The RPS algorithm to handle a class of computational models arising in computer science

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Abstract.
In this paper, we apply the residual power series technique to find out the solutions for second-order integro-differential equations (IDEs) system of Volterra type subject to given initial conditions. The new technique is effective and easy to use for solving linear and nonlinear IDEs without linearization, perturbation, or discretization. This approach provides the solutions in the form of a rapidly convergent series with easily computable components using symbolic computation software. The proposed technique obtains the Taylor expansions of the solutions and reproduces the exact solutions when the solutions are polynomials. The solutions obtained using the present method are tested by solving initial value problems of IDEs. Graphical results, tabulate data, and numerical comparisons are presented and discussed quantitatively to illustrate the solutions. Numerical results show the potentiality, the generality, and the superiority of our algorithm for solving such problems arising in computer science.

Key words:
RPS, IDEs

1. Introduction

Integro- differential equations of Volterra type for ordinary differential equations arise very frequently in many branches of applied mathematics and physics such as gas dynamics, nuclear physics, chemical reactions, atomic structures, atomic calculations, study of positive radial solutions of nonlinear elliptic equations, and so forth. In most cases, IDEs of Volterra type do not always have solutions which can be obtained using analytical methods. In fact, many of real physical phenomena encountered, are almost impossible to solve analytically, these problems must be attacked by various approximate and numerical methods.

The main advantage of the RPS method is the simplicity in computing the coefficients of terms of the series solutions by using the differential operators only and not as the other well-known analytic techniques that need the integration operators which is difficult in general [1-7]. Moreover, the proposed method can be easily applied in the spaces of higher dimension solutions and can be applied without any limitation on the nature of the systems and the type of classifications. Numerical techniques are widely used by scientists and engineers to solve their problems. A major advantage for numerical techniques is that a numerical answer can be obtained even when a problem has no analytical solution. On the other hand, many applications for different problems by using other numerical algorithms can be found in [8-15].

Here, we extend the application of the residual power series method to construct the solutions the following general form:

$$x_j^{(n)}(t) = f_j\big(t, x_1(t), x_2(t), \dots, x_n(t)\big) + \int_0^t k_j(t,\tau) g_j\big(x_1(\tau), x_2(\tau), \dots, x_n(\tau)\big) d\tau ,$$

subject to the constraints initial conditions

$$x_j(0) = \alpha_{0j}, x_j'(0) = \alpha_{1j}, \dots, x_j^{(n-1)}(0) = \alpha_{(n-1)j} \quad (2)$$

where $0 \leq \tau < t \leq 1$, $f_j: [0,1] \times \mathbb{R}^n \to \mathbb{R}$, $g_j: \mathbb{R}^n \to \mathbb{R}$ are analytic functions, $k_j(t,\tau)$ are continuous arbitrary kernel functions over the square $o \leq \tau < t \leq 1, j = 1,2, \dots, n$.

In this paper, we using the residual power series method (RPSM) to develop a new numerical method for obtaining smooth approximations to solutions and their derivatives for systems of IDEs of Volterra type. This paper is organized as follows. In Section 2, a short description for the RPS is presented. Also, we discuss the problem of the study. In Section 3, we present some numerical results. Finally, conclusions are given in Section 4.

2. The proposed algorithm

In this section, we construct solutions to such systems by substituting their residual power series expansion among their truncated residual functions. From the resulting equations recursion formulas for the computation of the coefficients are derived, while the coefficients in the expansions can be computed recursively by recurrent differentiating of the truncated residual functions by means of the symbolic computation software used [16-23].
The RPS technique is different from the traditional higher order Taylor series approach. The Taylor series approach is computationally expensive for large orders. By using residual error concept, we get series solutions, in practice truncated series solutions as well as other methods [23-38].
To apply the residual power series method, set the counter

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$i = 1,2,\dots,n$ and rewrite the system of IDEs (1) and (2) in the form of the following:

$$x_i^{(n)}(t) = f_i\big(t, x_1(t), x_2(t), \dots, x_n(t)\big), + \int_{t_0}^{t} k_i(t,\tau) g_i\big(x_1(\tau), x_2(\tau), \dots, x_n(\tau)\big) d\tau \quad (3)$$

subject to the constraints initial conditions

$$x_j(0) = \alpha_{0j}, x_j'(0) = \alpha_{1j}, \dots, x_j^{(n-1)}(0) = \alpha_{(n-1)j} \quad (4)$$

The residual power series technique consists in expressing the solutions of IDE (3) and (4) as a power series expansion about the initial point $t = t_0$. To achieve our goal, we suppose that these solutions take the form

$$x_i(t) = \sum_{m=0}^{\infty} x_{i,m}(t),$$

where $x_{i,m}$ are terms of approximations and are given as $x_{i,m}(t) = c_{i,m}(t - t_0)^m$.

