Quotient of Function Spaces in Topology Groups

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Summary
Topological groups are objects that combine two separate structures; the structure of a topological space and the algebraic structure of a group—linked by the requirement that the group operations are continuous with respect to the underlying topology. Many of the natural infinite groups one encounters in mathematics are in fact topological. With regard to this definition, it is easy to see that oddly enough, if a set is not open, it does not mean that it is necessarily closed. It is possible for a set to be neither closed nor open, or both closed and open at the same time. In fact, we are guaranteed two such sets in the definition of a topology τ. Both X and the empty set are guaranteed to be open, and because they are each other's complements, they are both guaranteed to be closed as well.
Keywords:
Topology; Quotient; Function spaces

1. Introduction

New procedures can be created by gluing edges of the flexible square. For example, by gluing the up and down edges, cylinder is obtained and a tube will be created by gluing the edges of cylinder together. The construction is called unification or quotient. Creating a quotient is a procedure for simplification. Content begins with an equivalence relation and each equivalent level specifies a one point. Mathematics is full of quotient structures. If the field set is topological space, there is always the possibility that the topological quotient set be induced, so that, the mapping of natural image will be successive. However, if the area is eligible the manifold structure will be formed, but it often happens that the quotient space does not form the manifold structure.

Topology is an umbrella term that includes several fields of study. These include pointset topology, algebraic topology, and differential topology. Because of this it is difficult to credit a single mathematician with introducing topology. The following mathematicians all made key contributions to the subject: Georg Cantor, David Hilbert, Felix Hausdorff, Maurice Fr´echet, and Henri Poincar´e. In general, topology is a special kind of geometry, a geometry that doesn't include a notion of distance. Topology has many roots in graph theory. When Leonhard Euler was working on the famous K¨onigsberg bridge problem he was developing a type of geometry that did not rely on distance, but rather how different points are connected. This idea is at the heart of topology. A topological group is a set that has both a topological structure and an algebraic structure.

The aim of this paper is to create the conditions in which a space of quotient function applies to the groups of topology.

2. Theoretical Foundations

Definitions and basic concepts:

Definition:

The ≤ is a Partial Order on set X if it is reflexive, transitive and asymmetrical.

If ≤ is a partial order on X, then X with ≤ is called partial ordered set.

The partial order set (x, ≤) is called lattice, when for each subset (with two members) upper bound and lower bound of the set is available.

Definition: assume that x is a set and τ is a collection of subsets x. If τ applies to the following circumstance, it is called a topology on x. Therefore, (x,τ) is a topological space.

(1) $X \in T$ and $\varnothing \in T$

(2) If $U_1 \in T$ and $U_2 \in T$ then $\aleph$ $U_1 \cap U_2 \in T$

(3) If $T^{/} \subseteq T$ ', then $UT^{/} \in T$

Definition: the feature of a topological space X in x point is as follow:

$\chi(X,x) = \aleph_0 + \min\{|B_x|:\ B_x \text{ is a base for } x \text{ in } X\}$

$\chi(X) = \sup\{\chi(X,x),\ x \in X\}$

Feature of space X:

- X is First-countable if and only if $\chi(X) = \aleph_0$

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Proof: assume that X is first countable. So, it has base countable in x∈X. Therefore:

$\min\{|B_x|: x \in X\} \leq \aleph_0$, So, $\aleph_0 + \aleph_0 \leq \chi(X, x) \leq \aleph_0$. Therefore:

$$x(X) = \sup\{x(X,x); x \in X\} = NO$$

In contrast: if $\aleph_0 = \chi(X)$ then, $U_1 \in T$

$$\sup\{x(X,x); x \in X\} \leq NO$$

It means in each x∈X, due to $\aleph_0 \leq \chi(X, x) \leq \aleph_0$, therefore:

$$NO + \min\{|B_X| : x \in X\} \leq NO$$

Then $\min\{|B_x|: x \in X\} \leq \aleph_0$ so for each $x \in X$, base countable is in $x \in X$, therefore, X is a first countable. Each member of τ is called opened set and complementary opened set of closed set in the space x. If {x}∈τ , then x will be the single point of apace x.

Definition: A topological space is a set X together with a collection τ of subsets of X that satisfy the following conditions: (1) X, {} ∈ τ (2) The union of any sets in τ is in τ (3) The finite intersection of any sets in τ is in τ We will refer to the elements of X as points. We will also call τ a topology on X and we will refer to any element of τ as an open set [4].

