# Simulation of Moment, Cumulant, Kurtosis and the Characteristics Function of Dagum Distribution

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**Abstract**
The Dagum distribution is a special case of Generalized Beta II distribution with the parameter q=1, has 3 parameters, namely a and p as shape parameters and b as a scale parameter. In this study some properties of the distribution: moment, cumulant, Kurtosis and the characteristics function of the Dagum distribution will be discussed. The moment and cumulant will be discussed through Moment Generating Function. The characteristics function will be discussed through the expectation of the definition of characteristics function. Simulation studies will be presented. Some behaviors of the distribution due to the variation values of the parameters are discussed.

***Key words:***
*Dagum distribution, moment, cumulant, kurtosis, characteristics function.*

## 1. Introduction

Nowadays the development of the statistical theory is very advance, especially the theory of distribution. One of the distributions that used a lot in practice is Dagum distribution. This distribution is a special case of Generalized Beta II Distribution (GB 2) with the parameter at GB2 is q=1. This distribution has three parameters, namely (a, p) as shape parameters and b as a scale parameter. There are many studies in the theory of distribution discuss the characteristics function from a distribution. Through the characteristics function we can discuss the mean, variance and standard deviation, also we can discuss the moment, cumulant and characteristics function. Moment is one of the importance characteristics of a distribution; there are two kinds of moment, moment with respect to origin and moment with respect to mean. Moment with respect to the mean can be used to see the skewness and kurtosis which can explain the behavior of the distribution of the data through graph. Moment with respect to the origin is the moment which can be used to evaluate the mean. Cumulant is one of the importance characteristics alternative of moment of a distribution, in other words the moment can define cumulant. Besides moment and cumulant, the characteristics function also very importance. The study of Dagum distribution will be discussed in this paper. The Dagum distribution was introduced by Camilo Dagum 1970's [1, 2, 3], this distribution is a special case of Generalized Beta II distribution. Dagum (1977, 1980) distribution has been used in studies of income and wage distribution as well as wealth distribution. In this context its features have been extensively analyzed by many authors, for excellent

survey on the genesis and on empirical applications see [4]. His proposals enable the development of statistical

distribution used to empirical income and wealth data that could accommodate both heavy tails in empirical income and wealth distributions, and also permit interior mode [5]. A random variable X is said to have a Dagum distribution with the parameters a, b and p, if and only if the density function of X is as follows [4]:

$$f(x \mid a,b,p) = \begin{cases} \dfrac{ap\,x^{ap-1}}{b^{ap}\left[1+\left(\dfrac{x}{a}\right)^{a}\right]^{p+1}}, & x>0, \quad a,b,p>0 \\ 0 & \text{Otherwise.} \end{cases}$$

where $(a, p)$ are shape parameters and b is a scale parameter. The mean and variance of Dagum distribution are as follows:

$$E(x) = b\,\frac{\Gamma\left(p+\frac{1}{a}\right)\Gamma\left(1-\frac{1}{a}\right)}{\Gamma(p)}$$

and the variance

$$= b^2\,\frac{\Gamma(p)\Gamma\left(p+\frac{2}{a}\right)\Gamma\left(1-\frac{2}{a}\right)-\Gamma^2\left(p+\frac{1}{a}\right)\Gamma\left(1-\frac{1}{a}\right)}{\Gamma^2(p)}$$

## 2. Method of analysis

### 2.1 Moment Generating Function(Mgf)

Moment generating function of Dagum distribution is

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$$Mx(t) = \sum_{n=0}^{k} \frac{t^n b^n}{n!} \frac{\Gamma\left(\mathrm{p} + \frac{\mathrm{n}}{\mathrm{a}}\right)\Gamma\left(1 - \frac{\mathrm{n}}{\mathrm{a}}\right)}{\Gamma(\mathrm{p})}$$