If we choose $x_{i,0}(t) = x_i(t_0)$ as initial guesses approximations of $x_i(t)$, then we can calculate $x_{i,m}(t)$ for $m = 1,2,\dots$ and approximate the solutions $x_i(t)$ of IDE (3) and (4) by the kth-truncated series

$$x_i^k(t) = \sum_{m=0}^{k} c_{i,m}(t - t_0)^m. \quad (6)$$

Prior to applying the residual power series technique, we rewrite IDEs (3) and (4) in the form of the following:

$$x_i^{(n)}(t) - f_i\big(t, x_1(t), x_2(t), \dots, x_n(t)\big) - \int_{t_0}^{t} k_i(t,\tau) g_i\big(x_1(\tau), x_2(\tau), \dots, x_n(\tau)\big) d\tau = 0. \quad (7)$$

The subsisting of kth-truncated series $x_i^k(t)$ into equation (7) will leads to the following definition for the kth residual functions:

$$\begin{aligned}\mathrm{Res}_i^k(t) &= \sum_{m=2}^{k} m(m-1) c_{i,m}(t - t_0)^{m-2} \\ &- f_i\left(t, \sum_{m=0}^{k} c_{1,m}(t - t_0)^m, \sum_{m=0}^{k} c_{2,m}(t - t_0)^m, \dots, \sum_{m=0}^{k} c_{n,m}(t - t_0)^m\right) \\ &- \int_{t_0}^{t} k_i(t,\tau) g_i\left(\sum_{m=0}^{k} c_{1,m}(\tau - t_0)^m, \sum_{m=0}^{k} c_{2,m}(\tau - t_0)^m, \dots, \sum_{m=0}^{k} c_{n,m}(\tau - t_0)^m\right) d\tau,\end{aligned} \quad (8)$$

and the following ∞ th residual functions: $\mathrm{Res}_i^{\infty}(t) = \lim_{k\to\infty} \mathrm{Res}_i^k(t)$.

It easy to see that, $\mathrm{Res}_i^{\infty}(t) = 0$ for each $t \in [t_0, t_0 + a]$. This show that $\mathrm{Res}_i^{\infty}(t)$ are infinitely many times differentiable at $t = t_0$. On the other hand, $\frac{d^s}{dt^s}\mathrm{Res}_i^{\infty}(t_0) = \frac{d^s}{dt^s}\mathrm{Res}_i^{k}(t_0) = 0$, for each $s = 1,2,\dots,k$. In fact, this relation is a fundamental rule in residual power series technique and its applications. Now, to obtain the 1st-approximate solutions, we put $k = 1$ and substitute $t = t_0$ into equation (8) and using the fact that $\mathrm{Res}_i^{\infty}(t_0) = \mathrm{Res}_i^{1}(t_0) = 0$, to conclude the following value:

$$c_{i,1} = f_i(t_0, c_{1.0}, c_{2.0}, \dots, c_{n.0}) = f_i(t_0, \alpha_1, \alpha_2, \dots, \alpha_n).$$

Thus, using 1st-truncated series the first approximation for IDEs (3) and (4) can be written as

$$f_i(t_0, \alpha_1, \alpha_2, \dots, \alpha_n)(t - t_0).$$

$$x_i^1(t) = x_i(t_0) + \quad (10)$$

Similarly, to find the 2nd approximation, we put $k = 2$ and $x_i^2(t) = \sum_{m=0}^{2} c_{i,m}(t - t_0)^m$. On the other hand, we differentiate both sides of equation (8) with respect to t and substitute $t = t_0$, to get

$$\begin{aligned}\frac{d}{dt}\mathrm{Res}_i^2(t_0) &= 2c_{i,2} - \frac{\partial}{\partial t} f_i(t_0, \alpha_1, \alpha_2, \dots, \alpha_n) \\ &- \sum_{j=1}^{n} c_{j,1} \frac{\partial}{\partial x_j^2} f_i(t_0, \alpha_1, \alpha_2, \dots, \alpha_n) \\ &- k_i(t_0, t_0) g_i(\alpha_1, \alpha_2, \dots, \alpha_n).\end{aligned} \quad (11)$$

