On the Locating Chromatic Number of Subdivision of Barbell Graphs Containing Generalized Petersen Graph

Asmiati[1], I Ketut Sadha Gunce Yana[2], Lyra Yulianti[3]

[1,2]Mathematics Department, Faculty of Mathematics and Natural Sciences, Lampung University, Jl. Brodjonegoro No.1 Bandar Lampung, Indonesia.
[3]Mathematics Department, Faculty of Mathematics and Natural Sciences, Andalas University, Kampus UNAND Limau Manis, Padang 25163, Indonesia.

Abstract
The locating chromatic number of a graph is the minimal color required so that it qualifies for some locating coloring. This paper will discuss about the locating chromatic number for the subdivision of barbell graph containing Petersen Graph.
Key words:
locating chromatic number, barbell graph, subdivision, Petersen graph.

1. Introduction

The locating chromatic number of a graph was firstly studied by Chartrand et al. [1] as some development of the concept of partition dimension[2] and graph coloring. Consider $G=(V,E)$ as the given connected graph and c as the proper coloring of G using k colors $1,2,\ldots,k$ for some positive integer *k*. We denote $\Pi=\{C_1,C_2,...,C_k\}$ as the partition of $V(G)$, where C_i is the color class, i.e the set of vertices given the *i*-th color, for $i\in[1,k]$. For an arbitrary vertex $v \in V(G)$, the color code $c_{\Pi}(v)$ is defined as the ordered k -tuple

$$c_\pi(v) = (d(v,C_1), d(v,C_2), \ldots, d(v,C_k)),$$

where $d(v,C_i)=\min\{d(v,x)\,|\,x\in C_i\}$ for $i\in[1,k]$. If for every two vertices $u,v\in V(G)$, their color codes are different, $c_\pi(u) \neq c_\pi(v)$, then c is defined as the locating coloring of G using k colors. The locating chromatic number of *G*, denoted by $\chi_L(G)$, is the minimum *k* such that *G* has some locating coloring.
There were some interesting results related to the determination of the locating chromatic number of some graphs. The results were obtained by focusing on some certain classes of graph. Chartrand et al. [3] has succeeded in constructing tree on n vertices, $n \geq 5$ with locating chromatic numbers varying from *3* to *n*, except for $(n-1)$. Moreover, Asmiati et al. [4] determined the locating chromatic number of homogeneous amalgamation of stars and their monotonicity properties. Recently, Behtoei and Omoomi [5] have obtained the locating chromatic number of the Kneser graph. Asmiati et al. [6] determined the locating chromatic number of Petersen graph and Syofyan et al. [7] trees with certain locating chromatic number.
The barbell graph is constructed by connecting two arbitrary connected graphs G and H by a bridge**.** Let $B_{P_{n,1}}$ for $n \geq 3$, be the barbell graph where *G* and *H* are two copies of generalized Petersen graphs $P_{n,1}$. The following definition of generalized Petersen graph is taken from [8]. Let $\{u_1, u_2, \ldots, u_n\}$ be the set of vertices in the outer cycle and $\{v_1, v_2, \ldots, v_n\}$ be the set of vertices in the inner cycle of the Petersen graph, for $n \geq 3$. Denote the generalized Petersen graph by $P_{n,k}$. From the definition, it is clear that for $n \geq 3$ and $1 \leq k \leq \lceil n-1)/2 \rceil$, the generalized Petersen graph has $2n$ vertices and $3n$ edges.
In [9], the locating chromatic number of the barbell graph containing two copies of generalized Petersen graphs $P_{n,1}$ has been obtained in the following theorem.
Theorem 1.1 [9] For $n \geq 3$, the locating chromatic number of barbell graph $B_{P_{n,1}}$ is 4 for odd n and 5 otherwise.
This paper will determine the locating chromatic of some graph constructed by subdividing the bridge of the barbell that contains the generalized Petersen graph, denoted by $B^{*s}_{P_{n,1}}$. This problem is inspired by the results of research Purwasih et. al [10] about the locating chromatic number for a subdivision of a graph on one edge.

2. Results and Discussion

In the following theorem, it is discussed about the locating chromatic number for subdivision of some barbell graph containing Petersen graph, denoted by $B^{*s}_{P_{n,1}}$.

2.1 Theorem

Let $B^{*s}_{P_{n,1}}$ *be a subdivision of barbell graph containing Petersen Graph for* $s \geq 1$. *Then the locating chromatic number of* $B^{*s}_{P_{n,1}}$ *is 4 for odd* n, $n \geq 3$ *or 5 for* n *even*, $n \geq 4$.

