Positive solutions of fuzzy fractional Volterra integro-differential equationswith the Fuzzy Caputo Fractional Derivative using the Jacobi polynomials operational matrix

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Abstract
In this paper study,we introduce fuzzy fractional Volterraintegro- differentional equations (FFVIDEs) under Caputo generaliuzed Hukuhara differentiability.There are many applications of the G- derivative ,but we give positive solutions for (FFVIDEs)under Riemann -Liouville gH-differentiability using theJacobi polynomials operational matrix. Weintroduce Riemann -Liouville gH-differentiability as a direct generalization of the concept of fuzzy Caputo differentiability in a deterministic sense for a fuzzy context.Wepropose a computational method based on the tau method with Jacobi polynomials for FFVIDEs of order $0<\beta<1$. The efficiency and applicability of the proposed method are demonstrated by several test examples. Finally,we give some illustrative examples.

Keywords:
Caputo H-differentiability, fuzzy-valued function, fuzzy fractional Volterra integro-differential equations, Jacobi polynomial operational matrix, Positive solution

1. Introduction

The theory of fractional calculus deals with the investigation and applications of derivatives and integrals of arbitrary order.The theory of fractional calculus developed mainnly as a purely theoretical field of mathematical،fractional calculus but the study of fuzzy fractional Volterra integro- differentional equations (FFVIDEs) has expanded rapidly in recent years due to considerable interest inboththeir mathematics and applications.The Caputo gH -differentiability concept based on Hukuhara differentiability can be employed to solve fuzzy fractional differential equations (FFDEs). Recently,Agarwal et al. [1] proposed the concept of solving fractional differential equations under Riemann-Liouville H-differentiability.but this concepts are not implementeexplicitly in the fractional case under Caputo H-differentiabinility. Moreover, a recent study, the generalized differentiability concepts was is introduced for fuzzy-valued functions. As a consequence, several properties of these concepts have been investigated and similar fuzzy differentiability connections have been compared between them. Hence, we extend the definitions of generalized gH-differentiability to the fractional case. In this study, we introduced the concept of Caputo H-differentiability as a direct generalization of fuzzy fractional Caputo derivative by usin the Hukuhara difference for (FFVIDEs) .We consider several possible definitions for the derivative of an interval valued function and the connections between them and their properties. Previously, generalized differentiability concepts is were introduced for fuzzy-valued functions. and several properties of these concepts were investigated by comparing the similar fuzzy differentiability connections between them.

In thes present study,،we also consider the properties of the well-known Jacobi polynomials operational matrix.We consider the existence and uniqueness of the solution to the initial value problem (*)

$(D^C_{a+})_\beta U(t)=F(t,\lambda u(t)) \qquad (D^C_{a+})_{\beta-1}u(t_0)=u_0^{\beta-1}\in E$

for FFVIDEs involving the Riemann-Liouville sequential fuzzy Caputo fractional G-derivative by using the Jacobi polynomials operational matrix method.

The, Mittag-Leffler function has a major role in the theory of fractional calculus for differential equations[2,3].

and it is stated as follows:

$$E_{\alpha,\beta}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)},\ \ t\in C,\alpha,\beta\neq 0$$

$$E_{\alpha}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+1)},\ \ t\in C., \qquad (1)$$

Recently monographs and research studies in the field of differential equations have provided solutions,such as those by Belmekki et al.[8], Komatik [8], Nito [4, 9], and Sequin [8]. Thes studiesemployed the fuzzy Laplace transforms method to solve fuzzy fractional differential equations. In Section [3].of the present study is obtained the exploit we obtain solutions of fractional equations under Riemann-Liouville Hukuhara differentiability using theJacobi polynomials operational matrix.

Previously,Two new uniqueness resultswere investigated for FFDE involving Riemann-Liouville generalized Hukuhara differentiability [4]. with the Kerion-type condition.Inthe present is study,we extend the Jacobi polynomials operational matrix method to solved

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(FFVIDEs).We need to convert thefollowing underlying (FFVIDEs) .

$(D^{C}_{a+})_{\beta}U(t) = F(t, \lambda u(t)) \qquad (D^{C}_{a+})_{\beta-1}u(t_0) = u_0^{\beta-1} \in E$

Where $0 < T < +\infty$, and $f \in C([0,T] \times E \times E)$

We consider the existence and uniqueness of the initial value problem for FFVIDEs involving Riemann-Liouville generalized Hukuhara differentiability by using the Jacobi polynomials operational matrix method.

Allahviranloo et al. [18] considered Fuzzy fractional differential equations under a generalizedfuzzy Caputo derivative.In addition,explicit solutionshave been presented for uncertain fractional differential equations under Riemann-LiouvilleH-differentiability [11]. Two new uniqueness resultswere alsoextended for fuzzy fractional differential equations involvingRiemann-Liouville generalized G-differentiability with fuzzy versions of the Nagumo and Krasnoselskiiconditions [12]. Another study investigated a fuzzy fractional integral equation with the Riemann-Liouville derivative and theexistence of the solutions for fuzzy fractional integral equations was established using theHausdorff measure of noncompactness [13]. The existence and uniqueness of solutions for nonlinearfuzzy integral equations of fractional order were established using the generalized Schauder theoremand contraction mapping principle [14] while compactly supported fuzzy sets were introduced in another study [15].The Caputo of FFVIDEsunder generalized Hukuhara (gH-)differentiability is introduced in the present study using the Jacobi polynomials operationalmatrix.follows. In Section 2, we recall some well-knowndefinitions of fuzzy numbers and wedefine some necessary concepts.

In Section [3-1], Riemann-Liouville gH-differentiability is defined and the Caputo of (FFVIDEs) is considered under fuzzy Caputo gH -differentiability .We study the existence and uniqueness of the solution for a Caputo of (FFVIDEs)using the initial value. In Section.4we solve some examples where the solutions are represented using the Jacobi polynomials operational matrix.

2. Preliminaries

In this section, we recall some definitions and introduce the necessary notation used throughout theisstudy.We denote the set of all real number by R,and the set of all fuzzy numbers on R is indicated by R_F.

Definition 1.[18] *A fuzzy number* $\tilde{u}$ *is a pair* $(\underline{u}(r), \overline{u}(r))$ *of functions* $\underline{u}(r), \overline{u}(r)$; $0 \le r \le 1$, *which satisfy the following requirements:*

- $\underline{u}(r)$ is a bounded monotonically increasing left continuous function;
- $\overline{u}(r)$ is a bounded monotonically decreasing left continuous function;
- $\underline{u}(r) \le \overline{u}(r),\ 0 \le r \le 1.$

Definition 2.*The metric* d *on* R_F *is given by* $d: R_F \times R_F \to R^+ \cup \{0\}$

$D(u,v) = sup\ d_H(u_r, v_r)$ (2)

where $d_H$is the Hausstdorff metric defined as follows.

$d_H(u_r, v_r) = \max\{(v_r^- - u_r^-), (v_r^+ - u_r^+)\}, \quad r \in [0,1]$ (3)

We list the following properties of d:

[(i)]

1. $d(u,v) = 0$
2. $d(u,v) = d(u+w, v+w)$
3. $d(\lambda u, \lambda v) = |\lambda| d(u,v),$
4. $d(u,v) \le d(u,w) + d(w,v)$

Ifwe have the$D$ metric on R_F.then(R_F, D) is a complete metric space.For all $u,v,w, \in R_F$ and $\lambda \in R$, we uniquely define the sum $u \oplus v$ and the product $\lambda \bullet u$ by

$[u \oplus v]_r = [u]^r + [v]^r \qquad ,[\lambda \bullet u]^r = \lambda [u]^r, \quad \forall r \in [0,1]$ (4)

Definition 3 .[8] *The generalized Hukuhara difference of two fuzzy number* $u, v \in R_F$ *is defined as follows*

$u\Theta_{gH} v = w \Leftrightarrow \quad (i) u = v + w \ \vee \ (ii) v = u + (-1)w$ (5)

Note that a function $f: [a,b] \to R_F$ is the so called fuzzy-valued function. The r level representation fuzzy-valued function f is expressed by:$f_r(t) = [f_r^-(t), f_r^+(t)], \quad t \in [a,b],\ r \in [0,1]$

Definition 4 .[18] *The generalized Hukuhara difference is the fuzzy number* w, *so it is easy to show that* (i) *and* (ii) *are both valid if and only if* w *is a crisp number. In terms of* r*-cuts* ,

Definition 5.[6] Let $x, y \in E$If $z \in E$ exist such that $x = y + z$, then z is called the H-difference of x and y, which is denoted by $x\Theta y \ne x + (-1)y$.

The generalized difference (g-difference for short)of two fuzzy numbers u,v $\in E$ *is given by the followinge expression [26].*

$$[u\Theta_g v]_r = \left[inf_{\beta \ge r}\{min(U_r^- - V_r^-, u_r^+ - v_r^+)\},\ sup_{\beta \ge r}\{max(u_r^- - v_r^-, u_r^+ - v_\alpha^+)\}\right]$$

For any fuzzy numbers u,v $\in E$ *the g-difference* $u\Theta_g v$ *exists and it is a fuzzy number*.In this study,‹we consider the definition of fuzzy differentiability given by Bede and Gal[26].

