Computational and Analytical solution of Fractional order Linear Partial Differential equations using Sumudu Transform and its properties

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Summary
In this work, an analytical outcome of fractional order Linear Partial differential equations is solved, which is collective result of the Sumudu Transform and its differential and integral properties and shows its ability on fractional order Linear Partial differential equations. In this method the outcomes are achieved in the form of quickly convergent infinite series with simply computable terms and approximate solutions of fractional orders are compared graphically. It is mention that this technique removes linearization and biologically unrealistic assumptions and gives an effective solution. The achieved outcomes are calculated using the symbolic calculus software Maple 16. This technique was undoubtedly very effective and powerful scheme in finding the solutions of approximate and numerical solutions as well as exact solutions.

Keywords:
Time Fractional Diffusion equation, One-dimensional Fractional wave-like equation, Two dimensional Fractional heat-like equation, Fractional Sumudu Transform method, Fractional integrals and derivatives for Sumudu Transform.

1. Introduction

Mathematicians have introduced FDEqs beneficial in numerous fields. FDEqs have been the attention on several fields due to their frequent appearance on many studies such as engineering, physics and chemistry. The fractional derivative has been occurring in various physical and chemical fields. Recently fractional diffusion equations have attracted attention of many researchers due to its wide applicability both in the theory of mathematical science and technology. The latest work [1, 2, 3] on fractional diffusion equations are useful in this field. Schneider and Wyss [4], Dhaigude and Nikam [5] considered the time fractional diffusion equation and wave equation and obtained their solutions. The heat and wave-like models are the integral part of applied sciences and arise in numerous physical problems. Numerous techniques containing spectral, characteristic, modified variational iteration, ADM and He's polynomials have been applied for solving these problems. In this paper, Fractional Sumudu Transform method [6] is applied for solving Time fractional Diffusion equation, One-dimensional Fractional wave-like equation, Two dimensional Fractional heat-like equation.

2. Basic Definitions of Fractional Calculus

Definition: *A real function* $\psi(x),\ x>0$ *is in the space* K_μ , $\mu\,\varepsilon\,R$ *if there is a real number* $\lambda>\mu$ *such that* $\psi(x)=x^{\lambda}g(x),$ *where* $g(x)\ \varepsilon\ K[0,\infty)$ *and it is in the space* K^{m}_{μ} *if and only if* $\psi^{(m)}\varepsilon\ K_\mu$ *for* $m\ \varepsilon\ \mathrm{N}$.

Definition *The Riemann-Liouville fractional integral operator of order* q *of a function* $\psi(x)\ \varepsilon\ K_\mu$, $\mu\geq -1$ *is given as follows*

$$J^{q}\psi(x)=\begin{cases}\frac{1}{\Gamma(q)}\int_{0}^{x}(x-\tau)^{q-1}\psi(\tau)d\tau, & q>0,\ x>0\\ \psi(x), & q=0.\end{cases} \qquad 2.1$$

The operator J^{q} has some properties, for $q,\ r\geq 0,$ $\xi,\mu\geq -1$ and C a real constant:

$$J^{0}\psi(x)=\psi(x)\quad ,$$

$$J^{q}J^{r}\psi(x)=J^{q+r}\psi(x)\quad ,$$

$$J^{q}J^{r}\psi(x)=J^{r}J^{q}\psi(x)\quad ,$$

- $$J^{q}x^{\xi}=\frac{\Gamma(\xi+1)}{\Gamma(q+\xi+1)}x^{q+\xi}\quad ,$$

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- $J^q C = \frac{C}{\Gamma(q+1)} x^q$.