Manuscript received July 5, 2019
Manuscript revised July 20, 2019

Proof. Let $B^{*s}_{P_{n,1}}$ be a subdivision barbell graph for $n \geq 3, s \geq 1$ with the vertex set $V(B^{*s}_{P_{n,1}}) = \{u_i, u_{n+i}, w_i, w_{n+i} : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq s\}$, and edge set $E(B^{*s}_{P_{n,1}}) = \{u_i u_{i+1}, u_{n+i} u_{n+i+1}, w_i w_{i+1}, w_{n+i} w_{n+i+1} : 1 \leq i \leq n-1\} \cup \{u_n u_1, u_{2n} u_{n+1}, w_n w_1, w_{2n} w_{n+1}\} \cup \{u_i u_{n+1}, w_i w_{n+1} : 1 \leq i \leq n\} \cup \{u_n w_n\} \cup \{u_n v_1, v_s w_n\} \cup \{v_i v_{i+1} : 1 \leq i \leq s-1\}$.

Let us distinguish four cases.

Case 1. *n odd.* According to Theorem 1.1, it is clear that $\chi_L(B^{*s}_{P_{n,1}}) \geq 4$.

To determine the upper bound for the locating chromatic number of subdivision Petersen graph $B^{*s}_{P_{n,1}}$, construct some locating coloring c using 4 colors as follows.

For odd s, define the following coloring

$$c(u_i) = \begin{cases} 1 & , \text{for } i = 1 \\ 3 & , \text{for even } i, i \geq 2 \\ 4 & , \text{for odd } i, i \geq 3. \end{cases}$$

$$c(u_{n+i}) = \begin{cases} 2 & , \text{for } i = 1 \\ 3 & , \text{for odd } i, i \geq 3 \\ 4 & , \text{for even } i, i \geq 2. \end{cases}$$

$$c(w_i) = \begin{cases} 1 & , \text{for odd } i, i \leq n-2 \\ 2 & , \text{for even } i, i \leq n-1 \\ 4 & , \text{for } i = n. \end{cases}$$

$$c(w_{n+i}) = \begin{cases} 1 & , \text{for even } i, i \leq n-1 \\ 2 & , \text{for odd } i, i \leq n-2 \\ 3 & , \text{for } i = n. \end{cases}$$

$$c(v_i) = \begin{cases} 1 & , \text{for } i = 1 \\ 3 & , \text{for odd } i, i \geq 3 \\ 4 & , \text{for even } i, i \geq 2. \end{cases}$$

The color codes of $V(B^{*s}_{P_{n,1}})$ for odd n and s are

$$c_\Pi(u_i) = \begin{cases} i & , \text{for } 2^{nd} \text{component, } i \leq \frac{n+1}{2} \\ i-1 & , \text{for } 1^{st} \text{component, } i \leq \frac{n+1}{2} \\ n-i+1 & , \text{for } 1^{st} \text{component, } i > \frac{n+1}{2} \\ n-i+2 & , \text{for } 2^{nd} \text{component,} i > \frac{n+1}{2} \\ 0 & , \text{for } 3^{th} \text{component, } i \text{ even, } i \geq 2 \\ & \;\; \text{for } 4^{th} \text{component, } i \text{odd, } i \geq 3 \\ 1 & , \text{otherwise.} \end{cases}$$

$$c_\Pi(u_{n+i}) = \begin{cases} i & , \text{for } 1^{st} \text{component, } i \leq \frac{n+1}{2} \\ i-1 & , \text{for } 2^{nd} \text{component, } i \leq \frac{n+1}{2} \\ n-i+1 & , \text{for } 2^{nd} \text{component,} i > \frac{n+1}{2} \\ n-i+2 & , \text{for } 1^{st} \text{component,} i > \frac{n+1}{2} \\ 0 & , \text{for } 4^{th} \text{component, } i \text{ even, } i \geq 2 \\ & \;\; \text{for } 3^{th} \text{component, } i \text{ odd, } i \geq 3 \\ 1 & , \text{otherwise.} \end{cases}$$

$$c_\Pi(w_i) = \begin{cases} i & , \text{for } 4^{th} \text{component, } i \leq \frac{n-1}{2} \\ i+1 & , \text{for } 3^{th} \text{component,} i \leq \frac{n-1}{2} \\ n-i & , \text{for } 4^{th} \text{component,} i \geq \frac{n+1}{2} \\ n-i+1 & , \text{for } 3^{th} \text{component,} i \geq \frac{n+1}{2} \\ 0 & , \text{for } 2^{nd} \text{component, } i \text{ even, } i \leq n-1 \\ & \;\; \text{for } 1^{st} \text{component, } i \text{ odd, } i \leq n-2 \\ 1 & , \text{otherwise.} \end{cases}$$