Theorem: Let G be a topological group, the following statements are equivalent: (1) G is a T0 − space (2) G is a T1 − space (3) G is a Hausdorff space

Theorem: Let G be a topological group. The following maps are homeomorphisms from G to G for all a ∈ G. (1) the translation maps ra and la (2) the inverse map: $x \to x^{-1}$ (3) the inner automorphism map $x \to axa^{-1}$

Proof: The proof for the translation maps is the same as it was for semi-topological groups; we proved this in Proposition 3.6. For the inverse mapping: we know it is a bijection since G is a group (every element has a unique inverse), it is continuous by definition of a topological group. The inverse map of the inverse map is itself, so it has a continuous inverse. Thus the inverse map is a topological group. Lastly, we have the automorphism map. This is a composition of two homeomorphisms, x → ax and x → xa−1, and is thus a homeomorphism [14].

Definition of quotient space:

Let x be a topological space and ~ be an equivalence relation on X. x^* is defined to be the set of equivalence classes of elements of X. the definition of topological space on x^* is called the quotient space. Therefore:

$\pi: X \to X^*$

$\pi(x) = [x]$

That is, π(x) = the equivalence class containing x. This can be confusing, so say it over to yourself a few times: π is a function from X into the power set of X; it assigns to each point x ∈ X a certain subset of X, namely the equivalence class containing the point x. Since each x ∈ X is contained in exactly one equivalence class, the function x → [x] is well-defined. At the risk of belaboring the obvious, since each equivalence class has at least one member, the function π is surjective [12].

Theorem: assume that $\{X_\alpha\}_{\alpha\epsilon\Gamma}$ is a family of subsets of set X which $X = \cup_{\alpha\in\Gamma} X\alpha$. For each $\alpha \in \Gamma$ assume that $\mathcal{J}$ is a topology on X_α. for each pair $(\alpha,\beta) \in \Gamma \times \Gamma$ the following condition should be provided:

1) $X_\alpha \cap X_\beta$ in both topology $\mathcal{J}_\alpha$ and $\mathcal{J}_\beta$ should be open (close).

2) Induction Topologies should be equal for $X_\alpha \cap X_\beta$, $\mathcal{J}_\alpha$ and $\mathcal{J}_\beta$.

In tis topology $\mathcal{J}$ exist on the X, and for each α we have $\mathcal{J}|x_\alpha = \mathcal{J}_\alpha$.

Proof: we announce a subset U from X as an open subset when for each α, $X_\alpha \cap U$ is open in X_α. Thus, a topology on X can be defined. This topology is called $\mathcal{J}$. It is obvious that $\mathcal{J}|x_\alpha \subseteq \mathcal{J}_\alpha$. Assume that $u \in \mathcal{J}_\alpha$, therefore for each $\beta \in \Gamma$:

$U \cap X_\beta = U \cap (X_\alpha \cap X_\beta)$

Since $X_\beta \cap X_\alpha$ is open in X_α, therefore, $X_\beta \cap U$ is open in X_α, and because the induction topology of X_α and X_β is equal on $X_\alpha \cap X_\beta$, so $U \cap X_\beta$ is open on X_β. $\mathcal{J} \in U$ and $\mathcal{J}\alpha \subseteq \mathcal{J} |X_\alpha$ therefore $\mathcal{J} |X_\alpha = \mathcal{J}_\alpha$.

Theorem: Let G be a group and let H be a normal subgroup. Then the left cosets of H in G form a group denoted G/H. G/H is called the quotient of G modulo H. The rule of multiplication in G is defined as (aH)(bH) = abH. Furthermore, there is a natural surjective homomorphism φ: G −→ G/H, defined as φ(g) = gH. Moreover the kernel of φ is H. Proof. We have already checked that this rule of multiplication is welldefined.

Definition: Let G be a topological group acting on the left on a set X. A symmetric neighbourhood U of 1G gives rise to an entourage (x, y) ∈ X2 : x ∈ U · y , and these generate the right G-uniformity on X. When X is a topological space, the collection of bounded complex functions on X that are continuous with respect to the topology on X and right uniformly continuous with respect to the group action is denoted RUCBG(X) [2] .