Definition *The Caputo Fractional derivatives* D^q *of a function* $\psi(x)$ *of any real number* q *such that* $m-1<q\le m, m \,\varepsilon\, \mathrm{N}$, *for* $x>0$ *and* $\psi \varepsilon\, C_{-1}^m$ *as:*

$$D^q\psi(x)=\begin{cases}\frac{1}{\Gamma(m-q)}\int_0^x (x-\tau)^{m-q-1}\psi^{(m)}(\tau)d\tau \\ \frac{\partial^m \psi(x)}{\partial x^m}, & q=m,\end{cases} \quad 2.2$$

and has the following properties for

$m-1<q\le m, m \,\varepsilon\, \mathrm{N}, \mu \ge -1$ and $\psi \,\varepsilon\, C_\mu^m$

- $D^q J^q \psi(x)=\psi(x)$,
- $J^q D^q \psi(x)=\psi(x)-\sum_{k=0}^{m-1}\psi^{(k)}(0^+)\frac{x^k}{k!}$, *for* $x>0$.

Definition *The Mittag-Leffler function* $E_q(x)$ *with* $q>0$ *has developed by G. M. Mittag-Leffler, for one-parameter simplification of exponential function, which is represented by*

$$E_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(qn+1)} \quad 2.3$$

3. Sumudu Transform and its derivative and integral properties

The Sumudu transform is an integral transform, first proposed by Watugala in 1998, [8] to solve engineering problems [7,8,9].

The Sumudu transform is explained over the set of functions:

$$A=\left\{\psi(\tau): \exists M, t_1, t_2>0, |\psi(\tau)|<Me^{\frac{\tau}{t_j}}, \quad if\ \tau\,\varepsilon\,(-1)^j\times[0,\infty)\right\}, \quad 3.1$$

given by

$$F(u)=S[\psi(\tau)]=\int_0^\infty \frac{1}{u}e^{-\tau/u}\,\psi(\tau)\,d\tau . \quad 3.2$$

The existence and uniqueness was given in [10]. For more information and properties of Sumudu transform and its derivatives, [10,11,12,13].

Definition The Sumudu transform $S[\psi(\tau)]$ of the Riemann-Liouville fractional integral is given as below $[14-15]$

$$S[J^q\psi(\tau)]=u^q S[\psi(\tau)]. \quad 3.3$$

Definition *The Sumudu transform* $S[\psi(\tau)]$ *of the Caputo fractional derivative is given as below* $[14-15]$:

$$S[D^q\psi(\tau)]=u^{-q}S[\psi(\tau)]-\sum_{k=0}^{m-1}u^{-q+k}\psi^{(k)}(0), \quad m-1<q\le m, \quad 3.4$$

and the inverse sumudu transform of:

$$S^{-1}[\sum_{k=0}^{m-1}u^k\psi^{(k)}(0)]=\sum_{k=0}^{m-1}\frac{\tau^k\psi^{(k)}(0)}{\Gamma(k+1)}. \quad 3.5$$

4. Analysis of Fractional Sumudu Transform Method

To show the fundamental scheme of this technique, we let the following linear fractional PDE:

$$D_\tau^q\,\psi\,(x,\tau)+L[\psi\,(x,\tau)]=Q(x,\tau), \tau>0, m-1<q\le m, \quad 4.1$$

by the initial condition

$$\psi\,(x,0)=f_0(x), \quad 4.2$$

here $D_\tau^q=\partial^q/\partial\tau^q$ is the fractional Caputo derivative $,L$ is the linear differential operator and $Q(x,\tau)$ is the source term of the function $\psi\,(x,\tau)$.

Now, operating both sides the Sumudu transform (4.1),

$$S[D_\tau^q\,\psi(x,\tau)]+S[L[\psi(x,\tau)]]=S[Q(x,\tau)]. \quad 4.3$$

Applying the differential property of Sumudu transform,

$$S[\psi(x,\tau)]=[\psi(x,0)]+u^q S[Q(x,\tau)]-u^q S[L[\psi(x,\tau)]]. \quad 4.4$$