$$c_\Pi(w_{n+i}) = \begin{cases} i & , \text{for } 3^{th} \text{component,} i \leq \frac{n-1}{2} \\ i+1 & , \text{for } 4^{th} \text{component,} i \leq \frac{n-1}{2} \\ n-i & , \text{for } 3^{th} \text{component, } i \geq \frac{n+1}{2} \\ n-i+1 & , \text{for } 4^{th} \text{component, } i \geq \frac{n+1}{2} \\ 0 & , \text{for } 1^{st} \text{component, } i \text{even, } i \leq n-1 \\ & \;\; \text{for } 2^{nd} \text{component, } i \text{odd, } i \leq n-2 \\ 1 & , \text{otherwise.} \end{cases}$$

$$c_{\Pi}(v_i) = \begin{cases} i-1 & \text{, for } 1^{st}\text{component, } i \le \frac{s+1}{2} \\ i+2 & \text{, for } 2^{nd}\text{component,} i \le \frac{s-1}{2} \\ s-i+2 & \text{, for } 1^{st}\text{component,} i > \frac{s+1}{2} \\ & \text{for } 2^{nd}\text{component, } i \ge \frac{s+1}{2} \\ 2 & \text{, for } 3^{th}\text{component, } i = 1 \\ 0 & \text{, for } 4^{th}\text{component, } i \text{ even, } i \ge 2 \\ & \text{for } 3^{th}\text{component, } i \text{ odd, } i \ge 3 \\ 1 & \text{, otherwise.} \end{cases}$$

For even s, define the following coloring.

$$c(u_i) = \begin{cases} 1 & \text{, for} i = 1 \\ 3 & \text{, for even } i, i \ge 2 \\ 4 & \text{, for odd } i, i \ge 3 \end{cases}$$

$$c(u_{n+i}) = \begin{cases} 2 & \text{, for} i = 1 \\ 3 & \text{, for odd } i,\ i \ge 3 \\ 4 & \text{, for even } i,\ i \ge 2. \end{cases}$$

$$c(w_i) = \begin{cases} 1 & \text{, for odd } i,\ i \le n-2 \\ 2 & \text{, for even } i,\ i \le n-1 \\ 4 & \text{, for } i = n. \end{cases}$$

$$c(w_{n+i}) = \begin{cases} 1 & \text{, for even } i,\ i \le n-1 \\ 2 & \text{, for odd } i,\ i \le n-2 \\ 3 & \text{, for } i = n. \end{cases}$$

$$c(v_i) = \begin{cases} 1 & \text{, for } i = 1 \\ 3 & \text{, for even } i,\ i \ge 2 \\ 4 & \text{, for odd } i,\ i \ge 3. \end{cases}$$

The color codes of $V(B^{*s}_{P_{n,1}})$ for odd n and even s are

$$c_{\Pi}(u_i) = \begin{cases} i & \text{, for } 2^{nd}\text{component,} i \le \frac{n+1}{2} \\ i-1 & \text{, for } 1^{st}\text{component, } i \le \frac{n+1}{2} \\ n-i+1 & \text{, for } 1^{st}\text{component, } i > \frac{n+1}{2} \\ n-i+2 & \text{, for } 2^{nd}\text{component, } i > \frac{n+1}{2} \\ 0 & \text{, for } 3^{th}\text{component, } i \text{ even, } i \ge 2 \\ & \text{for } 4^{th}\text{component, } i \text{ odd, } i \ge 3 \\ 1 & \text{, otherwise.} \end{cases}$$

$$c_{\Pi}(u_{n+i}) = \begin{cases} i & \text{, for } 1^{st}\text{component, } i \le \frac{n+1}{2} \\ i-1 & \text{, for } 2^{nd}\text{component, } i \le \frac{n+1}{2} \\ n-i+1 & \text{, for } 2^{nd}\text{component, } i > \frac{n+1}{2} \\ n-i+2 & \text{, for } 1^{st}\text{component, } i > \frac{n+1}{2} \\ 0 & \text{, for } 4^{th}\text{component, } i \text{ even, } i \ge 2 \\ & \text{for } 3^{th}\text{component, } i \text{ odd, } i \ge 3 \\ 1 & \text{, otherwise.} \end{cases}$$