$[u\Theta_{GH} v]_r = [\min\{(u_r^- - v_r^-), (u_r^+ - v_r^+)\}, \max\{(u_r^- - v_r^-), (u_r^+ - v_r^+)\}]$ (6)

If $w = [u\Theta_{gH} v]$ exists as a fuzzy number, then its level cuts $[w_r^-, w_r^+]$ are obtained by

$w_r^- = \min\{(u_r^- - v_r^-)\}\max\{(u_r^- - v_r^-), (u_r^+ - v_r^+)\}$ and $w_r^+ = \max\{(u_r^- - v_r^-), (u_r^+ - v_r^+)\}$ for $r \in [0,1]$, and if the H -difference exists, then $u\Theta_{gH} v = u -_H v$; The conditions for the existence of $u\Theta_{gH} v = w \in R_F$ are as follows.

Case(i):

$w_r^- = u_r^- - v_r^-$ and $w_r^+ = u_r^+ - v_r^+ w_r^- \le w_r^+$ with w_r^- increasing
$w_r^+ = u_r^+ - v_r^+$ and $w_r^+ = u_r^- - v_r^- w_r^- \le w_r^+$ with w_r^+ decreasing (7)

Case (ii):
$w_r^- = u_r^- - v_r^-$ and $w_r^+ = u_r^+ - v_r^+, w_r^- \le w_r^+$ with w_r^- decreasing
$w_r^- = u_r^+ - v_r^+$ and $w_r^+ = u_r^- - v_r^-, w_r^- \le w_r^+$ with w_r^+ increasing (8)

We suppose that if either case (i) or (ii) exists ,then Definition 4. should hold for any $\alpha \in [0,1]$. Howeve, $w_0^- = u_0^- - v_0^- = 0 < w_0^+ = u_0^+ - v_0^+ = 1$ while $1 = w_1^- > w_1^+ = 0$, so either case (i) or (ii) is not true from Eq. (5).

Definition 6 . The g difference G-different for short of two numbers $u, v \in R_F$ is given by its level sets as

$$[u\Theta_G v]_r = \left[\min\left\{\inf_{\beta\ge r}(U_r^- - V_r^-), \inf_{\beta\ge r}(u_r^+ - v_r^+)\right\}, \quad \max\left\{\sup_{\beta\ge r}(u_r^- - v_r^-), \sup_{\beta\ge r}(u_r^+ - v_\alpha^+)\right\}\right] \quad (9)$$

$$[u\Theta_G v]_r = \mathrm{cl}(\cup_{\beta\ge r}([u]_\beta \Theta_{GH}[V]_\beta)), \forall r \in [0,1] \quad (10)$$

Where the $generalized\ Hukuhara\ difference\ (gH - difference)\ \Theta_{gH}$ is given with interval operands $[U_\beta]$ and $[V_r]$.

Lemma 1 .For any fuzzy number $u, v \in R_F$ the gH-difference exists and it is a fuzzy number,so $u\Theta_{gH}v$ is the smallest fuzzy number **w.**

$$\mathrm{cl}([u]_{\beta\ge r}\Theta_{gH}[v]_{\beta\ge r}) = [u\Theta_G V]_\beta = [W]_{\beta\ge r} r \in [0,1] \le \beta \quad (11)$$

Proof. If we denote $(w)^- = (u\Theta_{gH}V)^-$ and $(W)^+ = (u\Theta_{gH}V)^+$ we have

$$(w_\beta^-)_{\beta\ge r} = \inf_{\beta\ge r}\min\{(u_\beta^- - v_\beta^-), (u_\beta^+ - v_\beta^+)\} \le (w_\beta^+)$$
$$= \sup_{\beta\ge r}\max\{(u_\beta^- - v_\beta^-), (u_\beta^+ - v_\beta^+)\}$$

and it follows that

$$\inf_{\beta\ge r}\min\{(u_\beta^- - v\beta^-), (u_\beta^+ - v_\beta^+)\}, \sup_{\beta\ge r}\max\{(u_\beta^- - v_\beta^-), (u_\beta^+ - v_\beta^+)\}$$
$$= [u -_G v]_r \quad (12)$$
$$= \mathrm{cl}\bigcup_{\beta\ge r}([u]_\beta -_{gH}[V]_\beta)$$

We observe that $([u]_\beta -_{gH}[V]_\beta) \subseteq [w]_\beta$ and thus$(\cup_{\beta\ge r}[u]_\beta -_{gH}[V]_\beta) \subseteq [w]_\beta$ is closed,so we obtain $[u -_G v]_r = cl(\cup_{\beta\ge r}([u]_\beta -_{gH}[V]_\beta))$.

Definition 7. Let $t_o \in (a,b)$ and h be such that $t_o + h \in (a,b)$, then the level-wise gH-derivative of a function $f:(a,b) \to R_F$, at t_o is defined as the set of set-valued gH-derivatives, if they exist,as follows.

$$f_{gH}'^{L}(t_o; r) = \lim_{h\to 0}\frac{1}{h}\{[f'_-(r; t_o + h) -_{gH} f'_-(r; t_o)], [f'^-(r; t_o + h) -_{gH} f'^-(t_o)]\} \quad (13)$$

$f_{gH}'^{L}(t_o; r)$ is a compact interval for all $r \in [0,1]$,and we say that f is level-wise generalized Hukuhara differentiable at t_0 and the family of intervals $f_{gH}'^{L}(t_o; r) \in [0,1]$ is the LGH-derivative of f at t_o.

Definition 8.[18,19]The generalized Hukuhara derivative of a fuzzy -valued function $f:(a,b) \to R_F$ *at* t_o *is defined as follows.*

$$f'_{gH}(t_o) = \lim_{h\to 0}\frac{f(t_o+h) -_{gH} f(t_o)}{h} = \lim_{h\to 0}\frac{f(t_o) -_{gH} f(t_o - h)}{h} \quad (14)$$

$f'_{gH}(r, t_o) \in R_F$ and we say that f is (gH)-differentiable at t_oIn addition, we say that f is $[(i) - gH]$-differentiable at t_o if

$$f'_{GH}(r, t_o) = [f'_-(r, t_o), f'^-(r, t_o)], r \in [0,1] \quad (15)$$

and that f is $[(ii) - gH]$-differentiable at t_0

$$f'_{GH}(r, t_o) = [f'^-(r, t_o), f'_-(r, t_o)], r \in [0,1]. \quad (16)$$

Definition 9 .[8]. *We say that a point* $t_o \in (a,b)$ *is a switching point for the differentiability of f,ifthe points*$t_o \in (t_1, t_2)$ *exist in neighborhood* V *of* x_0s*uch that*
type (I) at $t_1(15)$ holdswhereas(16) does not hold andit holds at $t_2(16)$ holds and (15) dose not hold, or
type(II) at $t_1(16)$ holds whereas(15) dose not hold and it holds at $t_2(15)$but(16) does not hold.

3 .Riemann-Liouville and Caputo GH-differentiability

Now,we introduce the definition of fuzzy Caputo as well as the Riemann-Liouville integrals and derivatives under Hukuhara difference.We provide definitions and statements similar to the non-fractional case in a fuzzy context [1]. We propose definitions of Riemann-Liouville and Caputo differentiability in the fuzzycontext literature based on the Hukuhara difference.The definition is similar to the concept of the Caputo-type derivative in the crisp case [5]and it is a direct extension of strongly generalized H-differentiability [6] to the fractional context.

Definition 10.[3] L et $f:[a,b] \to E$, *and the fuzzy Riemann-Liouville integral of the fuzzy-valued function f is defined as follows:*

$$(I_{a+}^\beta)f(t) = \frac{1}{\Gamma(1-\beta)}\int_a^t \frac{f(s)}{(t-s)^\beta}ds$$
$$= \frac{1}{\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}f(s)ds, \quad x > a, \quad \beta \in (0,1] \quad (17)$$

Since ,$f(t,r) = [f_-(t;r)f^-(t,r)]$, for all $r \in [0,1]$ then we can indicate the fuzzy Riemann-Liouville integral of the fuzzy -valued function based on the lower and upper functions as follows:

$$(I_{a+}^\beta)f(t;r) = [(I_{a+}^\beta)f_-(t;r), (I_{a+}^\beta)f^-(t;r)], r \in [0,1] \quad (18)$$

$$(I_{a+}^\beta)f_-(t;r) = \frac{1}{\Gamma(1-\beta)}\int_a^t \frac{f_-(s;r)}{(t-s)^\beta}ds$$
$$= \frac{1}{\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}f_-(s;r)ds, x > a, \beta \in (0,1] \quad (19)$$

and

$$(I_{a+}^\beta)f^-(t;r) = \frac{1}{\Gamma(1-\beta)}\int_a^t \frac{f^+(s;r)}{(t-s)^\beta}ds$$
$$= \frac{1}{\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}f^+(s;r)ds, \quad x > a, \quad \beta \in (0,1]$$

