Diophantine Quadruple with D(100) property

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Abstract
In this contribution, it is proved that a Diophantine $4-$tuple the set of four positive integers with the property that the product of any 2 of them plus 100 is a perfect square, than generalization of the result is obtained.
Key words:
Diophantine ; $m-$tuple; $D(n)$;

1. Introduction

A set of positive integers like as $\{a_1, a_2, a_3, \ldots\ a_m\}$ is to be said or supposed with the property $D(n)$ $where\ n \in \mathbf{Z}-\{0\}$, such that the product of any two element increased by n $i.e$ $a_i a_j + n$ which becomes perfect square $\forall \quad 1 \le i < j \le m$ where is to be considered Diophantine $m-$ tuples with property $D(n)$ for any arbitrary integer n and also for any linear polynomials in n. More, numerous authors considered the connections of the problem of Diophantus, in this context, one may refer [4 - 15]
Diophantus of Alexandria worked on the problem of four ($+ve$ rational) numbers such that the product of any two of them adding by 1 is a perfect square He obtained the result $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ [2].
However the first Diophantine [1]quadruple $D(1)$ have been founded by Fermat [3] the set of four numbers {1, 3, 8, 120}, such that the product of any two members and increased with 1, becomes result of perfect square [4, 5], other hand it has been proved and found in 1969 by two mathematicians Baker and Davenport [4] that a fifth positive integer could be not be added to this set. However, Euler introduced the first set of Diophantine Quintuple by adding the rational number, $\frac{777480}{8288641}$ [4, 5]. The question of existence of (integer) Diophantine quintuples was one of the oldest outstanding unsolved issue problems in the field of the number theory, where as in 2004 Andrej Dujella proved that most of or grate number of finite number of Diophantine quintuples exist [4 , 7, 8], . In 2016 the problem was finally resolved by He, Togbé and Ziegler [9].

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Here is given the set of four positive integer numbers $\{r, s, t, u\}$ such that the product of any 2 numbers and adding with 100 becomes a perfect square.

Theorem 1. The following sets of positive integers

$$\{r, 16r+80, 25r+100,\ r(rs+100)+s(4r-1)+440\} \quad where\ \forall r \in \mathbf{Z}-\{0\}, \tag{1}$$

$$\{r, 16r+80, 25r+100, 16r^3+144r^2+404r+360\} \quad where\ \forall r \in \mathbf{Z}-\{0\} \tag{2}$$

have existence the property of Diophantine quadruple of order 100. [See table 1]

Proof:
Let $\{r, s\}$ be an arbitrary two numbers with the property $D(n)$, for an integer n. It means that

$$rs + n = v^2 \tag{3}$$

We can easily check that the set $\{r, s, r+s+2v\}$ shows the property$D(n)$.

$$r(r+s+2v)+n = (r+v)^2$$
$$s(r+s+2v)+n = (s+v)^2$$

We can obtain following set while applying this construction to the
Diophantine pair $\{s, r+s+2v\}$
The set $\{s, r+s+2v, 4r+4s+4v\}$ therefore, could be the set

$$\{r, b, r+s+2v, 4r+4s+4v\} \tag{4}$$

have the property $D(n)$. If the product of its 1st and 4th element adding by n is a perfect square, i.e. if it holds

$$\mathrm{r}(4r+4s+4v)+n = y^2 \tag{5}$$

we will try to solve this equation using as less as possible restriction of number n we have

$$r^2 + 4\,(v^2 - 2n) + 4rv + n = y^2$$
$$6n = (r+2v-y)(r+2v-y)$$

Let us consider following
1)

$$r + 2v - y = 6$$

$$r + 2v + y = n$$

Now from this

$n = 2r + 4v - 6$ and $2n = 4r + 8v - 12$ put from (3) it follows:

$$r(s + 4) = (v - 2)(v - 6)$$

Putting $x = rk + 2$, we get $s = rk^2 - 4k - 4$ *and* $k = \frac{23}{2}$, $n = 4r(1 + 2k) + 4$

$\{r, rk^2 - 4k - 4, r(1+k)^2 - 4k, r(1+2k)^2 - 16k - 8\}$ (6)

With the $D(4r(1 + 2k) + 4)$ property

By putting the value of $r = 1$ in eq: (6)

$\{1, k^2 - 4k - 4, k^2 - 2k + 1, 4k^2 - 12k - 7\}$ (7)

With $D(8 + 8k)$

Note: That the formulas obtained by putting $r = 1$ in (6) and (7) are equivalent.

By putting the value from (3) it follows: $r(s + 4) = (v - 2)(v - 6)$

$s = rk^2 + 4k - 4$, $n = (4r(1 + 2k) + 36)$. If $k = \frac{15}{2}$

$\{r, rk^2 + 4k - 4, r(1+k)^2 + 4k + 8, r(1+2k)^2 + 16k + 8\} D(4r(1 + 2k) + 36)$ (8)

with the property,

By putting the value of $r = 1$ in (8),

$\{1, k^2 + 4k - 4, k^2 + 6k + 9, 4k^2 + 20k + 9\}$ (9)

With $D(40 + 8k)$ the property,

Note: That the formulas obtained by putting $r = 1$ in (8) and (9) are equivalent

Now

2)

$$r + 2v - y = 1$$

$$r + 2v + y = 6n$$

From $6n = 2r + 4v - 1$ and it follows from (3) that $k = \frac{97}{4}$ and

$r(3s + 2) = (3v + 1)(v + 1)$

Let us put $v = rm + 1$

Now from $3s + 2 = m(3v - 1)$ and m is of the form $3k + 1$ by using we get

$s = r(3k + 1)^2 + 2k, n = (2r(2k + 1) + 1)$ and the set

$\{r, r(3k+1)^2 + 2k, r(3k+2)^2 + 2k + 2, 9r(2k+1)^2 + 8k + 4\}$ (10)

With the $D(2r(2k + 1) + 1)$ property

As we know that

Similar idea can be apply to the set

$\{r, s, r + s + 2v, r + s - 2v\}$ (11)

With the order of $D(100)$.

Table 1: Product of any two numbers adding 100 is perfect square.

r	s	t	u	$\sqrt{rs+100}$	$\sqrt{rt+100}$	$\sqrt{ru+100}$	$\sqrt{st+100}$	$\sqrt{su+100}$	$\sqrt{tu+100}$
1	96	125	924	14	15	32	110	298	340
2	112	150	1872	18	20	62	130	458	530
3	128	175	3300	22	25	100	150	650	760
4	144	200	5304	26	30	146	170	874	1030
5	160	225	7980	30	35	200	190	1130	1340
6	176	250	11424	34	40	262	210	1418	1690
7	192	275	15732	38	45	332	230	1738	2080
8	208	300	21000	42	50	410	250	2090	2510
9	224	325	27324	46	55	496	270	2474	2980
10	240	350	34800	50	60	590	290	2890	3490
11	256	375	43524	54	65	692	310	3338	4040
12	272	400	53592	58	70	802	330	3818	4630
13	288	425	65100	62	75	920	350	4330	5260

2. Conclusion

Diophantine Quadruple with $D(100)$ properties have been successfully proved with examples.

With the valuable features aforementioned, and expect that our Diophantine Quadruple

With $D(100)$ properties will be useful for applications in areas such Cryptography.

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