$$c_{\Pi}(w_i) = \begin{cases} i & \text{, for } 4^{th}\text{component, } i \le \frac{n-1}{2} \\ i+1 & \text{, for } 3^{th}\text{component,} i \le \frac{n-1}{2} \\ n-i & \text{, for } 4^{th}\text{component, } i \ge \frac{n+1}{2} \\ n-i+1 & \text{, for } 3^{th}\text{component, } i \ge \frac{n+1}{2} \\ 0 & \text{, for } 2^{nd}\text{component, } i \text{ even, } i \le n-1 \\ & \text{for } 1^{st}\text{component, } i \text{odd, } i \le n-2 \\ 1 & \text{, otherwise.} \end{cases}$$

$$c_{\Pi}(w_{n+i}) = \begin{cases} i & \text{, for } 3^{th}\text{component,} i \le \frac{n-1}{2} \\ i+1 & \text{, for } 4^{th}\text{component, } i \le \frac{n-1}{2} \\ n-i & \text{, for } 3^{th}\text{component, } i \ge \frac{n+1}{2} \\ n-i+1 & \text{, for } 4^{th}\text{component,} i \ge \frac{n+1}{2} \\ 0 & \text{,for } 1^{st}\text{component, } i \text{ even, } i \le n-1 \\ & \text{for } 2^{nd}\text{component, } i \text{ odd, } i \le n-2 \\ 1 & \text{, otherwise.} \end{cases}$$

$$c_\Pi(v_i) = \begin{cases} i-1 & \text{, for } 1^{st}\text{component, } i \le \left(\frac{s}{2}\right)+1 \\ i+2 & \text{, for } 2^{nd}\text{component,} i \le \frac{s}{2} \\ s-i+2 & \text{, for } 1^{st}\text{component, } i > \left(\frac{s}{2}\right)+1 \\ & \quad \text{for } 2^{nd}\text{component, } i > \frac{s}{2} \\ 2 & \text{, for } 3^{th}\text{component, } i=1 \\ 0 & \text{, for } 3^{th}\text{component, } i \text{ even, } i \ge 2 \\ & \quad \text{for } 4^{th}\text{component, } i \text{ odd, } i \ge 3 \\ 1 & \text{, otherwise} \end{cases}$$

Since all vertices in $B^{*s}_{P_{n,1}}$ for odd n have distinct color codes, then c is the locating coloring using 4 colors. So, $\chi_L\left(B^{*s}_{P_{n,1}}\right) \le 4$.

***Case 2* (*n* even).** By Theorem 1.1, it is clear that have $\chi_L(B_{P_{n,1}}) \ge 5$. Consider the following two sub cases. For s odd, let c be a coloring using 5 colors as follows.

$$c(u_i) = \begin{cases} 1 & \text{, for } i=1 \\ 3 & \text{, for even } i, 2 \le i \le n-2 \\ 4 & \text{, for odd } i, 3 \le i \le n-1 \\ 5 & \text{, for } i=n \end{cases}$$

$$c(u_{n+i}) = \begin{cases} 2 & \text{, for } i=1 \\ 3 & \text{, for odd } i, i \ge 3 \\ 4 & \text{, for even } i, i \ge 2 \end{cases}$$

$$c(w_i) = \begin{cases} 1 & \text{, for odd } i, i \le n-1 \\ 2 & \text{, for even } i, i \le n-2 \\ 3 & \text{, for } i = n-1 \\ 5 & \text{, for } i=n. \end{cases}$$

$$c(w_{n+i}) = \begin{cases} 1 & \text{, for even } i, i \le n-2 \\ 2 & \text{, for odd } i, i \le n-1 \\ 4 & \text{, for } i=n. \end{cases}$$

$$c(v_i) = \begin{cases} 4 & \text{, for odd } i, 1 \le i \le s \\ 5 & \text{, for even } i, 1 \le i \le s\text{-}1 \end{cases}$$