(20)

Definition .11 Let $f': [a,b] * E \to E$,and the fuzzy Caputo derivative of fuzzy-valued function f is defined as follows[19]*:*

$$(D_{a+}^{\beta})_{gH} f(t) = \frac{1}{\Gamma(1-\beta)} \int_a^t \frac{f'_{gH}(s)}{(t-s)^{\beta}} ds \quad (21)$$

or $(D_{a+}^{\beta})_{gH} f(t) = I_{a+}^{1-\beta} f'_{gH}(t)$,

and we also say that f is $[-i-gH]^C$-differentiable at t_0 if

$$(D_{a+}^{\beta})_{gH} f(t_0,r) = [(D_{a+}^{\beta})_{gH} f_-(t_0,r), (D_{a+}^{\beta})_{gH} f^-(t_0,r)], \quad (22)$$

and that f is $[-ii-gH]^C$ differentiable at t_0 if

$$(D_{a+}^{\beta})_{gH} f(t_0,r) = [(D_{a+}^{\beta})_{gH} f^-(t_0,r), (D_{a+}^{\beta})_{gH} f_-(t_0,r)]. \quad (23)$$

Definition .12 Let $f: [a,b] \to E$ and $t_0 \in (a,b)$ where $f'_-(t), f'^-(t)$ are both differentiable and real-valued functions at t_0 We say that f is $[(i)-gH]$-differentiable at t_0 if

$$(D_{a+}^{\beta})_{gH}^{L} f(t_0,r) = [(D_{a+}^{\beta})_{gH}^{L} f_-(t_0,r), (D_{a+}^{\beta})_{gH}^{L} f^-(t_0,r)] \quad (24)$$

F is $[(ii)-gH]$-differentiable at t_0 if

$$(D_{a+}^{\beta})_{gH}^{L} f(t_0,r) = [(D_{a+}^{\beta})_{gH}^{L} f^-(t_0,r), (D_{a+}^{\beta})_{gH}^{L} f_-(t_0,r)] \quad (25)$$

Definition 13 . Let $f' \in C^F[a,b] \cap L^F[a,b]$. The fractional generalized Hukuhara Caputo derivative of fuzzy-valued function f is defined as follows:

$$(D_{a+}^{\beta})_{gH}^{L} f(t) = \frac{1}{\Gamma(1-\beta)} \int_a^t \frac{f'^{L}_{gH}(s)}{(t-s)^{\beta}} ds = I_{a+}^{1-\beta} f'^{L}_{gH}(t) \quad (26)$$

and $t_0 \in (a,b)$

$$\varphi(t) = \frac{1}{\Gamma(\beta)} \int_a^t \frac{f'^{L}_{GH}(s)}{(t-s)^{1-\beta}} ds = I_{a+}^{1-\beta} f'^{L}_{GH}(t), \quad (27)$$

if an element $(D_{a+}^{\beta})_{GH}^{L} f(t_0) \in C^F$ exists such that for all $r \in [0,1]$, $h > 0$, either

$$(i)(D_{a+}^{\beta})_{gH}^{L} f(t_0) = \lim_{h\to 0+} \frac{\varphi_{\alpha}^{-}(t_0+h)\Theta_{gH}\varphi_{\alpha}^{-}(t_0)}{h}$$

$$(ii)(D_{a+}^{\beta})_{gH}^{L} f(t_0) = \lim_{h\to 0+} \frac{\varphi_{\alpha}^{+}(t_0+h)\Theta_{gH}\varphi_{\alpha}^{+}(t_0)}{h}. \quad (28)$$

For the sake of simplicity, we say that a fuzzy-valued function f is $[-i-\beta]^c$ c-differentiable, if it is differentiable as in definition of $11-12$ case (i), and that is $[-ii-\beta]^c$-differentiable if it is differentiable as in the definition of 13 case (ii).

Lemma 2 . Let $f: [a,b] \to E$ be a fuzzy continuous function and $[\beta]$-times differentiable in the independent variable t over the interval of differentiation (integration) $[0,t]$. Then the following relations holds:

$$(D_{a+}^{\beta})_{\beta} f(t;r) = (D_{a+}^{RL})_{\beta}((f(t;r) - \sum_{k=0}^{k=n} \frac{t^k}{k!} f_0^{(k)}(r)), \beta \in (n-1,n), n \in N \quad (29)$$

where

$$f_0^{(k)}(r) = \frac{d^k f(t;r)}{dt^k} |t_0$$

and $(D_{a+}^{\beta} f)$ is the Caputo derivative operator.$(D_{a+}^{\beta} f)$ is the more common Riemann-Liouville fractional derivative operator, which can be defined as follows:

$$(D_{a+}^{RL})_{\beta} = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_a^t \frac{f(s)ds}{(t-s)^{1-n+\beta}} \quad (30)$$

*Proof.*The Riemann-Liouville integral operator $I_{a+}^{\beta}, (D_{a+}^{RL})_{\beta}$ and the Caputo derivative operator D_{a+}^{β} are in the following relation :

$$(I_{a+}^{\beta} D_{a+}^{c}) f(t;r) = f(t;r) - \sum_{k=0}^{k=n} \frac{t^k}{k!} f_0^{(k)}(r), \beta \in (n-1,n), n \in N \quad (31)$$

and operators $(I_{a+}^{\beta}), (D_{a+}^{RL})_{\beta}$ are the inverse of each other;hence, the proof is completed.

Theorem 1 . *Suppose that* $f(t) \in C([0,T)E,E)$ *and* $(D_{a+}^{c}) f(t) \in ((o,T) * E, E)$

$$f(t;r) = f(t_0;r) + \frac{1}{\Gamma(\beta)} D_{a+}^{c} f(\tau,r)(t-t_0)^{\beta}, \tau \in [t_0,t] \quad (32)$$

and operators $I_{a+}^{\beta}, (D_{a+}^{RL})^{\beta}$ are Caputo fuzzy fractional derivatives of order $\beta > 0$

Proof. Suppose that $f(t) \in C([0,T) * E, E)$, and $(D_{a+}^{c}) f(t)((o,T) * E, E)$ for $\beta \in (o,1]$,then we have

$$(D_{a+}^{\beta}) f(t) = \frac{1}{(1-\beta)} \frac{d}{ds} \int_a^t f(s)(t-s)^{-\beta} ds = \frac{d}{ds} I_{a+}^{1-\beta} f(s), \quad (33)$$

$$(I_{a+}^{m} D_{a+}^{m})_{(\beta)} f(t,r) = (I_{a+}^{\beta} D_{a+}^{c})_{\beta} f(t,r)$$

$$(I_{a+}^{m} D_{a+}^{m})_{\beta} f(t,r) = f(t;r) - \sum_{k=0}^{k=m-1} \frac{f^{(k)}(t;r)}{k!} (t-t_0)^k, \beta \in (n-1,n), n \in N$$

For $\tau \in [0,t]$, we have $(I_{a+}^{\beta} D_{a+}^{c})_{(\beta)} f(t,r) = f(t;r) - f(t_0,r)$

$$(I_{a+}^{\beta} D_{a+}^{c})_{(\beta)} f(t,r) = f(t;r) - f(t_0;r) \quad (34)$$

then $(I_{a+}^{\beta} D_{a+}^{c})_{(\beta)} f(t,r) \geq 0$.

Theorem 2 . *Suppose that* $f(t) \in C([0,T) * E, E)$ *and* $(D_{a+}^{c}) f(t)((o,T) * E, E)$, *and* $(I_{a+}^{1-\beta} f')_{GH}^{L}(t) = \varphi(t)$, $(I_{a+}^{1-\beta} f')_{GH}(t) \in C^F[a,b]$, *then we have*

$$(I_{a+}^{\beta} D_{a+}^{\beta} f)_{GH}(t) = f(t) - \sum_{k=0}^{k=n} \frac{t^k}{k!} f^{(k)}(t)(t-t_0)^k \quad (35)$$

for case $[-i-\beta]^C$-differentiability, and we have

$$(I_{a+}^{\beta} D_{a+}^{\beta} f)_{GH}(t) = -(-f(t)) - \sum_{k=0}^{k=n} \frac{t^k}{k!} f^{(k)}(t)(t-t_0)^k, \quad (36)$$

for case $[-ii-\beta]^c$-differentiability.