In fact $\frac{d}{dt}\mathrm{Res}_i^2(t_0) = \frac{d}{dt}\mathrm{Res}_i^{\infty}(t_0) = 0$. Thus, one can write

$$\begin{aligned}c_{i,2} &= \frac{1}{2}\Big[\frac{\partial}{\partial t} f_i(t_0, \alpha_1, \alpha_2, \dots, \alpha_n) \\ &+ \sum_{j=1}^{n} c_{j,1} \frac{\partial}{\partial x_j^2} f_i(t_0, \alpha_1, \alpha_2, \dots, \alpha_n) \\ &+ k_i(t_0, t_0) g_i(\alpha_1, \alpha_2, \dots, \alpha_n)\Big].\end{aligned} \quad (12)$$

Hence, using 2 nd-truncated series the second approximation for system of IDEs (3) and (4) can be written as

$$\begin{aligned}x_i^2(t) &= x_i(t_0) + f_i(t_0, \alpha_1, \alpha_2, \dots, \alpha_n)(t - t_0) \\ &+ \frac{1}{2}\Big[\frac{\partial}{\partial t} f_i(t_0, \alpha_1, \alpha_2, \dots, \alpha_n) \\ &+ \sum_{j=1}^{n} c_{j,1} \frac{\partial}{\partial x_j^2} f_i(t_0, \alpha_1, \alpha_2, \dots, \alpha_n) \\ &+ k_i(t_0, t_0) g_i(\alpha_1, \alpha_2, \dots, \alpha_n)\Big](t - t_0)^2.\end{aligned} \quad (13)$$

This procedure can be repeated till the arbitrary order coefficients of the residual power series solutions of equations (3) and (4) are obtained.

3. Numerical Examples

In most real-life situations, the differential equation that models the problem is too complicated to solve exactly, and there is a practical need to approximate the solution. In the next two examples, the exact solutions cannot be found analytically.

xample: Consider the following nonlinear system of second-order IDEs:

$$\begin{aligned}x_1^{(4)}(t) &= f_1(t) + x_1(t) x_2(t) + \int_0^t \big(x_1^2(\tau) x_2^3(\tau)\big) d\tau, \\ x_2^{(4)}(t) &= f_2(t) + e^{x_1(t)} + x_2(t) - \int_0^t x_1^4(\tau) - x_2^4(\tau) d\tau,\end{aligned} \quad (14)$$

subject to the following initial conditions:

$$\begin{aligned} &x_1(0)=1, x_1'(0)=0, x_1''(0)=0, x_1'''(0)=0,\\ &x_2(0)=0, x_2'(0)=0, x_2''(0)=2, x_2'''(0)=0 \end{aligned} \quad (15)$$

where $f_1(t)$ and $f_2(t)$ are chosen such that the exact solutions are: $x_1(t)=\cos(t^2)$ and $x_3(t)=\sin(t^2)$.

If we select the initial guesses as $x_{1,0}(t)=1$, $x_{1,1}(t)=0$, $x_{1,2}(t)=0, x_{1,3}(t)=0$, $x_{2,0}(t)=0$, and $x_{2,1}(t)=0, x_{2,2}(t)=2, x_{2,3}(t)=0$, then the first few terms approximations for equations (14) and (15) are

$$\begin{aligned} &x_{1,2}(t)=0, x_{1,3}(t)=0, x_{1,4}(t)=-\frac{1}{2}t^4, x_{1,5}(t)=0, \dots,\\ &x_{2,2}(t)=t^2, x_{2,3}(t)=0, x_{2,4}(t)=0, x_{2,5}(t)=0, \dots. \end{aligned} \quad (16)$$