Applying both sides the Sumudu inverse (4.4)

$$\psi(x,\tau)=\psi(x,0)+S^{-1}[u^q S[Q(x,\tau)-L[\psi(x,\tau)]]], \quad 4.5$$

Operating with the integral property of Sumudu transform

$$\psi(x,\tau)=\psi(x,0)+J_\tau^q[Q(x,\tau)-L[\psi(x,\tau)]]. \quad 4.6$$

The technique shows a series solution for $\psi(x,\tau)$ defined as

$$\psi_0(x,\tau)=\psi(x,0)=f_0(x),$$
$$\psi_{n+1}(x,\tau)=J^q(Q(x,\tau))-J^q(L[\psi_n(x,\tau)])=\sum_{n=1}^{\infty} f_n(x)\frac{\tau^{nq}}{\Gamma(qn+1)}, \quad 4.7$$

then the terms $\psi_n(x,\tau)$ follows:

$$\psi(x,\tau)=\sum_{n=0}^{\infty}\psi_n(x,\tau)=\sum_{n=0}^{\infty} f_n(x)\frac{\tau^{nq}}{\Gamma(qn+1)}. \quad 4.8$$

5. Comparison and Implementation of Computational Results

In order to discuss the efficiency of our proposed technique, we will illustrate the applications of our algorithm and investigate its accuracy on some fractional order Linear PDEqs. The efficiency and simplicity of the proposed technique is discussed through the following three examples. Graphs are given for the purpose of comparison

Example 1: Solution of Time Fractional Diffusion equation

Let the following time fractional diffusion equation :

$$\frac{\partial^q\psi}{\partial\tau^q}=\frac{\partial^2\psi}{\partial x^2}+x\frac{\partial\psi}{\partial x}+\psi, \qquad 0<q\le 1 \quad 5.1$$

by initial condition

$$\psi(x,0)=x=f_0(x). \quad 5.2$$

Now, applying the New Fractional Sumudu transform technique with the initial condition,

$$S[\psi(x,\tau)]=\psi(x,0)+u^q S[\psi_{xx}+x\psi_x+\psi]. \quad 5.3$$

Using both sides the Inverse Sumudu in (5.3),

$$\psi(x,\tau)=\psi(x,0)+S^{-1}(u^q S[\psi_{xx}+x\psi_x+\psi]). \quad 5.4$$

Operating with the integral property of Sumudu transform,

$$\psi(x,\tau)=\psi(x,0)+J_\tau^q[\psi_{xx}+x\psi_x+\psi]. \quad 5.5$$

The method shows a series solution for $\psi(x,\tau)$

$$\psi_0(x,\tau)=\psi(x,0)=f_0(x),$$
$$\psi_{n+1}(x,\tau)=J_\tau^q[\psi_{nxx}+x\psi_{nx}+\psi_n]=\sum_{n=1}^{\infty} f_n(x)\frac{\tau^{nq}}{\Gamma(qn+1)}. \quad 5.6$$

and the functions(f_k) $k=0\ldots$ are given by:

$$f_0=\psi_0,$$
$$f_1=f_{0xx}+xf_{0x}+f_0,$$
$$f_2=f_{1xx}+xf_{1x}+f_1,$$
$$f_3=f_{2xx}+xf_{2x}+f_2,$$
$$\vdots$$
$$f_{n+1}=f_{nxx}+xf_{nx}+f_n. \quad 5.7$$

So that the solution $\psi(x,\tau)$ in series form is defined as:

$$\psi(x,\tau)=f_0+f_1\frac{\tau^q}{\Gamma(q+1)}+f_2\frac{\tau^{2q}}{\Gamma(2q+1)}+f_3\frac{\tau^{3q}}{\Gamma(3q+1)}+\cdots+f_n\frac{\tau^{nq}}{\Gamma(nq+1)}. \quad 5.8$$

Now the solution $\psi(x,\tau)$ in closed form is defined as

$$\psi(x,\tau)=\sum_{n=0}^{\infty}\psi_n(x,\tau)=x\sum_{n=0}^{\infty}\frac{2^n\tau^{nq}}{\Gamma(nq+1)}=xE_q(2\tau^q), \quad 5.9$$

Where $E_q(x)$ is Mittag-Leffler function in one parameter.

For special condition $q=1$,

$$\psi(x,\tau)=xe^{2\tau}, \quad 5.10$$

which is an exact solution of the above time fractional diffusion equation (5.1) for $q=1$.

The solution of $\psi(x,\tau)$ w.r.t x and τ when $q=0.2,0.4,0.7,1$ and $\tau=0.1$ is given in Figure 5.1.