Proof. Indeed, by direct computation for case of $[-i-\beta]^C$-differentiability,
we have:

$$(I_{a+}^{\beta} D_{a+}^{\beta}) f(t;r) = [(I_{a+}^{\beta} D_{a+}^{\beta} f_-(t;r), (I_{a+}^{\beta} D_{a+}^{\beta} f^-(t;r)], r \in [0,1] \quad (37)$$

$$[f_-(t;r) - \sum_{k=o}^{k=n} \frac{t^k}{k!} f_-^{(k)}(t;r)(t-t_0)^k,$$

$$f^-(t;r) - \sum_{k=o}^{k=n} \frac{t^k}{k!} f^{-(k)}(t;r)(t-t_0)^k], \beta \in (n-1,n), n \in N \quad (38)$$

and for $[-ii-\beta]^c$- differentiability, we have

$$(I_{a+}^{\beta} D_{a+}^{c})_{\beta} f(t;r) = [(I_{a+}^{\beta} D_{a+}^{c})_{\beta} f^-(t;r), (I_{a+}^{\beta} D_{a+}^{c} f_-(t;r)], r \in [0,1] \quad (39)$$

$[f^-(t;r) - \sum_{k=o}^{k=n} \frac{t^k}{k!} f^{-(k)}(t;r)(t-t_0)^k,$

$f_-(t;r) - \sum_{k=o}^{k=n} \frac{t^k}{k!} f_{(k)}(t;r)(t-t_0)^k], \beta \in (n-1,n), n \in N$ (40)

as well as for $[-ii-\beta]^c$ -differentiability for all $r \in [0,1]$, which completes the proof.

3.1FFVIDES under Caputo GH-differentiability

we consider the following fuzzy Caputo fractional differential equation:

$(D_{a+}^C)_\beta U(t) = F(t, \lambda u(t)) \qquad (D_{a+}^C)_{\beta-1} u(t_0) = u_0^{\beta-1} \in E$ (41)

Where $F:(a,b) * E \to E$ is continuous fuzzy -valued function and $t_0 \in [a,b]$. The following Lemma transform theFFDEs in to their corresponding FFVIDEs.

Lemma 3: *Let $r \in [0,1]$ and $t_0 \in R$,then the fuzzy fractional differential equation: (43) is equivalent to one of the following integral equations:*

$U(t) = u(t_0) + \frac{\lambda}{\Gamma(\beta)} \int_{t_0}^{t} \frac{f(s,u(s))ds}{(t-s)^{1-\beta}}, t \in [a,b]$ (42)

if U is differentiable,and

$U(t) = u(t_0) \Theta_{gH} \frac{-\lambda}{\Gamma(\beta)} \int_{t_0}^{t} \frac{f(s,u(s))ds}{(t-s)^{1-\beta}}, t \in [a,b]$ (43)

if U is $[-ii-\beta]^c -$ differentiable,provided that the H-difference exists.

Proof. Let us consider $f \in C^F[a,b]$, then we have the following:

$(I_{a+}^\beta D_{a+}^c)_\beta f(t;r) = [(I_{a+}^\beta D_{a+}^c)_\beta f^-(t;r), (I_{a+}^\beta D_{a+}^c f_-(t;r)], r \in [0,1]$
(44)

$(I_{a+}^\beta D_{a+}^c) f^-(t;r) = f^-(t;r) - f^-(t_0;r), (I_{a+}^\beta D_{a+}^c) f_-(t;r) = f_-(t;r) - f_-(t_0;r)$ (45)

For case $[-ii-\beta]^c$-differentiability. For case $[-i-\beta]^c$-differentiability, we have

$(I_{a+}^\beta D_{a+}^c)_\beta f(t;r) = [(I_{a+}^\beta D_{a+}^c)_\beta f_-(t;r), (I_{a+}^\beta D_{a+}^c) f^-(t;r)],$ (46)

Finally, we recall that for case $[-i-\beta]^c$-differentiability,

$(I_{a+}^\beta D_{a+}^c)_\beta f(t;r) = [f_-(t;r) - f_-(t_0;r), f^-(t;r) - f^-(t_0;r)],$ (47)

and for case $[-ii-\beta]^c -$ differentiability,we have

$(I_{a+}^\beta D_{a+}^c)_\beta f(t;r) = [f^-(t;r) - f^-(t_0;r), f_-(t;r) - f_-(t_0;r)],$ (48)

which completes the proof [8].

Theorem 3.[4] *We consider the following fuzzy Caputo fractional differential equation*

$(D_{a+}^C)_\beta U(t) - \lambda * c(r) * u(t) = f(t)$ (49)

let $f:[a,b] * (a,b) * E \to E$ be bounded continuous functions.Let the sequence$u_n:[a,b] \to E$ given by

$\lim_{t \to o+} (t^{1-\beta} D_{0+}^C) U(t) = u_0^{1-\beta} \in E$ (50)

$r \in [0,1]$,$\beta \in (0,1]$, $\lambda \in R$ have a unique solution given by (49)

$U(t) = \frac{1}{\Gamma(\beta)} (u_{0+}^{\beta-1} t^{\beta-1} E_{\beta,\beta}(\lambda t^\beta)) + \frac{\lambda}{\Gamma(\beta)} \int_{o+}^{t} \frac{f(s)ds}{(t-s)^{1-\beta}} E_{\beta,\beta}(\lambda(t-s)^\beta),$
(51)

For case $[-ii-\beta]^c$-differentiability and

$U(t) = \frac{1}{\Gamma(\beta)} (u_{0+}^{\beta-1} t^{\beta-1} E_{\beta,\beta}(\lambda t^\beta)) \Theta \frac{-\lambda}{\Gamma(\beta)} \int_{o+}^{t} \frac{f(s)ds}{(t-s)^{1-\beta}} E_{\beta,\beta}(\lambda(t-s)^\beta),$
(52)

Theorem 4.[4] *Let $f:[a,b] \to E$ be a fuzzy-valued function on $[a,b]$*

f is $[-ii-gH]^C$-differentiable at $C \in [a,b]$ iff

f is$[-ii-gH]^{CF}$*-differentiable at* C.

$f$is $[-i-gH]^C$-differentiable at $C \in [a,b]$ iff

f is $[-i-gH]^{CF}$-differentiable at C.

Lemma 4. *Let $f:[a,b] \to E$ be a fuzzy-valued function such that $F_{gH}'^L \in C^F[a,b] \cap L^F[a,b]$,*

$(I_{a+}^\beta D_{a+}^\beta)_{gH}^L f(t) = f(t) \Theta_{gH} f(t_0) = I_{a+}^{1-\beta} f_{gH}'^L(t),$ (53)

Proof :By using Definition (5) and (10) we have $(I_{a+}^\beta f')_{gH}(t) = (I_{a+}^\beta D_{a+}^\beta)_{gH} f(t) = \int_a^b f'_{gH}(S)ds,$such that

$\int_a^b f'_{gH}(S)ds = I_{a+}^\beta I_{a+}^{1-\beta} f'_{gH}(t)$ (54)

We consider that f is $[-i-gH]^{Cf}$ - differentiable. according Theorem (4) f is $[-i-gH]^C$ - differentiable.Then we have

$\int_a^b f'_{gH}(S)ds = [I_{a+}^\beta I_{a+}^{1-\beta} f'_{gH}(t)] = (I_{a+}^\beta)_{gH} f(t)$ (55)

$(I_{a+}^\beta D_{a+}^\beta)_{gH} f(t) = [\int_a^b (f')_\beta^-(s)ds, \int_a^b (f')_\beta^+(s)ds] = f_\beta(t) \Theta_{gH} f_\beta(t_0),$
(56)

According to Theorem $(4), i$ f is $[-ii-gH]^{Cf}$ - differentiable.then we have

$(I_{a+}^\beta D_{a+}^\beta)_{gH} f(t) = [\int_a^b (f')_\beta^+(s)ds, \int_a^b (f')_\beta^-(s)ds] = f_\beta(t) \Theta_{gH} f_\beta(t_0),$
(57)

For all $t \in [a,b]$, $r \in [0,1]$, $\beta \in (0,1]$, which proves the lemma.

Theorem 5. *Let $f:[a,b] * E * E \to E$ be a fuzzy-valued function such that $F_{GH}'^L \in C^F[a,b] \cap LF[a,b]$, Let the sequences$u_n:[a,b] \to E$ is be given by*

$u_0(t) = u_0, \qquad U_{n+1}(t) = u_0(t) \Theta_{gH} \frac{-\lambda}{\Gamma(\beta)} \int_{t_0}^{t} \frac{f(s,u_n(s))}{(t-s)^{1-\beta}} ds$ (58)

which is defined for any $n \in N$.Then the sequens u_n is convex sentence to the unique solution of problem (59) which is $[-ii-gH]^{cf}$-differentiable on $[a,b]$,provided that $\lambda < 1$.

