepsilon) > 1 - \lambda \) we have
\[
F_{x,y}((\varepsilon)) = F_{x,y}(0) > 1 - \varphi(\lambda)
\]
and similarly
\[
F_{x,y}(\varepsilon) > F_{x,y}(\varepsilon) > 1 - \varphi^n(\lambda) \quad \text{for every } n \in N
\]
Therefore, \( F_{x,y}(u) = 1 \), for every \( u > 0 \) (since \( \lim_{n \rightarrow \infty} \varphi^n(\lambda) = 0 \) which contradicts to \( x \neq y \).

In the second part, it is the turn of multi-valued case. Mihet in [10,11] introduced multi-valued \( b_n \)-contraction.

**Definition 3.2:** Let \((X, F)\) be a probabilistic metric space and \( (b_n)_{n \in N} \) an increasing sequence from \( (0, 1) \) such that \( \lim_{n \rightarrow \infty} b_n = 1 \), a mapping \( f : X \rightarrow X \) is multi-valued
\( b_n \)-contraction if for every \( n \in N \), there exists \( q_n \in (0, 1) \) and for all \( x, y \in X, \varepsilon > 0 \)
\[
F_{x,y}(\varepsilon) > b_n \Rightarrow \forall p \in fx \exists q \in fy : F_{p,q}(q_n \varepsilon) > b_n.
\]
Now, we will prove two new Theorems about multi-valued \( b_n \)-contraction by applying new conditions.

**Theorem 3.4:** Let \((S, F, T)\) be a complete Menger space with t-norm \( T \) such that
\( \sup \leq t < 1 \) \( T(t, t) = 1 \) and \( f : S \rightarrow C(S) \) be a multi-valued \( b_n \)-contraction. If there exist
\( p_0 \in S \) and \( p_1 \in f(p_0) \) such that for all \( \varepsilon > 0 \) and \( n \in N \), \( F_{p_0,p_1}(\varepsilon) > b_n \) and \( \sum_i^{\infty} q^n_i < \infty \)
and \( \lim_{n \rightarrow \infty} b_{n+1} = 1 \) then \( f \) has a fixed point.

**Proof:** Let \( \varepsilon > 0 \) be given and \( \delta \in (0, 1) \) be such that for
every \( \delta \leq \min \{\varepsilon, 1 - b_n\} \). If
Let \((S, F)\) be a probabilistic metric space, \(M\) a nonempty subset of \(S\) and \(f : M \rightarrow M\). By using the contraction relation, we can find \(p_1 \in f p_0\) such that

\[ F_{p_0, p_1}(q_{n_0} \varepsilon) > b_{n_0} \]

for every \(n_0 \in \mathbb{N}\), especially for \(n_0 = n\). Defining \(t_n = q_{n-1} \varepsilon\), we have \(s_{n+1} \geq t_i \cdot (t) \geq b_{n+1}\). On the other hand, \(\sum q^n < \varepsilon\), so \(\lim_{n \to \infty} T_{i=1} g_{n-i} \geq T_{i=1} b_{n-i} = 1\). Now we can apply Theorem 2.1 to find a fixed point of \(f\).

We need the definition of a large class of mappings called weakly decimomappigns. This definition is necessary for the next theorem.

**Definition 3.3** [17]: Let \((S, F)\) be a probabilistic metric space, \(M\) a nonempty subset of \(S\) and \(f : M \rightarrow 2^S \setminus \{\emptyset\}\), a mapping \(f\) is weakly decimomappig if for every sequence \((p_n)_{n \in \mathbb{N}}\) from \(M\) such that \(p_{n+1} \in f p_n\), for every \(n \in \mathbb{N}\) and \(\lim F_{p_{n+1}, p_n}(\varepsilon) = 1\), for every \(\varepsilon > 0\), there exists a convergent subsequence \((p_{n_j})_{j \in \mathbb{N}}\).

In the next theorem, we will not use the conditions of Theorem 3.4. We use the condition of weakly decimomappign for multi-valued \(b_n\)-contraction.

**Theorem 3.5**: Let \((S, F, T)\) be a complete Menger space, \(T\) a \(t\)-norm such that \(\sup \sigma \leq t < 1\), \(T(t, t) = 1\), \(M\) a non-empty and closed subset of \(S\), \(f : M \rightarrow C(M)\) be a multi valued \(b_n\)-contraction that is also weakly decimomappign. If there exist \(p_0 \in M\) and \(p_1 \in f p_0\) such that for all \(\varepsilon > 0\) and \(n \in \mathbb{N}\), \(F_{p_0, p_1}(\varepsilon) > b_n\) and \(\lim q_n = 0\), then \(f\) has a fixed point.

