Some polynomial formula of the Diophantine Quadruple with $D(n)$ property

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Abstract:
The original problem of quadruple was studied by the Since the Greek mathematician Diophantus of Alexandria $\frac{1}{16}$, $\frac{33}{16}$, $\frac{17}{4}$ and $\frac{105}{16}$ was the first set of quadruples found in 3^{rd} century $(b.c)$ in which having any product of two terms increasing the set increased by 1 is a perfect square. Later Fermat obtained a set from integers as {1,3,8,120}, later davenport and baker both to generalize the fourth member of the set is, {1,3,8, d}. Here in this work we will present a more generalized version of Diophantine quadruple, {p, q, r, s}, where any two of the product increased by n result will be a perfect square, i.e. $pq + n = x^2$ and, It is proved that the Diophantine $quadruple$, set of positive and negative integer number with the property that the product of any two of them plus n is a perfect square, than generalization of the result is obtained.

Keywords:
$diophantine\ quadruple$, $arithmetic progression$, $diophantine\ m - tuple$

1. Introduction

The original problem of quadruple was studied by the Since the Greek mathematician Diophantus of Alexandria $\frac{1}{16}, \frac{33}{16}, \frac{17}{4}$ and $\frac{105}{16}$ was the first set of quadruples found in 3^{rd} century $(b.c)$[1] in which the product of any two term has been increased by 1 having a perfect square $\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}$ with satisfying [2] Further Diophantine found to be set of Diophantine Quadruple on positive integer $\{1, 33, 68, 105\}$ by the $D(256)$ property. As like the product of the two distinct numbers increased by 256 is perfect square.

The French Mathematician Piere De Fermat (After that in 17th century) (1601-1665) found the $\{1, 3, 8, 120\}$,first set of Diophantine Quadruples on positive integers With satisfying the $D(1)$ property [3,5]

During the 18th century Euler give the General formula of Fermeat's set for Diophantine Quadruple $\{1, 3, 8, 120\}$, $\{a, b,\ a + b + 2x,\ 4x(a + x)(b + x)\}$

In 1996 Andrej Dujella Generalized the Fermat's set $\{1,3,8,120\}$

$a = n, \forall n \in Z^+$, $b = n + 2$, $c = 4(n + 1)$, $d = 4(n + 1)(2n + 1)$

Euler Shown that the rational Diophantine triple is extendable into a rational Diophantine Quintuple. Euler introduced the first set of Diophantine Quintuple by adding the rational number $\frac{777480}{8288641}$ in the Fermat's Diophantine Quadruple $\{1, 3, 8, 120\}$, the increased set of Euler's rational Diophantine Quintuple is $\left\{1, 3, 8, 120, \frac{777480}{8288641}\right\}$ with same poperty $D(1)$[4,6]

Davenport and Baker (In 1969) [21], showing in adding for that arrangement $\{1, 3, 8, 120\}$, there isn't any any kind of 5th integer that retains exactly the identical property $D(1)$ that attract the Fermat's Diophantine Quadruple into Diophantine Quintuple.

Within Nineteenth hundred seventy nine it had been demonstrated that each $three - tumle$ $\{a, b, c\}$ having increasable keen on four$-tumple$ $\{a, b, c, d\}$ by Arkin, Strauss and Hoggattt [17] . These people demonstrated when is really a Diophantine multiple, after that it may be prolonged if by take

$d = a + b + c + 2abc + 2rst$. Wherever, $ab + 1 = r^2$, $ac + 1 = s^2$,
$bc + 1 = t^2$ [23]

Diophantine quintuple of Euler's set $\{1, 3, 8, 120, \frac{777480}{8288641}\}$ remained later on generalized from the Hoggatt. Arkin, and Strauss [17].