Proof. Now we show that sequence u_n, (59) is a Cauchy sequence $inC^F[a,b]$. To do Thus,we have

$$d(u_1,u_0) = d(u_0\ominus\frac{-\lambda}{\Gamma(\beta)}\int_{t_0}^{t}(\frac{f(s,u_0(s))}{d}s(t-s)^{1-\beta},u_0) \leq \frac{\lambda}{\Gamma(\beta)}\int (t-s)^{\beta-1}d(f(s,u_0(s)),0^{\sim}) = \lambda t_0^{\beta}M \quad (59)$$

where $M = \sup d(f(s,u(s)),o^{\sim})$. Since f is Lipschitz continuous then by Definition (2) , we can assume $d(u_n(s),u_{n-1}(s)) \leq \mu_{n-1}$ and usingthis assumption, we have

$$d(u_{n+1}(s),u_n(s)) = \frac{\lambda}{\Gamma(\beta)}d(\int_{t_0}^{t}(t-s)^{\beta-1}f(s,u_n(s))ds,(t-s)^{\beta-1}f(s,u_{n-1}(s))) \leq \frac{\lambda}{\Gamma(\beta)}\int_{t_0}^{t}d((t-s)^{\beta-1}f(s,u_n(s)),(t-s)^{\beta-1}f(s,u_{n-1}(s)))ds \quad (60)$$

$$d(u_{n+1}(s),u_n(s)) \leq \frac{\lambda}{\Gamma(\beta)}\int_{t_0}^{t}((t-s)^{\beta-1}g(s,d(u_n(s),u_{n-1}(s)))ds \quad (61)$$

$$d(u_{n+1}(s),u_n(s)) = \mu_n(s) \quad (62)$$

Moreover $|(D_{a+}^{c})_{\beta}u_{n+1}(t)|=|g(s,u_n(s)| \leq M_1$; and thus, by the Ascoli-Arzela theorem and the monotonicity of e sequence u_n,we can conclude that $\lim_{n\to\infty}\mu_n(t) = \mu(t)$ uniformly on $[t_0,t_0+r]$ and

$$\mu(t) = \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\frac{g(s,u(s))ds}{(t-s)^{1-\beta}} \quad (63)$$

Thus ,by the inductive method ,we know that

$$d(u_{n+1}(s),u_n(s)) \leq \mu_n(s), \quad (64)$$

$\forall t \in [t_0,t_0+r], n = 0,1,2,3,\ldots$ so ,we have

$$d((D_{a+}^{c})_{\beta}u_{n+1}(t),(D_{a+}^{c})_{\beta}u_n(t)) = d(f(s,u_n(s),f(s,u_{n-1}(s))) \leq g(s,d(u_n(s),u_{n-1}(s))). \quad (65)$$

$$d((D_{a+}^{c})_{\beta}u_{n+1}(t),(D_{a+}^{c})_{\beta}u_n(t)) \leq g(s,d(u_n(s),u_{n-1}(s))) \quad (66)$$

4. Extension of the Jacobi polynomials operationalmatrixmethod for FFVIDEs

We introduce a suitable method for approximating the fuzzy solutions of linear fuzzy fractional integro-differential equations as shifted Jacobi functions based on the fuzzy residual of the problem where the Jacobi operational matrix is employed in the derivation of the proposed method. In addition, the approximate solution based on the shifted Jacobi polynomials $p_n^{\alpha,\beta}(t)(n \geq 0,\alpha,\beta > 0)$can be obtained in terms of the Jacobi parameters $\alpha$and β.In this section, we derive the fuzzy approximation function using the shifted Jacobi polynomials

.Moreover, the Jacobi operational matrix based on fuzzy shifted Jacobi polynomials is introduced in detail where this method can be employed for solving fuzzy linear fractional differential equations of order fuzzy linear fractional differential equations of order 0<β<1. It should be noted that this method is an the extension studies implemented in the crisp sense by Doha et al.
[20] and Kazem [21].

$$p_n^{\alpha,\beta}(t) = \sum_{i=0}^{n}p_i^{(n)}t^i, \qquad p_i^{(n)} = (-1)^{n-i}C_i^{n+\alpha+\beta+i}.C_{n-i}^{n+\alpha}, \; i = 0,1,2,\ldots\ldots,n \quad (67)$$

$$\int((J_n^{\alpha,\beta}(t),J_m^{\alpha,\beta}(t))_{\gamma}.\omega^{\alpha,\beta}(t)\,dt = \sum_{l=0}^{j}p_l^{(j)}B(\tau+l+\beta+1,\alpha+1)$$

Where B(s, t) is the Beta function defined as follows.

$$B(s,t) = \int_0^1 \tau^{s-1}(1-\tau)^{t-1}\,d\tau = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$$

Let $\sigma = (0,1)$ and $\{p_n^{\alpha,\beta}(t)\}_{n=0}^{\infty}$ generate the space $p_n^{m+1,.\alpha,\beta}$., A function *f* belonging to $L_2^W(\sigma)$, can be expanded in $p_n^{m+1,.\alpha,\beta}$, by

$$u(t) = \sum_{i=0}^{m-1}p_i^{\alpha,\beta}(t)\lambda_i$$

Where the coefficients $\lambda_i$are obtained by

$$\lambda_i = \frac{1}{v_i^{\alpha,\beta}}\int_0^1 p_i^{\alpha,\beta}(t).u(t).\omega^{\alpha,\beta}(t)\,dt \;, i = 0,1,\ldots\ldots$$

Lemma 5*.The fuzzy Caputo fractional derivative of order* 0<β<1*over the shifted Jacobi*
Functions can be obtained in the form of

$$I^{(\tau)}p_K^{\alpha,\beta}(t) = \sum_{i=0}^{k}p_i^k(t)x^{i+v}\frac{\Gamma(i+1)}{\Gamma(i+v+1)}$$

$$(D_{a+}^{C})_{\beta}p_K^{\alpha,\beta}(t) = \sum_{i=0}^{k}p'^{k}_{i}(t)x^{i-\beta}\frac{\Gamma(i+1)}{\Gamma(i+\beta+1)}$$

Where $p'^{k}_{i} = 0$,*fori* $< [\beta]$,*and* , $p_i^k = p'^{k}_{i}$ for i ≥ [β].

Proof. The proof is straightforward from based on Section 3-2 and the Caputo derivative of x^k.
The fuzzy Caputo operational matrix based on the shifted Jacobi polynomial $p_n^{\alpha,\beta}(t)$,a real is expressed by relation (5). So, we have the following.

$$(D_{a+}^{E})_{\beta}\emptyset(t) \cong (D_{a+}^{E}(\beta))\emptyset(t)$$

Definition 15[29]. For $u \in L_P^E[0,1] \cap C^E[0,1]$ and the shifted Jacobi polynomial $p_n^{\alpha,\beta}(t)$,a real value function over [0, 1], the fuzzy function is approximated by

$$u(t) \cong u_m(t) = \sum_{i=0}^{m-1}p_i^{\alpha,\beta}(t)\lambda_i \quad (68)$$

Where the fuzzy coefficients $\lambda_i$are obtained by

$$\lambda_i = \frac{1}{\vartheta i^{\alpha,\beta}}\int((J_n^{\alpha,\beta}(t),J_m^{\alpha,\beta}(t))_{\gamma}.\omega^{\alpha,\beta}(t)\,dt \;, i = 0,1,\ldots\ldots \quad (69)$$

Theorem6.The best approximation of a fuzzy function based on the Jacobi points exists and is unique, and only the first (m + 1)-terms of the shifted Jacobi polynomials

are considered. Then we have
$u(t) \cong u_m(t) = \sum_{i=0}^{m-1} p_i^{\alpha,\beta}(t)\lambda_i = F^T \emptyset(t)$, (70)
That

$$\emptyset_{m+1}(t) = [p_0^{\alpha,\beta}(t), p_1^{\alpha,\beta}(t), \dots\dots\dots . p_m^{\alpha,\beta}(t)]^T, \quad ©_{m+1}^T = [\lambda_0, \lambda_1, \lambda_2, \dots\dots\dots \lambda_m],$$

Where $\omega^{\alpha,\beta}(t) = (1-t)^\alpha * t^\beta$, $p_n^{\alpha,\beta}(t)$is as the same as the shifted Jacobi polynomials described in Section 2-1, and$\sum *$means denotes the addition with respect to Θ $in\ E$.
Thus, the following lemma provides the upper bound of the approximate function $f_{m+1}(t)$ using the shifted Jacobi polynomials. This error bound proves that the approximate function$f_m(t)$ converges to $f(t)$ based on the shifted Jacobi polynomials.
The proof is an immediate result of Theorem 4.2.1 in [23].