Definition: We say that a point x of a topological space X has a local G-base if there exists a base of neighbourhoods at x of the form U(x) = {Uα(x) : α ∈ N N}, where Uβ(x) ⊆ Uα(x) whenever α ≤ β for all α, β ∈ N N. The space X is said to have a local G-base if it has a G-base at each point [7].

Definition: Let G be a topological group and let H be a normal subgroup of G. From Definition 2.67 we know that G/H is a group. Let φ be the mapping from G to H by φ(x) = xH, we will refer to this function as the canonical mapping from G to G/H. We can define a topology on G/H as follows: U is open in G/H if and only if φ −1 (U) is open in G, we call this the quotient topology [12].

Theorem: Let G/H be a topological group with the quotient topology and φ as above. The following three statements hold: (1) φ is onto (2) φ is continuous (3) φ is open

Proof: Let gH ∈ G/H. It follows that φ maps g to gH, thus φ is onto. Proposition 2.20 tells us that φ : G → G/H is continuous when U is open if and only if φ −1 is open. This condition follows directly from the definition of the quotient topology. Let U be open in G. We need to show that φ(U) is open. We know that φ(U) = UH. By Proposition 3.7 we get that UH is open [8].

Definition: Let G be a topological group. A family B ⊆ V(1) is said to be a base of neighborhoods of 1 (or briefly, a base at 1) if for every U ∈ V(1) there exists a V ∈ B contained in U (such a family will necessarily be a filterbase).

Definition (Linear topologies) = Let V = {Ni: i ∈ I} be a filter base consisting of normal subgroups of a group G. Then V satisfies (a)–(c), hence generates a group topology on G having as basic neighborhoods of a point g ∈ G the family of cosets {gNi : i ∈ I}. Group topologies of this type will be called linear topologies. Let us see now various examples of linear topologies [8]

Example: let G be a group and let p be a prime:

• The pro-finite topology, with {Ni: i ∈ I} all normal subgroups of finite index of G

• The pro-p-finite topology, with {Ni: i ∈ I} all normal subgroups of G of finite index that is a power of p

• The p-adic topology, with I = N and for n ∈ N, Nn is the subgroup (necessarily normal) of G generated by all powers {g p n: g ∈ G}

• The natural topology (or Z-topology), with I = N and for n ∈ N, Nn is the subgroup (necessarily normal) of G generated by all powers {g n: g ∈ G}

• The pro-countable topology, with {Ni: i ∈ I} all normal subgroups of at most countable index [G: Ni]. The next simple construction belongs to Taimanov. Now neighborhoods of 1 are subgroups which are not necessarily normal.

Exercise 3.8: Let G be a group with trivial center. Then G can be considered as a subgroup of Aut (G) making use of the internal automorphisms. Identify Aut (G) with a subgroup of the power GG and equip Aut (G) with the group topology τ induced by the product topology of GG, where G carries the discrete topology. Prove that:

• The filter of all τ -neighborhoods of 1 has as base the family of centralizers {cG(F)}, where F runs over all finite subsets of G

• τ is Hausdorff; • τ is discrete iff there exists a finite subset of G with trivial centralizer.

Theorem: Let G be a topological group and let H be normal in G. The following statements hold:

(1) The canonical mapping φ(x) = xH is a continuous and open homomorphism.

(2) G/H with the quotient topology is a topological group

Proof: By Theorem 4.14 we have that φ is continuous and open. To show that φ is a homomorphism, let x, y ∈ G. To start, we have φ(xy) = xyH. From the properties of cosets we get xyH = xHyH. We can simplify to get φ(x)φ(y). Thus φ(xy) = φ(x)φ(y) and φ is a homomorphism

Definition: The quotient G\X is equipped with the quotient topology, so, that the quotient map π : X → G\X is continuous: a subset V ⊆ G\X is open if and only if π −1 (V) ⊆ X is open. The projection π is an open map, which is to say if U ⊆ X is open then π(U) = GU ⊆ G\X is open: indeed, if U is open then π −1 (π(U)) = S g∈G gU is open, so π(U) is open by definition of the topology [10]

Lemma: Let G be a topological group. Let A(G) be a C ∗ -subalgebra of `∞(G) with 1 ∈ A(G) and T ⊆ G

(i) T is an A(G)-interpolation set if and only if A is injective on T and there is a homeomorphism between T A and βTd, the StoneCech-compactification of ˇ T equipped with the discrete topology, that leaves the points of T fixed.