## 2.2 Characteristics Function

A characteristic function is unique for a distribution and the characteristic function can be used to find the moment of a random variable. If the random variable X and the characteristic function denoted by $\Phi$ for all $t \epsilon \mathcal{R}$, then [6]

$$\Phi\mathrm{c}(\mathrm{t}) = \mathrm{E}(\mathrm{e}^{\mathrm{itx}})$$

The function $\Phi\mathrm{c}(\mathrm{t})$ generally is complex, where: $\mathrm{E}(\mathrm{e}^{\mathrm{itx}}) = \mathrm{E}\{\cos \mathrm{tx} + \mathrm{i} \sin \mathrm{tx}\}$

$$\Phi\mathrm{c}(\mathrm{t}) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k} b^{2k}}{2k!} \frac{\Gamma\left(\mathrm{p} + \frac{2\mathrm{k}}{\mathrm{a}}\right)\Gamma\left(1 - \frac{2\mathrm{k}}{\mathrm{a}}\right)}{\Gamma(\mathrm{p})} + \sum_{k=0}^{\infty} (-1)^k \frac{it^{2k+1} b^{2k+1}}{2k+1!} \frac{\Gamma\left(\mathrm{p} + \frac{2k+1}{\mathrm{a}}\right)\Gamma\left(1 - \frac{2k+1}{\mathrm{a}}\right)}{\Gamma(\mathrm{p})}$$

where i is complex number, i=$\sqrt{-1}$ .

## 2.3. The-r moment with respect to origin

If X is a random variable and r is a positive integer, then the-r moment of the random variable X with respect to the origin in defined as [7] :

$$\mu'_r = E(X^r)$$

The-r moment with respect to the origin can be found from the characteristics function as follows [8]:

$$\mu'_{\mathrm{r}} = (-\mathrm{i})^{\mathrm{r}} \left(\frac{\mathrm{d^r C(t)}}{\mathrm{dt^r}}\right) \text{ for } \mathrm{t} = 0$$

where $\mathrm{C(t)}$ is characteristic function.

So that we found the first moment is:

$$= b\, \frac{\Gamma\left(\mathrm{p}+\frac{1}{\mathrm{a}}\right)\Gamma\left(1-\frac{1}{\mathrm{a}}\right)}{\Gamma(\mathrm{p})},$$

Second moment is:

$$= b^2\, \frac{\Gamma\left(\mathrm{p}+\frac{2}{\mathrm{a}}\right)\Gamma\left(1-\frac{2}{\mathrm{a}}\right)}{\Gamma(\mathrm{p})}, \ldots, \text{ and}$$

the-r moment is:

$$= b^r\, \frac{\Gamma\left(\mathrm{p} + \frac{\mathrm{r}}{\mathrm{a}}\right)\Gamma\left(1 - \frac{\mathrm{r}}{\mathrm{a}}\right)}{\Gamma(\mathrm{p})}$$

## 2.4 The-r moment with respect to mean

Let k be a positive integer and C is a constant, and then $E(X-c)^k$ is the moment of order k. If $c = E(X) = \mu$, then $E(X-\mu)^k$ is the central moment of order k. The moment with respect to mean is defined by $\mu_k$, and [7, 9]

$$\mu_k = E(X-\mu)^k$$

Then the first central moment of the Dagum distribution is:

$$\mu_1 = \mu'_1 - \mu'_1 = 0\ ,$$

The second central Moment is:

$$\mu_2 = \mu'_2 - {\mu'_1}^2 = b^2 \left\{\frac{\Gamma(\mathrm{p})\Gamma\left(\mathrm{p}+\frac{2}{\mathrm{a}}\right)\Gamma\left(1-\frac{2}{\mathrm{a}}\right) - \Gamma^2\left(\mathrm{p}+\frac{1}{\mathrm{a}}\right)\Gamma^2\left(1-\frac{1}{\mathrm{a}}\right)}{\Gamma^2(\mathrm{p})}\right\},$$

The third Central Moment is: $\mu_3 = \mu'_3 - 3\mu'_1\mu'_2 + 2{\mu'}_1^3$

$$= b^3 \left\{\frac{\Gamma^2(\mathrm{p})\Gamma\left(\mathrm{p}+\frac{3}{\mathrm{a}}\right)\Gamma\left(1-\frac{3}{\mathrm{a}}\right)}{\Gamma^3(\mathrm{p})} - \frac{3\Gamma(\mathrm{p})\Gamma\left(\mathrm{p}+\frac{2}{\mathrm{a}}\right)\Gamma\left(1-\frac{2}{\mathrm{a}}\right)\Gamma\left(\mathrm{p}+\frac{1}{\mathrm{a}}\right)\Gamma\left(1-\frac{1}{\mathrm{a}}\right)}{\Gamma^3(\mathrm{p})} + \frac{2\Gamma^3\left(\mathrm{p}+\frac{1}{\mathrm{a}}\right)\Gamma^3\left(1-\frac{1}{\mathrm{a}}\right)}{\Gamma^3(\mathrm{p})}\right\}, \ldots,$$