Lemma *6. u(t)has a unique best approximation from* $p_n^{m+1,.\alpha,\beta}(t)$ *,say* $u_m(t) \in p_n^{m+1,.\alpha,\beta}$ that i.e $\forall\, v \in p_n^{m+1,.\alpha,\beta} ||u(t) - u_m(t)||_\omega \le ||u(t) - v||_\omega$

Lemma7.[22]Let $\emptyset(t)$ be the shifted Jacobi vector defined inEq.(4) and let β>0.
Then

$$(D_{a+}^E)_\beta \emptyset(t) \cong (D_{a+}^E(\beta))\emptyset(t) \quad (71)$$

where$(D_{a+}^E)_{\boldsymbol{\beta}}$ *is the* $(m+1)\times(m+1)$ *Operational matrix of derivatives of order* β *in the Caputo sense which defined by*:

$$D^{(\beta)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & & 0 \\ & & \vdots & & \ddots & \vdots \\ \Delta([\beta],0) & \Delta([\beta],1) & \Delta([\beta],2) & 0 & & \Delta([\beta],n) \\ \Delta(i,0) & \Delta(i,1) & \Delta(i,2) & 0 & \cdots & \Delta(i,N) \\ \Delta(m,0) & \Delta(m,1) & \Delta(m,2) & 0 & & \Delta(m,m) \end{bmatrix}$$

Where

$$\Delta_{(\beta)}(i,j) = \sum_{k=[\beta]}^{i} \delta ijk$$

And δijk is given as follows.

$$\delta ijk = \frac{(-1)^{i-k}\Gamma(j+\beta+1)\Gamma(i+\beta+1)\Gamma(i+\alpha+k+\beta+1)}{\Gamma(j+k+\alpha+\beta+1)\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)\Gamma(k-\tau+1)\tau_j} \times \frac{(-1)^{j-l}\Gamma(j+\beta+l+\alpha+1)\Gamma(l+k+\beta-\tau+1)\Gamma(\alpha+1)}{\Gamma(l+k+\alpha+\beta-\tau+2)\Gamma(l+\beta+1)(j-l)!l!}$$

(Note that in $D^{(\beta)}$*, the first* $[\beta]$*rows, are all zeros).*

Where $(D_{a+}^E)_\beta$ is the $(m+1)$-square operational matrix of the fuzzy fractional Caputo derivative of the shifted Jacobi polynomials and$(D_{a+}^E(\beta))\emptyset \in C^E[0,1]$.Thus, by using(70) and (68) we can approximate the fuzzy fractional Caputo derivative as follows.

$$u(t) \cong u^{\sim}_{m+1}(t) = \sum_{i=0}^{m} p_i^{\alpha,\beta}(t) * f_i = F_{m+1}^T * \emptyset_{m+1} \quad (72)$$

$$\boldsymbol{u(t,r)} \cong \boldsymbol{u_{m+1}^{\sim}(t,r)} = \left[\sum_{i=0}^{m} \boldsymbol{p_i^{\alpha,\beta}(t)} * \boldsymbol{\lambda_{i_-}^{r}}, \sum_{i=0}^{m} \boldsymbol{p_i^{\alpha,\beta}(t)} * \boldsymbol{\lambda_{i_+}^{r}}\right], \quad \boldsymbol{0} < r \le 1$$

$$_E^\beta D\, u(t,r) \cong {}_E^{(\beta)}D\, \boldsymbol{u_{m+1}^{\sim}(t,r)} = \left[\sum_{i=0}^{m} {}_E^{(\beta)}\boldsymbol{D}\boldsymbol{p_i^{\alpha,\beta}(t)} * \boldsymbol{\lambda_{i_-}^{r}}, \sum_{i=0}^{m} {}_E^{(\beta)}\boldsymbol{D}\boldsymbol{p_i^{\alpha,\beta}(t)} * \boldsymbol{\lambda_{i_+}^{r}}\right] \quad (73)$$

The subsequent property of the product of the fuzzy Jacobi function vectors is also utilized where *gijk* is given by
gijk=$\int (\langle J_n^{\alpha,\beta}(t), J_m^{\alpha,\beta}(t)\rangle_\gamma . \omega^{\alpha,\beta}(t)\, dt$, $i = 0,1,\dots\dots$

The error bound of the fuzzy Caputo fractional differential operator is considered in the next theorem for 0<β<1. Therefore, we defineE_β^Eas follows.

$$E_{k,v} = ({}_E^{\boldsymbol{\beta}}\boldsymbol{D})p_k^{\alpha,\beta}(t) - \sum_{i=0}^{m}({}_E^{(\beta)}\boldsymbol{D_{kj}})\boldsymbol{p_j^{\alpha,\beta}(t)} \quad (74)$$

Subsequently, by replacing Eq. (72) in the initial condition of the problem
U (0) =$\sum_{j=0}^{m} \lambda_j^{(r)} * p_j^{(\alpha,\beta)}(0) = u_0$
And from the above equation with Eq. (73), the (*m*+ 1)-fuzzy linear algebraic equations are generated.
It is obvious that the unknown fuzzy coefficients are obtained by solving this fuzzy system using the method presented for the example in [24].

$$U_{n+1}(t) = U_0(t)\Theta_{gH}\frac{-1}{\Gamma(\beta)}\int_{0+}^{t}\frac{F(s,u_n(s))ds}{(t-s)^{1-\beta}}\Theta\frac{-\lambda}{\Gamma(\beta)}\int_{0+}^{t}\frac{u_n(s))ds}{(t-s)^{1-\beta}}, \quad (75)$$

5. Example

Example 1 In this section,we present examples of the solution of FFVIDEs under Caputo H-differentiability in order to show the application of the positive solutionsobtained. Let us consider the followingexample.

$$\left({}_{gH}^{C}D_{0+}^{\vartheta}u\right)_\beta(t) + \lambda u = te^{-t}, \quad 0 \le t \le 1, \quad 0 < \vartheta \le 1$$

(76)

$$u(0,r) = [-1+r, 1-r]$$

Again, according to case (i) in Definition(13) and Theorem 5, we can determine the
parametric form of (72) asfollows.

$$\left({}_{gH}^{C}D_{0+}^{\vartheta}u_-\right)_\beta(t,r) + \lambda u_-(t,r) = te^{-t}, \quad u_-(0,r) = -1+r, \ 0 \le t \le 1, \quad 0 < \vartheta \le 1$$

and

$$\left({}_{gH}^{C}D_{0+}^{\vartheta}u^+\right)_\beta(t,r) + \lambda u^+(t,r) = te^{-t}, \quad u^+(0,r) = 1-r, \ 0 \le t \le 1, \quad 0 < \vartheta \le 1$$

Usingthe presented method in Section 3-2, we can obtain following system of fuzzy equations:

$$\sum_{j=0}^{m} f_{j,-}^{r}\, p_j^{\alpha,\beta}(t) = \sum_{j=0}^{m} \theta_{j,-}^{r}[\Delta_\gamma(i,j) + I]p_j^{\alpha,\beta}(t),$$

$$\sum_{j=0}^{m} f_{j,+}^{r}\, p_j^{\alpha,\beta}(t) = \sum_{j=0}^{m} \theta_{j,+}^{r}[\Delta_{\gamma}(i,j) + I]p_j^{\alpha,\beta}(t)$$

where

$$\boldsymbol{f_i = \frac{1}{\vartheta i^{\alpha,\beta}} \int_0^1 p_i^{\alpha,\beta}(t) * te^{-t} * \omega^{\alpha,\beta}(t)dt}, \qquad i = 0,1,2,\ldots\ldots,m$$

Next, bysubstituting Eq. (72) in the initial condition of Eq. (76) yields

u(0)=$\sum_{j=0}^{m} \lambda_j^{(r)} * p_j^{(\alpha,\beta)}(0) = u_0,\ u(0,r) = [-1+r, 1-r]$ (77)

ByTaking$m$ = 2, v = 0.95, α = 0.0, $\boldsymbol{\beta}$=0.5 and applying theproposed method, obtain

$$D^{.95} = \begin{pmatrix} 0 & 0 & -0.0478 \\ 2.4852 & 0.1137 & \\ 0.3655 & 5.8573 & 0.2754 \end{pmatrix},\ p_j^{\alpha,\beta}(t)$$

$$= \begin{cases} 1, \\ \frac{-3}{2} + \frac{5}{2\,t} \\ \frac{15}{8} - \frac{35}{4\,t} + \frac{63}{8\,t^2} \end{cases}$$

$[©_{3,-}, ©_{3,+}]$
$= [(-0.1783, 0.1947, -0.0158), (0.3806, -0.0385, 0.0329)]$

and by, putting *entering* $D^{.95}$ and $p_j^{(\alpha,\beta)}$(t) inEqs. (75) and(76), we can obtain the fuzzy unknowncoefficients asfollows.

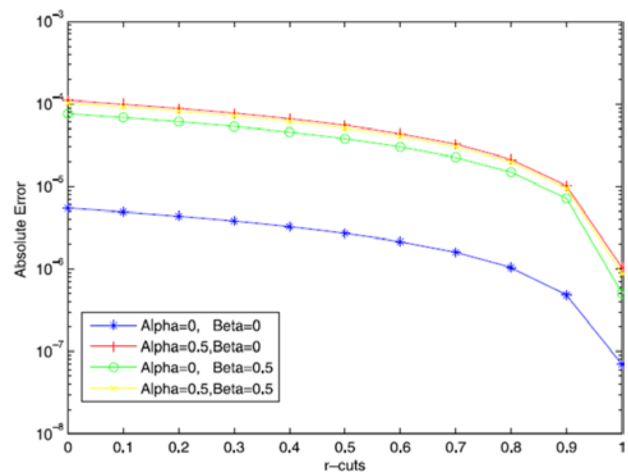
$©_{3,-} = (-0.1783, 0.1947, -0.0158)$, $©_{3,+} = (0.3806, -0.0385, 0.0329)$

Table 1. Absolute error using the proposed method for Example 1. with different values of α,β,and m = 9