The color codes of $V(B^{*s}_{P_{n,1}})$ for even n and odd s are

$$c_\Pi(u_i) = \begin{cases} i & \text{, for } 2^{nd}\text{componentand} 5^{th}, i \le \frac{n}{2} \\ i-1 & \text{, for } 1^{st}\text{component, } i \le \frac{n}{2} \\ n-i & \text{, for } 5^{th}\text{component, } i > \frac{n}{2} \\ n-i+1 & \text{, for } 1^{st}\text{component, } i > \frac{n}{2} \\ n-i+2 & \text{, for } 2^{nd}\text{component, } i > \frac{n}{2} \\ 0 & \text{, for } 3^{th}\text{component, } i \text{ even, } 2 \le i \le n-2 \\ & \quad \text{for } 4^{th}\text{component, } i \text{ odd, } 2 \le i \le n-1 \\ 2 & \text{, for } 4^{th}\text{component, } i=1 \\ & \quad \text{for } 3^{th}\text{component, } i=n \\ 1 & \text{, otherwise.} \end{cases}$$

$$c_\Pi(u_{n+i}) = \begin{cases} i & \text{, for } 1^{st}\text{component, } i \le \frac{n}{2} \\ i-1 & \text{, for } 2^{nd}\text{component, } i \le \frac{n}{2} \\ i+1 & \text{, for } 5^{th}\text{component, } i \le \frac{n}{2} \\ n-i+1 & \text{, for } 2^{nd}\text{component and } 5^{th}, i > \frac{n}{2} \\ n-i+2 & \text{, for } 1^{st}\text{component, } i > \frac{n}{2} \\ 0 & \text{, for } 3^{th}\text{component, } i \text{ odd, } 3 \le i \le n-1 \\ & \quad \text{for } 4^{th}\text{component, } i \text{ even, } 2 \le i \le n \\ 2 & \text{, for } 3^{th}\text{component, } i=1 \\ 1 & \text{, otherwise.} \end{cases}$$

$$c_\Pi(w_i) = \begin{cases} i & \text{, for } 5^{th}\text{component, } i \le \frac{n}{2} \\ i+1 & \text{, for } 4^{th}\text{component, } i \le \frac{n}{2} \\ & \quad \text{for } 3^{th}\text{component, } i \le \left(\frac{n}{2}\right)-1 \\ n-i & \text{, for } 5^{th}\text{component, } i > \frac{n}{2} \\ n-i+1 & \text{, for } 4^{th}\text{component, } i > \frac{n}{2} \\ n-i-1 & \text{, for } 3^{th}\text{component, } \frac{n}{2} \le i \le n-1 \\ 0 & \text{, for } 1^{st}\text{component, } i \text{ odd, } i \le n-1 \\ & \quad \text{for } 2^{nd}\text{component, } i \text{ even, } i \le n-2 \\ 2 & \text{, for } 1^{st}\text{component,} i = n-1 \\ & \quad \text{for } 2^{nd}\text{component, } i=n \\ 1 & \text{, otherwise.} \end{cases}$$

$$c_{\Pi}(w_{n+i}) = \begin{cases} i & \text{, for } 4^{th}\text{component, } i \le \frac{n}{2} \\ i+1 & \text{, for } 5^{th}\text{component, } i \le \frac{n}{2} \\ i+2 & \text{, for } 3^{th}\text{component, } i \le \left(\frac{n}{2}\right)-1 \\ n-i & \text{, for } 3^{th}\text{component, } \frac{n}{2} \le i \le n-1 \\ & \quad \text{for } 4^{th}\text{component, } i > \frac{n}{2} \\ n-i+1 & \text{, for } 5^{th}\text{component,} i > \frac{n}{2} \\ 0 & \text{, for } 1^{st}\text{component, } i \text{ even, } i \le n-2 \\ & \quad \text{for } 2^{nd}\text{component, } i\text{odd, } i \le n-1 \\ 2 & \text{, for } 1^{st}\text{component and } 3^{th}\text{, } i = n \\ 1 & \text{, otherwise.} \end{cases}$$

$$c_{\Pi}(v_i) = \begin{cases} i+1 & \text{, for } 1^{st}\text{component, } i \le \frac{s+1}{2} \\ i+2 & \text{, for } 2^{nd}\text{component, } i \le \frac{s+1}{2} \\ & \quad \text{for } 3^{th}\text{component, } i \le \frac{s-1}{2} \\ s-i+2 & \text{, for } 1^{st}\text{component, } i > \frac{s+1}{2} \\ & \quad \text{for } 3^{th}\text{component, } i \ge \frac{s+1}{2} \\ s-i+3 & \text{, for } 2^{nd}\text{component, } i > \frac{s+1}{2} \\ 0 & \text{, for } 4^{th}\text{component, } i\text{odd, } 1 \le i \le s \\ & \quad \text{for } 5^{th}\text{component, } i \text{ even, } 1 \le i \le s\text{-}1 \\ 1 & \text{, otherwise.} \end{cases}$$