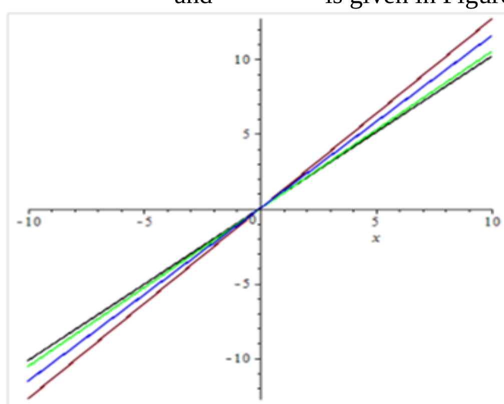

Fig. 5.1 The figure shows the solution of time fractional diffusion eq when $\tau=0.1$ and $q=0.2,0.4,0.7,1$

The solution of $\psi(x,\tau)$ with respect to x and τ when $q=0.5$ using $0\le x\le 5$ and $0\le\tau\le 5$ is given in Figure 5.2.

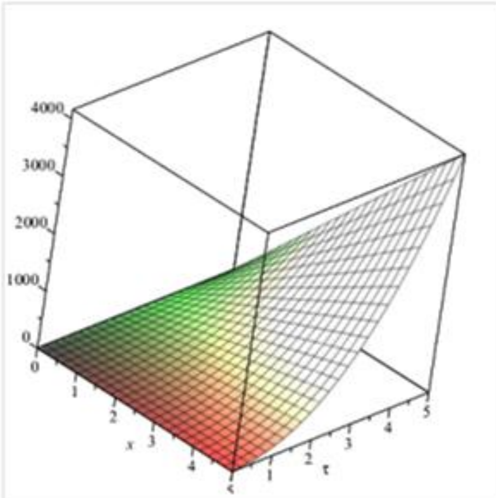

Fig. 5.2 The figure shows the solution of time fractional diffusion eq when $q = 0.5,$ using $0 \le x \le 5$ *and* $0 \le \tau \le 5$

Table 5.1: Comparison between derivitaives of different non-integer orders of time fractional diffusion equation with different values of q.

x	τ	$q=1$	$q=0.7$	$q=0.4$	$q=0.2$
0.25	0.1	0.3053506667	0.4006403836	0.8693237140	3.085196185
0.50	0.1	0.6107013333	0.8012807673	1.738647427	6.170392371
0.75	0.1	0.9160520000	1.201921152	2.607971142	9.255588556
0.90	0.1	1.099262400	1.442305381	3.129565370	11.10670627
1	0.1	1.221402667	1.602561534	3.477294855	12.34078474

Example 2: Solution of One-dimensional fractional wave-like equation

Let the following one-dimensional fractional wave-like equation:

$$\frac{\partial^q \psi}{\partial \tau^q} = \frac{1}{2} x^2 \frac{\partial^2 \psi}{\partial x^2}, \qquad 0 < q \le 1 \qquad 5.11$$

subject to initial condition

$$\psi(x,0) = x^2 = f_0(x). \qquad 5.12$$

Now, applying the New Fractional Sumudu transform method with the initial condition,

$$S[\psi(x,\tau)] = \psi(x,0) + u^q S[\frac{1}{2} x^2 \psi_{xx}]. \qquad 5.13$$

Using both sides the Inverse Sumudu in (5.13),

$$\psi(x,\tau) = \psi(x,0) + S^{-1}(u^q S[\frac{1}{2} x^2 \psi_{xx}]). \qquad 5.14$$