and the-r central moment is

$$= \sum_{i=0}^{r} \binom{r}{i} \frac{b^{r-i}\Gamma\left(\mathrm{p}+\frac{\mathrm{r-i}}{\mathrm{a}}\right)\Gamma\left(1-\frac{\mathrm{r-i}}{\mathrm{a}}\right)}{\Gamma(\mathrm{p})} \left(-\frac{b\Gamma(\mathrm{p}+1)\Gamma\left(1-\frac{1}{\mathrm{a}}\right)}{\Gamma(\mathrm{p})}\right)^i.$$

## 2.5 Cumulant

Cumulant of a random variable X is defined by [8, 9] :

$$K(t) = \log(Mx(t))$$

where $Mx(t)$ is moment generating function of a distribution, and its coefficient of the Taylor series is cumulant, namely [9]:

$$\log(Mx(t)) = k_n \frac{t^n}{n!}$$

By using Taylor series we have:

$$\log(Mx(t)) = t\mu'_1 + \frac{1}{2}t^2(\mu'_2 - {\mu'_1}^2) + \frac{1}{3}t^3(2\mu'_1 - 3\mu'_1\mu'_2 + \mu'_3) + \cdots$$

Then we have:

$k_1 = \mu'_1$,

$k_2 = (\mu'_2 - {\mu'_1}^2)$,

$k_3 = (2\mu'_1 - 3\mu'_1\mu'_2 + \mu'_3)$, and so on.

Cumulant also can be found from moment with respect to origin.

$$k_r = \mu'_r - \sum_{n=1}^{r-1} \binom{r-1}{n-1} k_n \mu'_{r-n}$$

The first cumulant of the Dagum distribution is:

$$k_1 = \mu'_1$$

$$= b\frac{\Gamma\left(p+\frac{1}{a}\right)\Gamma\left(1-\frac{1}{a}\right)}{\Gamma(p)}$$

The second cumulant is:

$$k_2 = \mu'_2 - \mu'^2_1$$

$$= b^2\left\{\frac{\Gamma(p)\Gamma\left(p+\frac{2}{a}\right)\Gamma\left(1-\frac{2}{a}\right) - \Gamma^2\left(p+\frac{1}{a}\right)\Gamma^2\left(1-\frac{1}{a}\right)}{\Gamma^2(p)}\right\}$$

The third cumulant is:

$$k_3 = \mu'_3 - 3\mu'_1\,\mu'_2 + 2\,\mu'^3_1$$

$$= b^3\left\{\frac{\Gamma^2(p)\Gamma\left(p+\frac{3}{a}\right)\Gamma\left(1-\frac{3}{a}\right)}{\Gamma^3(p)} - \frac{3\Gamma(p)\Gamma\left(p+\frac{1}{a}\right)\Gamma\left(1-\frac{1}{a}\right)\Gamma\left(p+\frac{2}{a}\right)\Gamma\left(1-\frac{2}{a}\right)}{\Gamma^3(p)} + \frac{2\Gamma^3\left(p+\frac{1}{a}\right)\Gamma^3\left(1-\frac{1}{a}\right)}{\Gamma^3(p)}\right\}$$

,...., and the-r cumulant is:

$$\left\{b^r\frac{\Gamma^{r-1}(p)\Gamma\left(p+\frac{r}{a}\right)\Gamma\left(1-\frac{r}{a}\right)}{\Gamma^r(p)} - \sum_{n=1}^{r-1}\binom{r-1}{n-1}\frac{\Gamma^{r-n}(p)}{b^n}k^n\frac{\Gamma\left(p+\frac{r-n}{a}\right)\Gamma\left(1-\frac{r-n}{a}\right)}{\Gamma^{r-n}(p)}\right\}$$

## 2.6 Skewness

Skewness is the degree of asymmetric of a distribution. A symmetric distribution is a distribution which has the same mean, median and mode values. When the mean, median and mode of a distribution are not the same, it will imply that the distribution is not symmetric. This property can be seen in the skewness of a distribution. When the moment with respect to the mean has been found, the skewness can be defined as follows [10]:

$$\mathrm{Skew(X)} = \frac{\mathrm{E(X-\mu)^k}}{\sigma^3} = \frac{\mu^3}{\mu_2^{\frac{3}{2}}}$$

A distribution is symmetric if the graph of the distribution is symmetric around the central point.