If we collect the above results, then the 20th-truncated series of the residual power series solution for $x_1(t)$ and $x_2(t)$ are as follows:

$$\begin{aligned} x_1^{20}(t) &= 1-\frac{1}{2}t^4+\frac{1}{24}t^8-\frac{1}{720}t^{12}+\frac{1}{40320}t^{16}-\frac{1}{3628800}t^{20}\\ &+\frac{1}{479001600}t^{24}-\frac{1}{87178291200}t^{28}+\frac{1}{20922789888000}t^{32}\\ &-\frac{1}{6402373705728000}t^{36}+\frac{1}{2432902008176640000}t^{40}\\ &=\sum_{j=0}^{10}(-1)^j\frac{(t^2)^{2j}}{(2j)!} \end{aligned} \quad (17)$$

$$\begin{aligned} x_2^{20}(t) &= t^2-\frac{1}{6}t^6+\frac{1}{120}t^{10}-\frac{1}{5040}t^{14}+\frac{1}{362880}t^{18}-\frac{1}{6}t^{22}\\ &+\frac{1}{6227020800}t^{26}-\frac{1}{1307674368000}t^{30}\\ &+\frac{1}{355687428096000}t^{34}-\frac{1}{121645100408832000}t^{38}=\sum_{j=0}^{9}(-1)^j\frac{(t^2)^{1+2j}}{(1+2j)!} \end{aligned} \quad (18)$$

Thus, the exact solutions of equations (14) and (15) have the general form

$$x_1(t)=\cos(t^2),\ x_2(t)=\sin(t^2). \quad (19)$$

In Figure 1, we plot $\mathrm{Ext}_1^k(t)$ and $\mathrm{Ext}_2^k(t)$ for $k=5,10,15,20$ which are approaching the axis $y=0$ as the number of iterations increase. Figure 2 shows $\mathrm{Res}_1^k(t)$ and $\mathrm{Res}_2^k(t)$ for $k=5,10,15,20$.

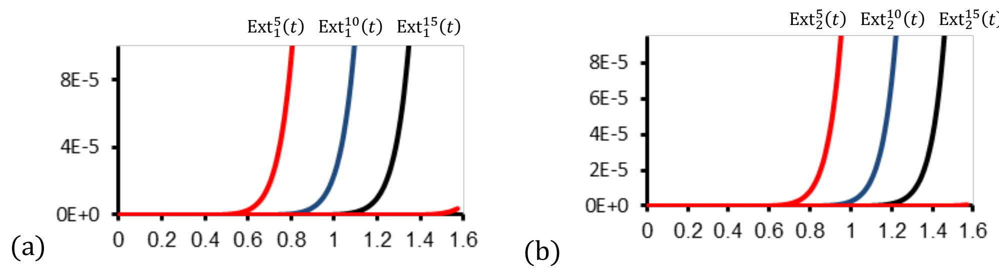

Fig. 1 Plots of exact error functions for equations (14) and (15).

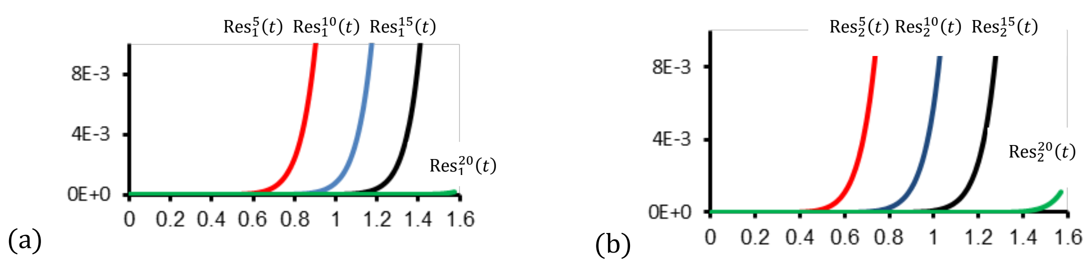

Fig. 2 Plots of residual error functions for equations (14) and (15)

4. Conclusion

Here, we can conclude that the RPS method is powerful and efficient technique in finding approximate solution for higher-order nonlinear systems of IDEs. There is an important point to make here, the results obtained by the RPS technique are very effective and convenient in nonlinear cases with less computational work and time. This confirms our belief that the efficiency of our technique gives it much wider applicability for general classes of nonlinear problems.

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