**Proof**: We can construct a sequence \((p_n)_{n \in \mathbb{N}}\) from \(M\), such that \(p_1 \in f p_0\) and \(p_{n+1} \in f p_n\).

Given \(t > 0\) and \(\lambda \in (0, 1)\), we will show that

\[ \lim_{n \to \infty} F_{p_{n+1}, p_n}(t) = 1 \quad (3) \]

Indeed, by assumption there exist \(p_0 \in M\) and \(p_1 \in f p_0\) such that for all \(\varepsilon > 0\) and \(n_0 \in \mathbb{N}\), \(F_{p_0, p_1}(\varepsilon) > b_{n_0}\). By using the contraction relation we can find \(p_2 \in f p_1\) such that \(F_{p_1, p_2}(q_{n_0} \varepsilon) > b_{n_0}\) and by induction \(p_{n+1} \in f p_n\) such that \(F_{p_{n+1}, p_n}(q_{n_0} \varepsilon) > b_{n_0}\) for every \(n_0 \in \mathbb{N}\), especially for \(n_0 = n\). Since \(\lim q_n = 0\) and \(\lim b_n = 1\), for all \(t > 0\) and \(\lambda \in (0, 1)\) by choosing enough large \(n\), \(q^n < t\) and \(b_n > 1 - \lambda\), so \(F_{p_{n+1}, p_n}(t) > 1 - \lambda\), the proof of (3) is complete. By definition 3.3, there exist a subsequence \((p_{n_j})_{j \in \mathbb{N}}\) such that \(\lim p_{n_j}\) exists. We shall prove that \(x = \lim p_{n_j}\) is a fixed point of \(f\). Since \(f x = f x\) and therefore, it remains to prove that \(x \in f x\), i.e., for all \(\varepsilon > 0\) and \(\lambda \in (0, 1)\), there exists \(b'(\varepsilon, \lambda)\), such that \(F_{x, b'(\varepsilon, \lambda)}(\varepsilon) > 1 - \lambda\). From the condition \(\sup \sigma \leq t < 1\), \(T(t, t) = 1\), it follows that there exists \(\eta(\lambda) \in (0, 1)\) such that

\[ u > 1 - \eta(\lambda) \Rightarrow T(u, u) > 1 - \lambda. \]

Let \(j_1(\varepsilon, \lambda) \in \mathbb{N}\) be such that \(F_{p_{n_j}, x}(\varepsilon) > b_n\) for every \(j \geq j_1(\varepsilon, \lambda)\) and enough large \(n\). Since \(x = \lim p_{n_j}\) such a number \(j_1(\varepsilon, \lambda)\) exists. As \(f\) is multi-valued \((b_n)\)-contraction, for \(p_{n_j+1} \in f p_{n_j}\) there exists \(b'(\varepsilon, \lambda) \in \mathbb{N}\) such that \(F_{p_{n_j+1}, b'(\varepsilon, \lambda)}(\varepsilon) > b_n > 1 - \eta(\lambda)\) for all \(j \geq j_1(\varepsilon, \lambda)\) and enough large \(n\). From (3) it follows that \(\lim p_{n_j+1} = x\) and therefore there exists \(j_1(\varepsilon, \lambda) \in \mathbb{N}\) such that \(F_{p_{n_j+1}, x}(\varepsilon) > 1 - \eta(\lambda)\) for every \(j \geq j_1(\varepsilon, \lambda)\). Let \(j_3(\varepsilon, \lambda) = \max \{ j_1(\varepsilon, \lambda), j_2(\varepsilon, \lambda) \}\), then for every \(j \geq j_3(\varepsilon, \lambda)\) we have \(F_{x, b'(\varepsilon, \lambda)}(\varepsilon) = T(F_{p_{n_j+1}, x}(\varepsilon), F_{p_{n_j+1}, b'(\varepsilon, \lambda)}(\varepsilon)) > 1 - \lambda\). Hence, if \(j > j_3(\varepsilon, \lambda)\), then we can choose \(b'(\varepsilon, \lambda) = b'(\varepsilon, \lambda) \in f x\). The proof is complete.

**References**