If the $(X) = 0$ , then the distribution is symmetric at the central point or the mean; if $\mathrm{Skew(X)} > 0$ then the distribution is skew to the right, and if $\mathrm{Skew(X)} < 0$ then the distribution is skew to the left. The skewness of the Dagum distribution is

$$\mathrm{Skew(X)} = \frac{\mu_3}{\mu_2{}^{3/2}} = \frac{\Gamma^2(p)\Gamma\left(p+\frac{3}{a}\right)\Gamma\left(1-\frac{3}{a}\right) - 3(p)\Gamma(p)\left(p+\frac{1}{a}\right)\Gamma\left(1-\frac{1}{a}\right)}{\Gamma^{3/2}(p)\Gamma^{3/2}\left(p+\frac{2}{a}\right)\Gamma^{3/2}\left(1-\frac{2}{a}\right) - \Gamma^3\left(p+\frac{1}{a}\right)\Gamma^3\left(1-\frac{1}{a}\right)} + \frac{+2\Gamma^3\left(p+\frac{1}{a}\right)\Gamma^3\left(1-\frac{1}{a}\right)}{\Gamma^{3/2}(p)\Gamma^{3/2}\left(p+\frac{2}{a}\right)\Gamma^{3/2}\left(1-\frac{2}{a}\right) - \Gamma^3\left(p+\frac{1}{a}\right)\Gamma^3\left(1-\frac{1}{a}\right)}$$

$$\frac{+6\Gamma(p)\Gamma\left(p+\frac{2}{a}\right)\Gamma\left(1-\frac{2}{a}\right)\Gamma^2\left(p+\frac{1}{a}\right)\Gamma^2\left(1-\frac{1}{a}\right) - 3\Gamma^4\left(p+\frac{1}{a}\right)\Gamma^4\left(1-\frac{1}{a}\right)}{+\Gamma^4(p)\Gamma^4\left(p+\frac{1}{a}\right)\Gamma^4\left(1-\frac{1}{a}\right)}$$

## 2.7 Kurtosis

Kurtosis is a measurement to see the sharpness of a peak of a distribution. Kurtosis itself measures the high and low of its peak and is measured relative to the normal distribution. Like the skewness, kurtosis also can be seen from the graph of a distribution. When the moment with respect to the mean have been found, $\mu_k = E(X-\mu)^k$ , then the Kurtosis is

$$\mathrm{Kurto(X)} = \frac{\mathrm{E(X-\mu)^4}}{\sigma^4} = \frac{\mu_4}{\mu_2{}^2}$$

A distribution which has Kurtosis more sharper, if it has $\mathrm{Kurto(X)} > 3$. In this case is called leptokurtic, namely the peak of the distribution relatively high. On the other hand, $\mathrm{Kurto(X)} < 3$ is called platocurtic, where the peak of a distribution almost flat. If $\mathrm{Kurto(X)} = 3$ is called mesocurtic, where the peak of the distribution is relatively not high and not flat. Kurtosis of Dagum distribution is:

$$Kurto(X) = \frac{\mu_4}{\mu_2{}^2}$$

$$\frac{\Gamma^3(p)\Gamma\left(p+\frac{4}{a}\right)\Gamma\left(1-\frac{4}{a}\right) - 4\Gamma^2(p)\Gamma\left(p+\frac{3}{a}\right)\Gamma\left(1-\frac{3}{a}\right)\Gamma\left(p+\frac{1}{a}\right)\Gamma\left(1-\frac{1}{a}\right)}{\Gamma^2(p)\Gamma^2\left(p+\frac{2}{a}\right)\Gamma^2\left(1-\frac{2}{a}\right) - 2\left[(\Gamma(p)\Gamma\left(p+\frac{2}{a}\right)\Gamma\left(1-\frac{2}{a}\right)\Gamma^2\left(p+\frac{1}{a}\right)\Gamma^2\left(1-\frac{1}{a}\right)\right]}$$

# 3. Simulation

## 3.1 Probability Density Function(Pdf)

To see the change of the shape of a distribution, we can see through the graph of probability density function of Dagum distribution by changing the values of the parameters.