(ii) T is an A(G)-interpolation set if and only if for every pair of subsets T1, T2 ⊂ T, T1 ∩ T2 = ∅ implies T1 A ∩ T2 A = ∅.

(iii) If T is an A(G)-interpolation set and f : T → C is a bounded function, then f has an extension f ∈ A(G) with kfk∞ = kfkT .

(iv) If T is an approximable A(G)-interpolation set, then for every bounded function h: T → C and every neighbourhood U of the identity, there is f ∈ A(G) such that f T = h, f(G \ UT) = {0} and kfk∞ = khkT .

Proof: First observe that A is injective on every A(G)-interpolation set

T: if t1 6= t2 ∈ T, there is f ∈ `∞(T) with f(t1) 6= f(t2). Take ¯f ∈ A(G) extending f.

By (1) ¯f = ¯f A ◦ A, hence A(t1) 6= A(t2).

Assertion (i) follows then from the universal property defining the StoneCech compactification of a discrete space. In fact, the restriction of the ˇ evaluation map A to T gives a homeomorphism of the discrete set Td onto its image in GA. So T A is a (topological) compactification of Td, and we may apply.

Assertion (ii) follows also directly from a well-known characterization of the Stone-Cech compactification of a discrete space, see for instance.

To prove (iii), let f: T → C with kfkT = M be given. If BM is the closed disc of radius M centered at 0 (in C), we can use (i) and the universal property of βTd to find a continuous function f β: T A → BM with f β T = f.

Then, by Tietze's extension theorem, f β can be extended to a continuous function f A: GA → BM, the restriction f A G is then the desired extension. To prove (iv), let T be an approximable A(G)-interpolation set. First, we find, using (iii), f1 ∈ A with f1 T = h and kf1k∞ = khkT . The definition of approximable A(G)-interpolation sets provides two neighbourhoods V1, V2 with V1 ⊆ V2 ⊆ U and f2 ∈ A such that f2(V1T) = {1} and f2(G \ V2T) = {0} [8]

Proposition 1: Let H be an open subgroup of a topological group G. The uniform structure Ul(G/H) is discrete

Proof: Since H is a neighborhood of the identity of G, the image of the entourage Hl of the diagonal under (π × π) is the set {(fH, gH):f, g ∈ G, f −1g ∈ H}={(fH, gH) : f, g ∈ G, g ∈ fH} = {(fH, fH) : f ∈ G}. The latter set is the diagonal of G/H ×G/H. Hence, Ul(G/H) is just the discrete uniform structure

Assume that X be a set. Let SX denote the symmetric group on X, consisting of all self-bijections of the set X and equipped with the composition of bijections as the group law. Let γ be a partition of X. Define St γ = {f ∈ SX: ∀B ∈ γ, f(B) = B}. Plainly, St γ is a subgroup of SX [1].

Proposition: The subgroups St γ, as γ runs over all partitions of X with |γ| ≤ c, form a neighborhood basis for a Hausdorff group topology on SX, which we will denote by τc.

Proof: Let Γ = {γ: γ is a partition of X with |γ| ≤ c}. Since, for any γ ∈ Γ, St γ is a group, St 2 γ = St γ = St −1 γ . Let γ ∈ Γ and g ∈ St X. Define a cover β of X as follows: B ∈ β if and only if B = g(A), for some A in γ. Evidently β is in Γ and g −1St βg ⊆ St γ. If γ, β ∈ Γ, then define a cover α of X as follows: A set D is in α if and only if D = A ∩ B for some A ∈ γ and B ∈ β. Clearly, α is in Γ and St α ⊆ St γ ∩ St β. Now the topology can be defined by taking the set **Ω** = {St γ: γ is a partition of X, |γ| ≤ c}, as a basis of open neighborhoods of the identity. If ι is the identity of SX and f ∈ SX not equal to ι, then there exists x ∈ X such that f(x) 6= x. Put γ = {{x}, {f(x)}, X\{x, f(x)}}. Then clearly, γ ∈ Γ and f 6∈ St γ. This implies T γ∈Γ St γ = ι. Hence, SX is Hausdorf [5].