Operating with the integral property of Sumudu transform,

$$\psi(x,\tau) = \psi(x,0) + J_\tau^q[\frac{1}{2} x^2 \psi_{xx}]. \qquad 5.15$$

The method assumes a series solution for $\psi(x,\tau)$

$$\psi_0(x,\tau) = \psi(x,0) = f_0(x),$$

$$\psi_{n+1}(x,\tau) = J_\tau^q[\frac{1}{2} x^2 \psi_{nxx}] = \sum_{n=1}^{\infty} f_n(x) \frac{\tau^{nq}}{\Gamma(qn+1)}. \qquad 5.16$$

and the functions(f_k) $k=0...$ are given by:

$$f_0 = \psi_0,$$

$$f_1 = \frac{1}{2} x^2 f_{0xx},$$

$$f_2 = \frac{1}{2} x^2 f_{1xx},$$

$$f_3 = \frac{1}{2} x^2 f_{2xx},$$

$$\vdots$$

$$f_{n+1} = \frac{1}{2} x^2 f_{nxx}. \qquad 5.17$$

So that the solution $\psi(x,\tau)$ in series form is defined as

$$\psi(x,\tau) = f_0 + f_1 \frac{\tau^q}{\Gamma(q+1)} + f_2 \frac{\tau^{2q}}{\Gamma(2q+1)} + f_3 \frac{\tau^{3q}}{\Gamma(3q+1)} + \cdots + f_n \frac{\tau^{nq}}{\Gamma(nq+1)}. \qquad 5.18$$

Now the solution $\psi(x,\tau)$ is defined as

$$\psi(x,\tau) = \sum_{n=0}^{\infty} \psi_n(x,\tau) = x^2 \sum_{n=0}^{\infty} \frac{\tau^{nq}}{\Gamma(nq+1)} = x^2 E_q(\tau^q), \qquad 5.19$$

Where $E_q(x)$ is Mittag-Leffler function in one parameter. For special condition $q=1$, we have

$$\psi(x,\tau) = x^2 e^\tau. \qquad 5.20$$

which is an exact solution of the above one-dimensional fractional wave-like equation (5.11) for $q=1$.

The solution of $\psi(x,\tau)$ with respect to x and τ when $q = 0.05, 0.25, 0.75, 1$ and $\tau = 0.5$ is given in Figure 5.3.

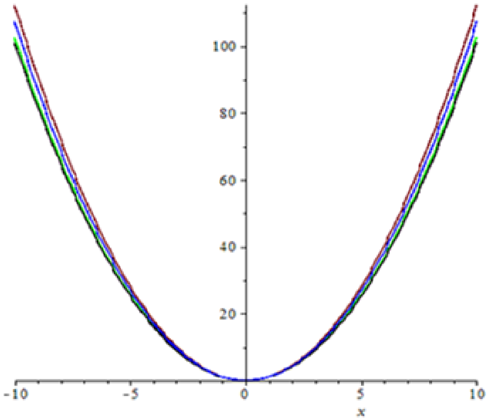

Fig. 5.3 The figure shows the solution of one dimensional wave-like eq when

$q = 0.05, 0.25, 0.75, 1 \quad and \quad \tau = 0.5$

The solution of $\psi(x,\tau)$ with respect to x and τ when $q = 0.5$ using $-5 \le x \le 5$ and $-5 \le \tau \le 5$ is given in Figure 5.4.

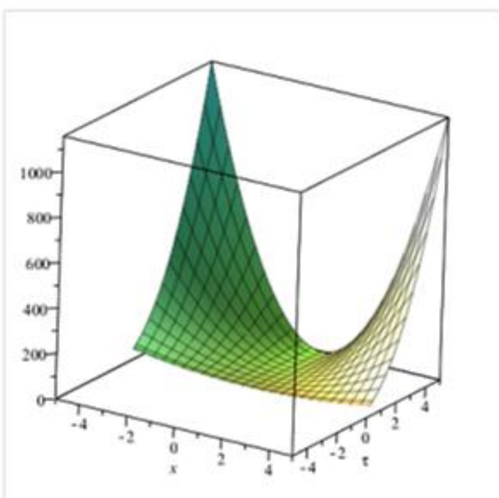

Fig. 5.4 The figure represents the solution of one-dimensional fractional wave-like eq when $q = 0.5,$ using $-5 \le x \le 5$ and $-5 \le \tau \le 5$

Table 5.2: Comparison between derivitaive of different non-integer orders of one-dimensional fractional wave-like equation with different values of q.