1. Parameter *a* fixed, *b* fixed, and *p* decrease
   *a* = 26, *b* = 12 and *p* = 78, 67,45, 34, 20, 17.

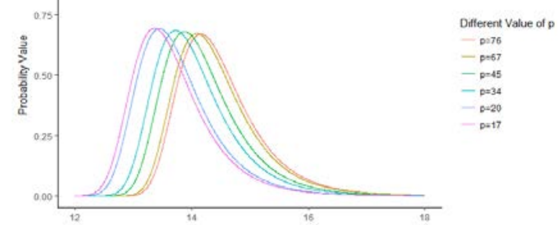

Fig.1 The graph of the pdf of Dagum distribution with the parameter values a fixed, b fixed and p decrease

Figure.1 shows that the graph of the pdf of Dagum distribution has the same sharpness, but has different sloping, the scale of the graph of Dagum distribution has the same scale. But for the values of the parameter p getting smaller, the graph shifted to the left or move toward negative in accordance with the change values of parameter p, the smaller the value of p, the more sloping it is.

2. Parameter a fixed, b decrease and p fixed a = 26, b = 14, 13.75, 13, 12.55, 12, 11 and p = 67

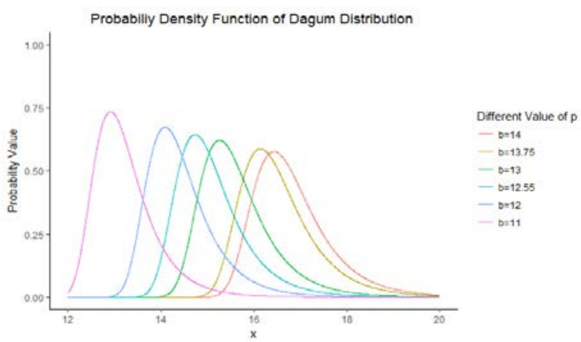


Fig. 2 The graph of pdf of Dagum distribution where the values of a fixed, b decrease and p fixed.

Figure 2 shows that at the values of a = 26, b = 14, 13.75, 13, 12.55, 12, 11 and p = 67, the graph of the pdf of Dagum distribution has the different shape of sharpness and sloping but not significance due to a and p fixed. But the change of the parameter b implies the change of the scale or the shape of the distribution. This is due to the values of b getting smaller.

3. Parameter a increase, b fixed, and p fixed a = 26, 45, 50, 65, 80, 100, b = 12 and p = 67.

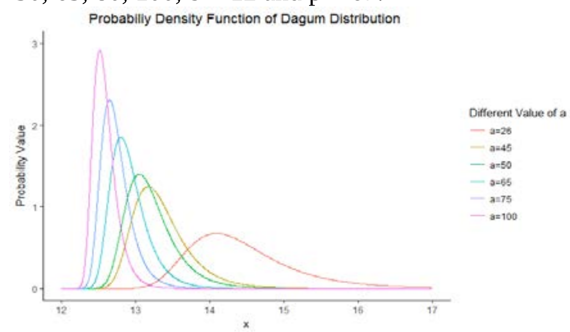


Fig. 3 The graph of pdf of Dagum distribution the variation values of parameter a increase, b and p fixed

Figure 3. is the graph of Dagum distribution with the variation values of parameters a = 26, 45, 50, 65, 80, 100, b = 12, and p = 67. It shows that the graph of pdf Dagum distribution has the shape of peak sharpness and its sloping changing by the change of the values of parameter a. For the values of parameter a getting larger, it implies that the sloping is getting larger and the graph getting high and sharpness.

## 3.2 Skewness

Parameter a = seq(50, 55, 0.1) and p = seq(1, 5, 0.1)

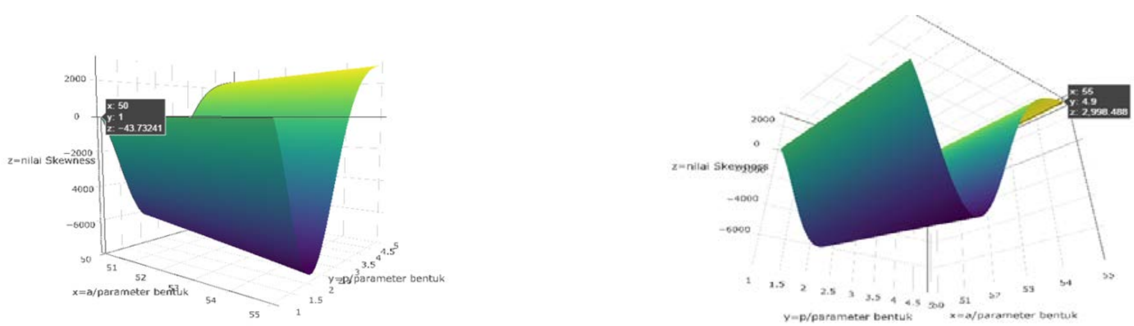


Fig. 4 The graph of Skewness of the Dagum distribution with the variation values of parameters a and p.