$(\boldsymbol{\alpha},\boldsymbol{\beta})$	**(0,0)**	**(0.5,0)**	**(0,0.5)**	**(0.5,0.5)**	**(0,0)**	**(0.5,0)**	**(0,0.5)**	**(0.5,0.5)**
$\boldsymbol{r}$	$\boldsymbol{E_{.95}^{1}}$	$\boldsymbol{E_{.95}^{2}}$	$\boldsymbol{E_{.95}^{3}}$	$\boldsymbol{E_{.95}^{4}}$	$\boldsymbol{E_{.95}^{1}}$	$\boldsymbol{E_{.95}^{3}}$	$\boldsymbol{E_{.95}^{4}}$	$\boldsymbol{E_{.95}^{4}}$
0	5.2340e-2	1.4530e-3	7.6606e-4	1.0543e-4	4.856e-4	3.467e-3	3.873e-3	3.455e-4
.1	5.3455e-3	6.6595e-5	6.7854e-6	3.0321e-6	3.043e-3	4.873e-4	2.489e-3	2.435e-5
.2	5.6543e-7	8.6543e-5	5.0345e-7	5.0872e-6	4.075e-3	8.734e-2	8.345e-5	2.764e-4
.3	4.3589e-7	7.6523e-5	4.5467e-3	6.5427e-6	4.034e-3	9.023e-3	3.934e-7	5.236e-5
.4	4.5674e-8	7.9065e-7	3.7789e-5	3.0854e-3	5.854e-7	2.876e-3	2.439e-7	4.245e-5
.5	4.7654e-4	6.8704e-8	7.0432e-6	2.0543e-7	3.548e-4	5.034e-3	4.549e-2	5.732e-3
.6	5.6743e-8	4.7690e-5	2.4045e-2	5.5003e-6	1.349e-4	8.640e-5	5.593e-2	3.459e-5
.7	2.8754e-4	3.4573e-4	7.0342e-7	9.0432e-2	2.595e-6	6.002e-5	5.945e-5	6.495e-3
.8	4.7643e-2	4.4532e-6	6.9432e-4	8.0320e-4	7.984e-7	8.034e-5	4.795e-6	4.503e-7
.9	5.5436e-5	5.0912e-6	5.0432e-6	9.0346e-2	4.048e-7	3.980e-4	3.498e-5	8.543e-6

Comparison of the absolute errors forExample5.using different values of α, $\boldsymbol{\beta}$ *at* t =1 *in Table1.Table 1shows clearly that the proposed method achieves better accuracy with* α = $\boldsymbol{\beta}$ *=0 so according to this assumption the method is in agreement with the Legendre tau method proposed previously [25]. These results are confirmed by*Figure 1. *The problem is a fuzzy fractional oscillation equation, thismethod successfullyobtains asuitableapproximation thatbecause its precion is increases progressively with the increasing of the number of Jacobi functions according to Figure2.In addition, the approximate fuzzy solution is shown in Figure 2 for different fractional orders***v**.

The estimated CPU time is shown in Table 3.where the values were usingmathematica verion 7.0



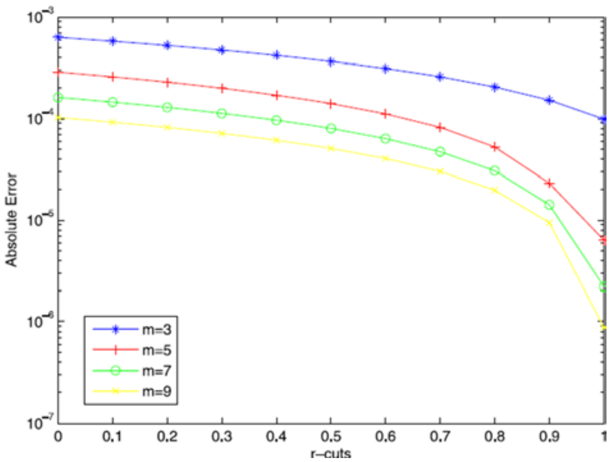


Figure 1. Absolute error with different values (α,βin Example.1, v = 0.95, m = 9.

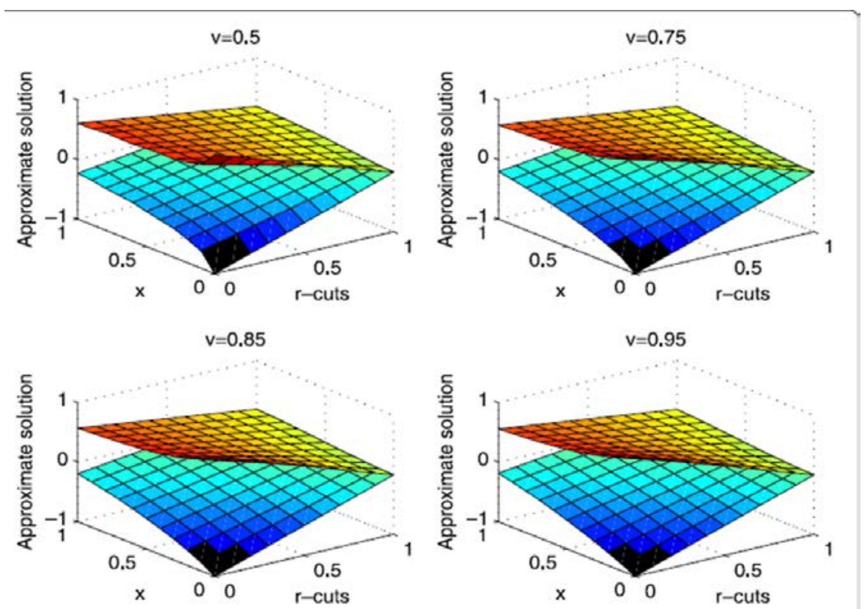


Figure 2.Fuzzy approximate solution forExample 1.usingdifferent fractional orders v, α = 0.5,β= 0.5, m = 9

Example 2.We Consider the following FFDE:

$$\left({}_{gH}^{C}D_{0+}^{\vartheta}u\right)_{\beta}(t) = \lambda.u(t) + (t+1), \quad 0 < \beta \leq 1, \quad 0 < r \leq 1,$$
$$u(o,r) = [u_{-}^{r}(0), u_{+}^{r}(0)] = \ [0.5 + 0.5\,r, 1.5 - 0.5\,r]; \qquad (77)$$

where we, suppose that $\boldsymbol{\lambda} = -1 \in R^{-} = (-\infty, 0)$.Using [i--β]-differentiability and Theorem 4. we haveobtain the following parametric form:

$$\left({}_{gH}^{C}D_{0+}^{\vartheta}u_{-}\right)_{\beta}(t,r) = -1.u_{-}(t,r) + t + 1,$$
$$u_{-}^{r}(0) = u_{0-}^{r} \quad 0 < \beta \leq 1, \quad 0 < r \leq 1, \ (78)$$

and

$$\left({}_{gH}^{C}D_{0+}^{\vartheta}u^{+}\right)_{\beta}(t,r) = -1.u^{+}(t,r) + t + 1,$$
$$u_{+}^{r}(0) = u_{0+}^{r} \quad 0 < \beta \leq 1, \quad 0 < r \leq 1, \ (79)$$

where $[u_{-}^{r}(0), u_{+}^{r}(0)] = \ [0.5 + 0.5\,r, 1.5 - 0.5\,r]$. The analytical solution to the problem (75) can be obtained using Eqs. (77) and (76) asfollows.

$$u_{-}(t,r) = (0.5 + 0.5\,r)E_{\beta,1}[t^{\beta}] + \int_{0}^{t} (t-s)^{\beta-1}E_{\beta,\beta}[-(t-s)^{\beta}](t+1)ds,$$

$$u^{-}(t,r) = (1.5 - 0.5\,r)E_{\beta,1}[t^{\beta}] + \int_{0}^{t} (t-s)^{\beta-1}E_{\beta,\beta}[-(t-s)^{\beta}](t+1)ds,$$

Using the method explained in Section 3-2, the equation is obtained in matrix
form as:

$©^T_{m+1,-}[D^\beta + I]\emptyset(t) = F^T_{m+1,-}\emptyset(t),$ (80)
$©^T_{m+1,+}[D^\beta + I]\emptyset(t) = F^T_{m+1,+}\emptyset(t)$

where the values of vector $F^T$are obtained by Eq. (72). deriving the fuzzy residual function
and multiplying it by $p_j^{(\alpha\beta,)}(t)*\omega^{\alpha,\beta}(t)$, $j = 0,1,2,\dots\dots,m-1$ we generate the following (m)-fuzzy algebraic equations.
$©^T_{m+1,-}[D^\beta + I] = F^T_{m+1,-}$, $©^T_{m+1,+}[D^\beta + I] = F^T_{m+1,+}$
(81)

In addition, for $u(o,r) = [u^r_-(0), u^r_+(0)]$, we have

$u^r_-(0) \cong ©^T_{m+1,-}\emptyset_{m+1} = (0.5 + 0.5\,r)$, (82)

$u^r_+(0) \cong ©^T_{m+1,+}\emptyset_{m+1} = (1.5 - 0.5\,r)$

Finally, Eqs. (78) and (79) yield the (m + 1)-fuzzy linear equations to provide uswith the unknown
fuzzy coefficients $\lambda_j$after solving this system.