The extended Diophantine quintuple is $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}, \frac{549120}{10201}\right\}$ with $D(1)$ property [19]

It was further more demonstrated simply by Dujella and also Peth"o inside 1998 a pair {1, 3} can prolong directly into four-Tuple {1, 3, 8, 120} through integers 8 as well as 120 this is not able to lengthen this right into a Diophantine quintuple through any kind of integers [18,22]. Within 2004 Andrej Dujella demonstrated which for the most part the finite number of Diophantine quintuples can be found [4,7,8]. Within 2016 the issue had been lastly solved through Bo He, Volker Ziegler as well as Alain Togbé [9]. Fermat's Diophantine Four-tuple {1, 3, 8, 120} is actually increase able just through rational number. However, it is not able to lengthen through any kind of integers [10]

Manuscript received April 5, 2019
Manuscript revised April 20, 2019

Within Nineteenth hundred ninety-nine Gibbs was given first time a set of Six-tuple about the positive rational numbers.
$\left\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\right\}$ Having the similar property $D(1)$ Except this and later on Gibbs given different set of six-tuples as like in this reference [20].

2. diophantine m − tuple

A set of positive integers like as $\{a_1, a_2, a_3, \ldots\ a_m\}$ remains to be the property $D(n), n \in \mathrm{Z} - \{0\}$, that product of any two element increased by $n\ (a_i a_j + n)$ which becomes perfect square for all $1 \leq i < j \leq m$ where is to considered Diophantine m-tuples by property $D(n)$ for any arbitrary integer n and also for any linear polynomials in n. Further, numerous authors considered the links of the problem of Diophantus. In this context, one may refer [11-16],

3. Results and Discussions

In this work, we have found four positive and negative integer numbers $\{p, q, r, s\}$ such that product of any two numbers adding with n becomes a perfect square
The set of positive integers
$\{1, \mathrm{k}^2 - 4\mathrm{k} - 4, \mathrm{k}^2 - 2\mathrm{k} + 1, 4\mathrm{k}^2 - 12\mathrm{k} - 7\}$ (1)
With the $\mathrm{D}(8\mathrm{k} + 8)$ property.
$\{1, k^2 + 4k - 4, k^2 + 6k + 9, 4k^2 + 20k + 9\}$ (2)
With the $D(8k + 40)$ property
Now We resolve deal with the like idea, let {p, q} be an arbitrary pair with the property
$pq + n = x^2$ (3)
It is easy cheek that the set
$\{p, q, p + q + 2x\}$ also has the property $\mathrm{D}(n)$, really
$p\ (p + q + 2x) = n = (p + x)^2$
$q(p + \mathrm{q} + 2x) + n = (\mathrm{q} + x)^2$
Applying this creation to the Diophantine pair $\{q, p + q + 2x\}$ we get the set
$\{q, p + q + 2x,\ p + 4q + 4x\}$, hence, the set
$\{p, q\ , p + q + 2x, p + 4q + 4x\}$ (4)
Has the property D(n) if the product of its first and fourth element increased by n is a perfect square i.e. if holds
$p(p + 4q + 4x) + n = \mathrm{x}^2$ (5)
Now putting the value of $n = 2\mathrm{k}$ in the equation (3) then we obtain,
$pq + 2k = x^2$ (6)
We will try to solve this equation using as less as possible restrictions on number n we have $p^2 + 4(x^2 - 2k) + 4px + n = y^2$, and $(\mathrm{p} + 2\mathrm{x} - \mathrm{y})(\mathrm{p} + 2\mathrm{x} +) = 6\mathrm{k}$
Let us consider two following cases