Figure 4. The graph of skewness of the Dagum distribution with the values of the parameter a at the x-axis and the values of parameter p at the y-axis has the values of skewness negative and positive at the z-axis. For the green layer where the values of the parameter a around 50 up to 50.8 with the values of parameter p from 1 to 1.3 has the value of skewness greater than 0 (positive), this means that it has skew to the right. For the yellow layer where the values of the parameter a around 50 up to 55 with the values of parameter p from 2 to 3 has the value of skewness greater than 0 (positive), this means that it has skew to the right. For the purple layer where the value of

the parameter a around 50 to 55 with the values of parameter p from 3 to 5 has the value of skewness less than 0 (negative), this means that it has skew to the left.

### 3.3 Kurtosis

Parameter a = seq(1, 4, 0.1) and b = seq(1, 70, 0.1)

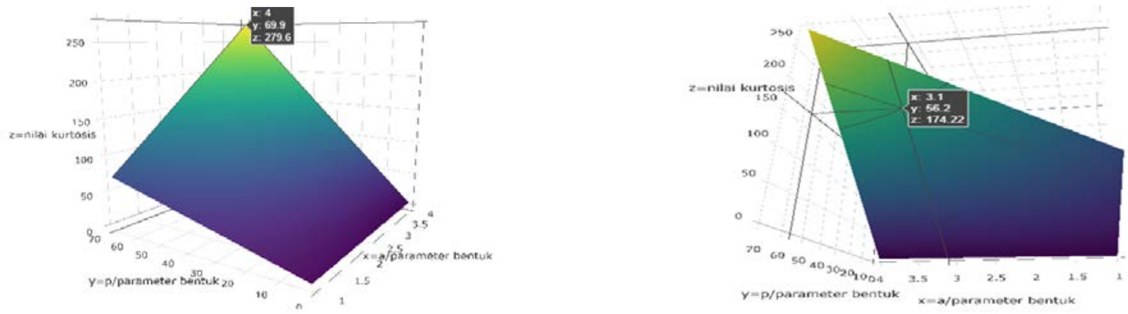


Fig. 5 The Kurtosis graph of the Dagum distribution with the variation values parameters a and p.

Figure 5. kurtosis of the Dagum distribution with the values of the parameter a at the x-axis, and the values of parameter p at the y-axis, has the values of skewness negative and positive at the z-axis. For the green layer, yellow, blue and purple are the values of kurtosis with the parameter a from 1 to 4 and the values of p from 1 to 70 has the values of Kurtosis positive, this mean that the distribution has sharp peak and relatively high and is called leptokurtic.

### 3.4 Moment Generating Function (MGF)

Parameter a = seq(26, 36, 0.5), p = seq(67, 77, 0.5) and b fixed = 12

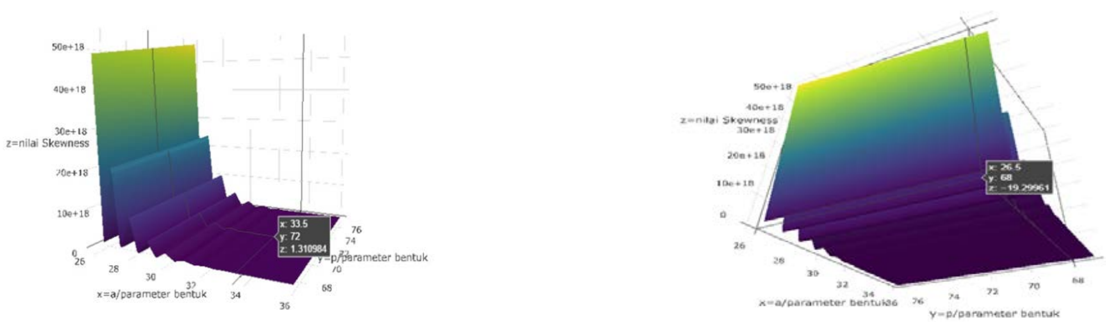


Fig. 6 The graph of MGF Dagum distribution with the variation value of parameter a, p and b fixed.