Using$m$ = 2, α = β=0.5 and v = 0.75, we have

$$D^{.75} = \begin{pmatrix} 0 & 0 & 0 \\ \Delta_{0.75}(1,0) & \Delta_{0.75}(1,1) & \Delta_{0.75}(1,2) \\ \Delta_{0.75}(2,0) & \Delta_{0.75}(2,1) & \Delta_{0.75}(2,2) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -0.1755 \\ 2.9629 & 0.5524 & \\ -1.2429 & 4.2241 & 1.1048 \end{pmatrix}$$

and under the assumption that r-cut = 1,we have the following.

$$F_3 = \begin{pmatrix} 1.50 \\ 0.25 \\ 1.0 \end{pmatrix}, \qquad ©_3 = [1.1550, 0.1384, 0.0281]$$

Thus,by considering these two matrices and substituting them into Eqs. (100) and (102), we can obtain the fuzzy coefficients as
Fromshown in Table 2.Therefore, we can obtain a good approximation of the exact solution
using the proposed method. Table 2 shows, the results obtained at *t=1*, The method
was also tested with different values of α, $\boldsymbol{\beta}$and the results are depicted are depicted in Figure3. The results were more
accurate with α =0, $\boldsymbol{\beta}$= 0.5.According to Figure 4. the absolute error decreased as *m* increased.
Finally, the approximate fuzzy solution is illustrated in Figure5. for different values of *v* whichshows theproposed approach can effectively solve FFDEs of different fractional orders.

Table 2.Absolutebsolute error using the proposedmethod for Example 2. with different values of α,$\beta$and m = 5

(α,β)	**(0,0)**	**(0.5,0)**	**(0,0.5)**	**(0.5,0.5)**	**(0,0)**	**(0.5,0)**	**(0,0.5)**	**(0.5,0.5)**
r	E^1_{95}	E^2_{95}	E^3_{95}	E^4_{95}	E^1_{95}	E^3_{95}	E^4_{95}	E^4_{95}
0	6.2340e-4	3.4530e-6	6.6606e-3	6.0543e-4	4.856e-4	4.467e-3	4.873e-3	3.455e-4
.1	4.3455e-3	8.6595e-7	5.7854e-2	6.0321e-6	9.043e-3	4.873e-4	2.489e-3	2.435e-5
.2	5.6543e-7	2.6543e-5	4.0345e-7	6.0872e-6	4.075e-3	8.734e-2	8.345e-5	2.764e-4
.3	7.3589e-7	7.6523e-5	4.5467e-3	6.5427e-6	4.034e-3	9.023e-3	3.934e-7	5.236e-5
.4	2.5674e-8	3.9065e-7	3.7789e-5	3.0854e-3	5.854e-7	2.876e-3	2.439e-7	8.245e-5
.5	7.7654e-4	7.8704e-8	7.0432e-6	5.0543e-7	3.548e-4	5.034e-3	5.549e-2	5.732e-3
.6	4.6743e-8	3.7690e-4	2.4045e-5	5.5003e-6	1.349e-4	8.640e-5	5.593e-2	3.459e-5
.7	6.8754e-4	5.4573e-4	3.0342e-7	9.0432e-2	2.595e-6	5.002e-5	5.945e-5	3.495e-3
.8	2.7643e-2	3.4532e-8	4.9432e-4	3.0320e-4	7.984e-7	2.034e-5	5.795e-6	4.503e-7
.9	3.5436e-5	4.0912e-6	4.0432e-6	4.0346e-2	4.048e-7	7.980e-4	3.498e-5	2.345e-6

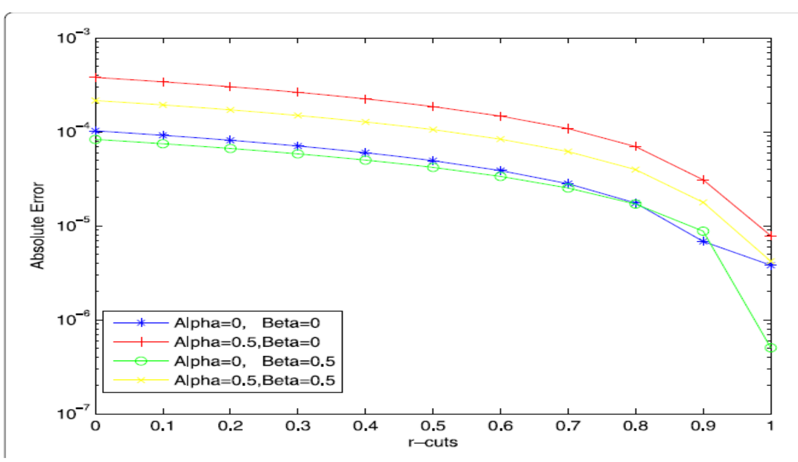


Figure 3.Absolute error for different values of (α,β) in Example2, v = 0.85, m = 7.

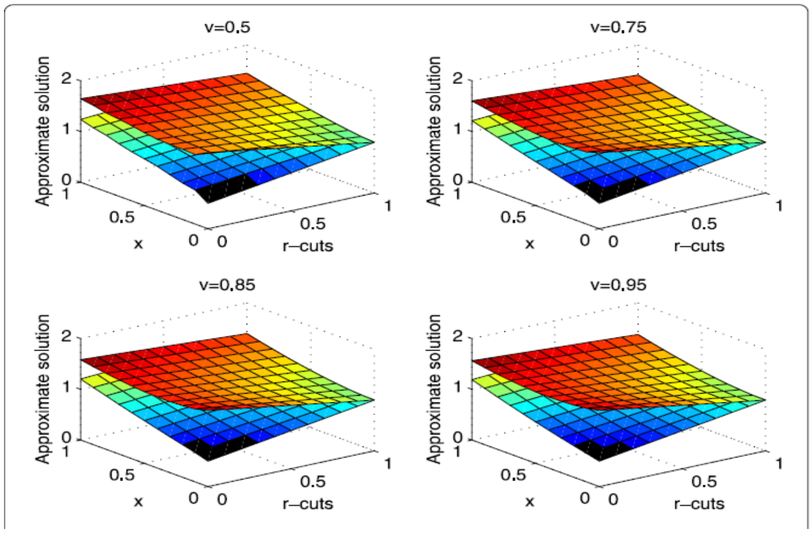


Figure 4.Fuzzy approximate solution for Example2, using different fractional orders v, α = β= 0, m = 7.

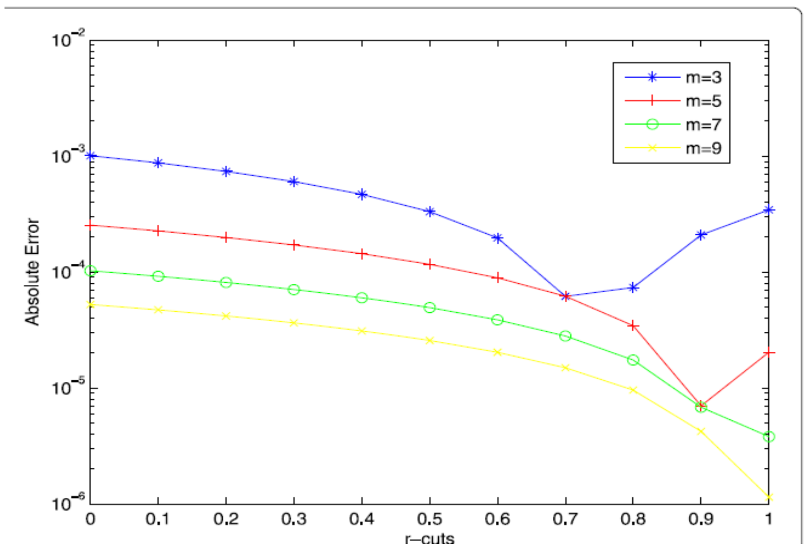


Figure 5.Absolute error for different values ofm in Example 2, v = 0.85, α = β= 0.

6. Conclusions

In this study,we investigated the positive solutions of FFVIDEs under Riemann-Liouville H-differentiability and gH-differentiability Previously, Agarwal et al. [1], studied the solutions of UFFDEs,but they did not explain how they can be obtained. Thus, the present study is the first to derive the positive solutions of FFVIDEs under Caputo H-differentiability.We considered two new solutions results for Mittag-Leffler FFDEs involving Caputo generalized H-differentiability with fuzzy versions of Mittag-Leffler functions and the Jacobi polynomials operational matrix. In future researches,we will obtain positive solutionsfor FFVIDEs with fuzzy Caputo gH -differentiability and fuzzy Caputo Hukuhara differentiability in order to investigate the convergence of this set of equations. We planto extend thismethod to solvingmultilinear and nonlinear problems as well as solving FFVIDEs of the order 0<β< 1. Furthermore,we will attempt to extend the proposed method to othe types of fuzzy derivatives such as Riemann-Liouville differentiability[4]. The proposed method can also be investigated with other equations.

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