Lemma: Let G be a topological group, A1(G) ⊆ A2(G) ⊆ `∞(G) be two C ∗ -subalgebras with 1 ∈ A1(G), and let (Tη)η< κ and every φ ∈ A1(G)

Proof: Let T = S η< κ, a function hη : G → [−1, 1] supported on Tη with hη(T1,η) = {1} and hη(T2,η) = {−1}. Then consider the function h: G → [−1, 1] supported on T and given by h(t) = hη(t) if t ∈ Tη for some η < κ.

By Statement (iv) of Lemma 2.3, there is a a function f ∈ A2(G) such that f(G \ UT) = 0, f T = h and kfk∞ = khkT = 1. Let now φ be any function in A1(G), and take ε > 0. Given η < κ, we are going to prove that kf − φkTη ≥ 1 − ε. Take pη ∈ T1,η A1 ∩ T2,η A1 and pick t1,η ∈ T1,η and t2,η ∈ T2,η with |φ(t1,η) − φ A1 (pη)| < ε and |φ(t2,η) − φ A1 (pη)| < ε, where φ A1 denotes the extension of φ to GA1 . Then (2) 2 = |hη(t1,η) − hη(t2,η)| = |h(t1,η) − h(t2,η)| = |f(t1,η) − f(t2,η)| ≤ |f(t1,η) − φ(t1,η)| + |φ(t1,η) − φ A1 (pη)| + |φ A1 (pη) − φ(t2,η)| + |φ(t2,η) − f(t2,η)|.

It follows that either |f(t1,η) − φ(t1,η)| ≥ 1 − ε or |f(t2,η) − φ(t2,η)| ≥ 1 − ε. Since ε > 0 was arbitrary, we find that kf − φkTη ≥ 1. Since kfk∞ = 1 and f(G \ UT) = {0}, we see that f is the required function [3].

Proposition: Let M be a closed subspace of a normed linear space X. Then the following statements hold.

(a) π is continuous, with kπ(f)k = kf + Mk ≤ kfk for each f ∈ X.

(b) Let BX r (f) denote the open ball of radius r in X centered at f, and let B X/M r (f +M) denote the open ball of radius r in X/M centered at f + M. Then for any f ∈ X and r > 0 we have π B X r (f) = B X/M r (f + M).

(c) W ⊆ X/M is open in X/M if and only if π −1 (W) = {f ∈ X : f + M ∈ W} is open in X.

(d) π is an open mapping, i.e., if U is open in X then π(U) is open in X/M. Proof.

(a) Choose any f ∈ X. Since 0 is one of the elements of M, we have kπ(f)k = kf + Mk = inf m∈M kf − mk ≤ kf − 0k = kfk

(b) First consider the case f = 0 and r > 0. Suppose that g + M ∈ π BX r (0) . Then g + M = h + M for some h ∈ BX r (0), i.e., khk < r. Hence kg + Mk = kh + Mk ≤ khk < r, so g + M ∈ B X/M r (0 + M). Now suppose that g + M ∈ B X/M r (0 + M). Then infm∈M kg − mk = kg + Mk < r. Hence there exists m ∈ M such that kg − mk < r. Thus g − m ∈ BX r (0), so g + M = g − m + M = π(g − m) ∈ π B X r (0) [5].

3. Conclusion

Lemma: Every two separated compact set in a Hausdorff space have separate neighborhoods.

Proof: Consider two compressed and distinct sets A and B. According to aforementioned Lemma, for each $X \in B$, there are neighborhoods with distinct V_X and U_X from X and A. As aforementioned proof, we can find finite cover like $\{V_{X1} \ldots V_{Xn}\}$ for compressed set B due to the neighborhoods $V = \bigcup_{I=1}^{n} v_{x_i}$ and $U = \prod_{i=1}^{n} u_{x_i}$ are distinct from A and B. Therefore, it is proved.

Lemma: in Hausdorff space X, each compressed set has a distinct open neighborhood with each point in complementary set. Specially, when in a Hausdorff space, each compressed set is close.

Proof: consider the compressed set F and point P which P is not $P \in F$. Since the space is Hausdorff, for each x∈X, V_X and U_X distinct open neighborhoods are exist from X and P, respectively. Since $\{V_X\}_{x \in X}$ is a open cover for the compressed set F, there is a finite sub-cover like $\{V_{X1} \ldots V_{Xn}\}$ for it. $V = \bigcup_{I=1}^{n} v_{x_i}$ and $U = \prod_{i=1}^{n} u_{x_i}$ are distinct open neighborhoods of P and F. Therefore, it is proved.

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