Figure 6. MGF of Dagum distribution with the values of the shape parameter a at the x-axis, and the values of the parameter p at the y-axis have the values of MGF at the z-axis. For every values of MGF of the Dagum distribution at the parameters a, b and p has the values positive and negative. The graph for the value of a= 33,5 and 26,5 parameter p = 72 and 68 and b = 12 the values of MGF are 1,310984 and -19,299 . For the value of the parameter a and b getting larger imply that the value of MGF positive.

### 3.5 Cumulant

Parameter a = seq(26, 36, 0.5) , b = seq(12, 22, 0.5) and p=2 fixed.


Fig. 7 The graph of Cumulant of the Dagum distribution with the variation values of parameters a, b and p fixed.

Figure 7. cumulant of the Dagum distribution with the values of parameter a at the x-axis, and the values of parameter b at the y-axis have the values of cumulant at the z-axis. For every value of cumulant of the Dagum distribution at the values of parameters a,b, and p have values positive and negative. For the value of a=26.5, b=13 and p=2 the value of cumulant is -214.926. But for the value of parameter a = 36, b = 22 and p = 2 the cumulant is positive. For the values of parameter a and b getting larger, the values of cumulant are getting positive.

### 3.6 Characteristic Function

Parameter a = seq(26,36,0.5), b=seq(12,22,0 has values positive and negative) and p = 67

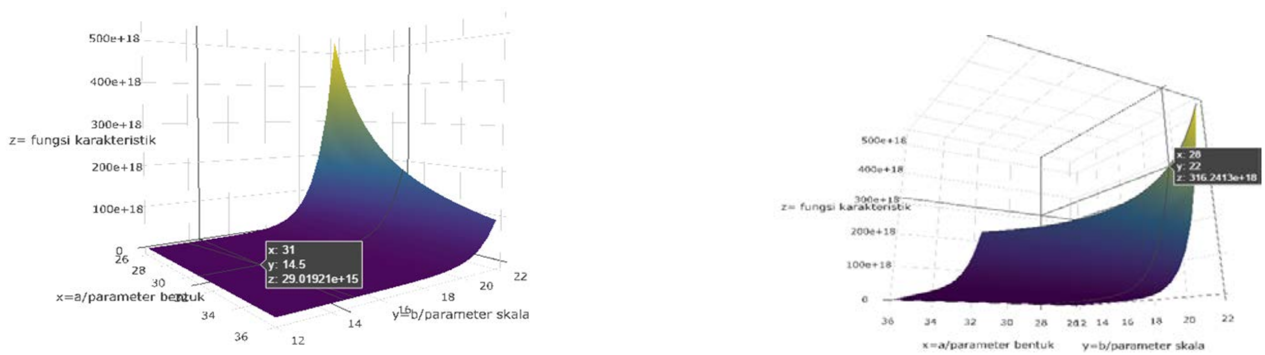


Fig. 8 The graph of the characteristic function of the Dagum distribution with variation values of the parameters a, b and p fixed

Figure 8 the characteristic function of the Dagum distribution with the value of the parameter a at the x-axis and the value of the parameter b at the y-axis has the value of the characteristic function at the z-axis. The characteristic values of the Dagum distribution are positive for each values of the parameters a, b, and p . For the value of a = 28, b = 22 and p = 67 the value of cumulant is z = 316.2413 $e^{18}$.

## 4. **Conclusion**

Based on the results of analysis and simulation we conclude that: The parameters of the Dagum distribution (a, b, p) with two shape parameters and one scale parameter have difference function to depict the graph of the Dagum distribution. The shape parameter p used to see the change of the graph at the right side of the Dagum distribution and also to see the movement of the graph toward the positive and negative direction. The shape parameter a used to see the change of the graph at both side's right and left of the graph, the shape of either the peak sharpness or sloping can be seen clearly. The scale parameter b used to see the distribution of the data due to the change in the values of scale parameter b, it's also has impact in the change of either the shape of the peak sharpness or sloping of the graph of Dagum distribution.

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