x	τ	$q=1$	$q=0.75$	$q=0.25$	$q=0.05$
0.25	0.5	0.1030436197	0.1262942710	0.2652295926	0.3634229493
0.50	0.5	0.4121744792	0.5051770843	1.060918371	1.453691797
0.75	0.5	0.9273925782	1.136648439	2.387066334	3.270806543
0.90	0.5	1.335445312	1.636773753	3.437375520	4.709961422
1	0.5	1.648697917	2.020708337	4.243673482	5.814767188

Example 3: Solution of Two dimensional fractional heat-like equation

Consider the following two dimensional fractional heat-like equation:

$$\frac{\partial^q \psi}{\partial \tau^q} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}, \qquad 0 < q \le 1$$

$$x > 0, \ \ \tau > 0, \ \ y < 2\pi \qquad 5.21$$

by initial condition

$$\psi(x,y,0) = \sin(x)\sin(y) = f(x), \qquad 5.22$$

Now, applying the New Fractional Sumudu transform technique with the initial condition,

$$S[\psi(x,y,\tau)] = \psi(x,y,0) + u^q S[\psi_{xx} + \psi_{yy}]. \qquad 5.23$$

Using both sides the Inverse Sumudu in (5.23),

$$\psi(x,y,\tau) = \psi(x,y,0) + S^{-1}(u^q S[\psi_{xx} + \psi_{yy}]). \qquad 5.24$$

Operating with the integral property of Sumudu transform,

$$\psi(x,y,\tau) = \psi(x,y,0) + J_\tau^q[\psi_{xx} + \psi_{yy}]. \qquad 5.25$$

The method assumes a series solution for

$$\psi(x,y,\tau)$$
$$\psi_0(x,y,\tau) = \psi(x,y,0) = f(x),$$
$$\psi_{n+1}(x,y,\tau) = J_\tau^q[\psi_{nxx} + \psi_{nyy}] = \sum_{n=1}^{\infty} f_n(x)\frac{\tau^{nq}}{\Gamma(qn+1)}, \qquad 5.26$$

and the functions(f_k)$_{k=0...}$ are given by:

$$f_0 = \psi_0,$$
$$f_1 = f_{0xx} + f_{0yy},$$
$$f_2 = f_{1xx} + f_{1yy},$$
$$f_3 = f_{2xx} + f_{2yy},$$
$$\vdots$$
$$f_{n+1} = f_{nxx} + f_{nyy}. \qquad 5.27$$

So that the solution $\psi(x,y,\tau)$ in series form is defined as

$$\psi(x,y,\tau) = f_0 + f_1\frac{\tau^q}{\Gamma(q+1)} + f_2\frac{\tau^{2q}}{\Gamma(2q+1)} + f_3\frac{\tau^{3q}}{\Gamma(3q+1)} + \dots$$
$$\dots + f_n\frac{\tau^{nq}}{\Gamma(nq+1)}. \qquad 5.28$$

So that the solution $\psi(x,y,\tau)$ is represented as

$$\psi(x,y,\tau)=\sum_{n=0}^{\infty}\psi_n(x,y,\tau)=\sin(x)\sin(y)\sum_{n=0}^{\infty}\frac{(-2)^n\tau^{nq}}{\Gamma(nq+1)}$$
$$=\sin(x)\sin(y)E_q(-2\tau^q), \qquad 5.29$$

where $E_q(x)$ is Mittag-Leffler function in one parameter. For special condition $q=1$, we have

$$\psi(x,y,\tau)=\sin(x)\sin(y)e^{-2\tau}. \qquad 5.30$$

which is an exact solution of the above two dimensional fractional heat-like equation (5.21) for $q=1$.

The solution of $\psi(x,y,\tau)$ with respect to x and τ when $q=0.25,0.50,0.75,1$ and $\tau=5$ is given in Figure 5.5, 5.6, 5.7, 5.8.