(1) $p + 2\mathrm{x} - \mathrm{y} = 6$
$p + 2\mathrm{x} + \mathrm{y} = \mathrm{k}$
From this $k = 2\mathrm{p} + 4\mathrm{x} - 6$, $then\ 2k = 4p + 8x - 12$, and from the value of $2k$ from in (3) $it\ follow$: $\mathrm{p}(\mathrm{q} + 2) = (\mathrm{x} - 2)(\mathrm{x} - 6)$,
now Putting $\mathrm{x} = \mathrm{pk} + 2$ We get
q as $= \mathrm{pk}^2 - 4\mathrm{k} - 4$, $2\mathrm{k} = 4\mathrm{p}(1 + 2\mathrm{k}) + 4$ hence, we get the set
, $\{\mathrm{p},\ \mathrm{pk}^2 - 4\mathrm{k} - 4,\ \mathrm{q}(1 + \mathrm{k})^2 - 4\mathrm{k},\ \mathrm{q}(1 + 2k)^2 - 16\mathrm{k} - 8\}$ (7)
having the $\mathrm{D}(4\mathrm{p}(1 + 2\mathrm{k}) + 4)$ property
Now put the value of $p = 1$ in(7)
$\{1, \mathrm{k}^2 - 4\mathrm{k} - 4, \mathrm{k}^2 - 2\mathrm{k} + 1, 4\mathrm{k}^2 - 12\mathrm{k} - 7\}$ (8)
With the $\mathrm{D}(8\mathrm{k} + 8)$ property.
Similar now by putting $x = pk + 6$, We obtain the value of t as
$q = pk^2 + 4k - 4$ and $2k = 4p(1 + 2k) + 36$ hence, we get the set
$\{p,\ pk^2 + 4k - 4,\ q(1 + k)^2 + 4k + 8,\ q(1 + 2k)^2 + 16k + 8\}$ (9)
With the $D(4p(1 + 2k) + 36)$ $property$
Has the property $D(4p(1 + 2k) + 36)$ Now Put $p = 1$ in (9)
$\{1, k^2 + 4k - 4, k^2 + 6k + 9, 4k^2 + 20k + 9\}$ (10)
With the $D(8k + 40)$ property

3.1 Theorm 1

$\{\mathrm{m}, \mathrm{mk}^2 - 4\mathrm{k} - 4, \mathrm{m}(1 + \mathrm{k})^2 - 4\mathrm{k}, \mathrm{m}(1 + 2k)^2 - 16\mathrm{k} - 8\}$
has the property $\mathrm{D}(4\mathrm{m}(1 + 2\mathrm{k}) + 4)$ and the set
$\{m, mk^2 + 4k - 4, m(1 + k)^2 + 4k + 8, m(1 + 2k)^2 + 16k + 8\}$
$has\ \ the\ property$ $D(4m(1 + 2k) + 36)$ and the set
The similar idea can be applied to the set
$\{p, q, p + q + 2x, p + q - 2x\}$ (11)
This set has the property $D(n)$ if and only if the product of its third and fourth element increased by n is a perfect square
$(p + q + 2x)(p + q - 2x) + n = y^2$
Hence, $6k = (p - q - y)(p - q + y)$, again we are successful to discuss two cases
(2)
$p - q - y = 6$
$p - q + y = k$
From this, $k = 2q - 2s - 6$, Then
$2k = 4q - 4p - 12$ and from (3) it follow,
$(p + 4)(q - 4) = (x + 2)(x - 2)$, Putting $x = (p + 4)k + 2$, we $get\ the\ value\ of\ q, as$
$q = pk^2 + 4(k^2 + k + 1)$, and
$2k = 4p(k^2 - 1) + (4k + 2)^2$
Then, we have set

$\{p, pk^2 + 4(k^2 + k + 1), p(1 + k)^2 + (k + 2)(4k + 4), p(1 - k)^2 + 4k(k - 1)\}$ (12)
having the $D(4p(k^2 - 1)(4k + 2)^2$.property
by put$ting\ the\ value\ \ p = 1\ in\ eq$ (12)
$\{1, 5k^2 + 4k + 4, 5k^2 + 14k + 9, 5k^2 - 6k + 1\}$ (13)
$with\ the\ \ D(20k^2 + 16)$ property

3.2 Theorm 2

$\{m, mk^2 + 4(k^2 + k + 1), m(1 + k)^2 + (k + 2)(4k + 4), m(1 - k)^2 + 4k(k - 1)\}$
has the property $D(4m(k^2 - 1)(4k + 2)^2$ and the set

4 Conclusion

The generalized Diophantine Quadruple with $D(n)$ property, the Diophantine Quadruple with $D(n)$ properties have been successfully proved with a few formulas some polynomial formulas for the positive integers {p, q, r, s } by an arithmetic progression are to be generalized ,it is concluded that there are an infinite Diophantine quadruple D(n) property , also some systematically generalized sets are to be found with the valuable features aforementioned, and expect that our Diophantine Quadruple ,With $D(n)$ properties will be useful for applications in areas such Cryptography.

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