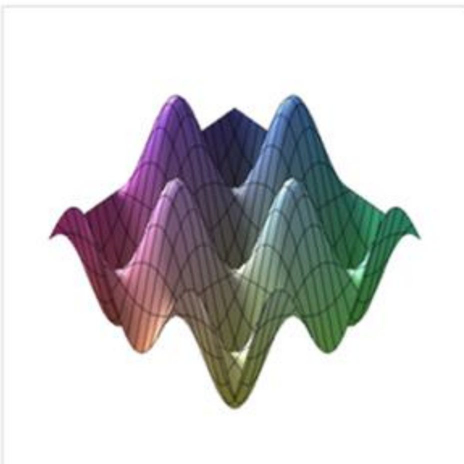

Fig. 5.5 The figure shows the solution of two dimensional fractional heat-like eq when $q=0.25$ and $\tau=5$

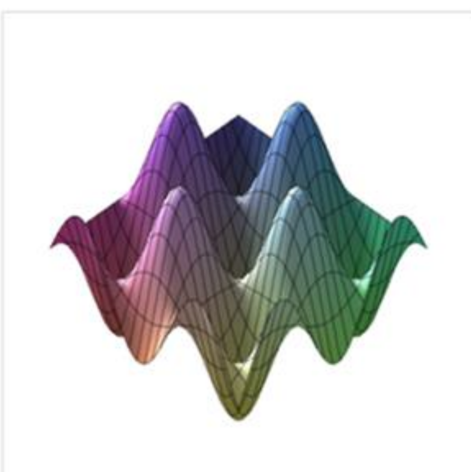

Fig. 5.6 The figure represents the solution of two dimensional fractional heat-like eq when $q=0.50$ and $\tau=5$

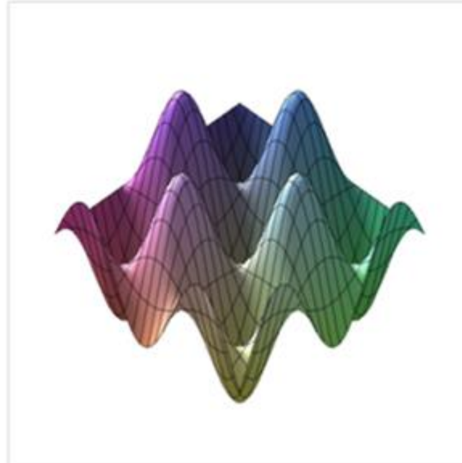

Fig. 5.7 The figure represents the solution of two dimensional fractional heat-like eq when $q=0.75$ and $\tau=5$

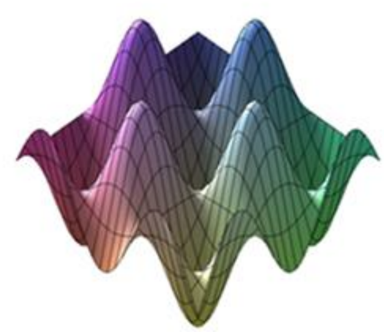

Fig. 5.8 The figure represents the solution of two dimensional fractional heat-like eq when $q=1$ and $\tau=5$

Table 5.3: Comparison between derivatives of different non-integer orders of two dimensional fractional heat-like equation with different values of q.

x	y_π	τ	$q=1$	$q=0.75$	$q=0.50$	$q=0.25$
0.25	$\frac{4}{\pi}$	0.01	0.1714769530	0.1634118951	0.1415290070	0.0960447315
0.50	$\frac{4}{\pi}$	0.01	0.3322922997	0.3166636303	0.2742584262	0.1861178673
0.75	$\frac{4}{\pi}$	0.01	0.4724473205	0.4502267544	0.3899357846	0.2646190951
0.90	$\frac{4}{\pi}$	0.01	0.5429278985	0.5173924265	0.4481071368	0.3040954684
1	4	0.01	0.5832278552	0.5557969598	0.4813688246	0.3266675892

6. Conclusion

In this paper, a theory of the Sumudu transform and its derivatives is successfully applied for fractional order linear partial differential equations. Three examples from the literature [16-17] are presented to determine the accuracy and simplicity of this proposed technique. It is mention that when we compare our work with the literature [16-17], this technique removes linearization and biologically unrealistic assumptions and gives an effective numerical solution than other techniques. The achieved outcomes are calculated using the symbolic calculus software Maple 16. This technique was undoubtedly very effective and powerful scheme in finding the solutions of approximate and numerical solutions as well as exact solutions.

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