For sub case s even, we have

$$c(u_i) = \begin{cases} 1 & \text{, for } i = 1 \\ 3 & \text{, for even } i,\ 2 \le i \le n-2 \\ 4 & \text{, for odd } i,\ 2 \le i \le n-1 \\ 5 & \text{, for} i = n. \end{cases}$$

$$c(u_{n+i}) = \begin{cases} 2 & \text{, for } i = 1 \\ 3 & \text{, for odd } i,\ i \ge 3 \\ 4 & \text{, for even } i,\ i \ge 2. \end{cases}$$

$$c(w_i) = \begin{cases} 1 & \text{, for odd } i,\ i \le n-1 \\ 2 & \text{, for even } i,\ i \le n-2 \\ 3 & \text{, for } i = n-1 \\ 4 & \text{, for } i = n. \end{cases}$$

$$c(w_{n+i}) = \begin{cases} 1 & \text{, for even } i,\ i \le n-2 \\ 2 & \text{, for odd } i,\ i \le n-1 \\ 5 & \text{, for } i = n. \end{cases}$$

$$c(v_i) = \begin{cases} 4 & \text{, for odd } i,\ 1 \le i \le s\text{-}1 \\ 5 & \text{, for even } i,\ 1 \le i \le s \end{cases}$$

The color codes of $V(B^{*s}_{P_{n,1}})$ for even n and s are

$$c_{\Pi}(u_i) = \begin{cases} i & \text{, for } 2^{nd}\text{componentand } 5^{th}\text{, } i \le \frac{n}{2} \\ i-1 & \text{, for } 1^{st}\text{component, } i \le \frac{n}{2} \\ n-i & \text{, for } 5^{th}\text{component, } i > \frac{n}{2} \\ n-i+1 & \text{, for } 1^{st}\text{component, } i > \frac{n}{2} \\ n-i+2 & \text{,for } 2^{nd}\text{component, } i > \frac{n}{2} \\ 0 & \text{, for } 3^{th}\text{component, } i \text{ even, } 2 \le i \le n-2 \\ & \quad \text{for } 4^{th}\text{component, } i\text{odd, } 2 \le i \le n-1 \\ 2 & \text{, for } 4^{th}\text{component, } i = 1 \\ & \quad \text{for } 3^{th}\text{component, } i = n \\ 1 & \text{, otherwise.} \end{cases}$$

$$c_{\Pi}(u_{n+i}) = \begin{cases} i & \text{, for } 1^{st}\text{component,} i \le \frac{n}{2} \\ i-1 & \text{, for } 2^{nd}\text{component,} i \le \frac{n}{2} \\ i+1 & \text{, for } 5^{th}\text{component, } i \le \frac{n}{2} \\ n-i+1 & \text{, for } 2^{nd}\text{componentand } 5^{th}\text{, } i > \frac{n}{2} \\ n-i+2 & \text{, for } 1^{st}\text{component, } i > \frac{n}{2} \\ 0 & \text{, for } 3^{th}\text{component, } i \text{ odd, } 2 \le i \le n\text{-}1 \\ & \quad \text{for } 4^{th}\text{component, } i \text{ even, } 2 \le i \le n \\ 2 & \text{, for } 3^{th}\text{component, } i = 1 \\ 1 & \text{, otherwise.} \end{cases}$$

$$c_\Pi(w_i)=\begin{cases} i & \text{,for } 4^{th}\text{component, } i\le \frac{n}{2} \\ i+1 & \text{, for } 5^{th}\text{component,} i\le \frac{n}{2} \\ & \text{for } 3^{th}\text{component, } i\le \left(\frac{n}{2}\right)-1 \\ n-i & \text{, for } 4^{th}\text{component, } i>\frac{n}{2} \\ n-i+1 & \text{, for } 5^{th}\text{component,} i>\frac{n}{2} \\ n-i-1 & \text{, for } 3^{th}\text{component, } \frac{n}{2}\le i\le n-1 \\ 0 & \text{, for } 1^{st}\text{component, } i\text{odd, } i\le n-1 \\ & \text{for } 2^{nd}\text{component, } i \text{ even, } i\le n-2 \\ 2 & \text{, for } 1^{st}\text{component, } i=n-1 \\ & \text{for } 2^{nd}\text{component, } i=n \\ 1 & \text{, otherwise.} \end{cases}$$

$$c_\Pi(w_{n+i})=\begin{cases} i & \text{,for } 5^{th}\text{component, } i\le \frac{n}{2} \\ i+1 & \text{, for } 4^{th}\text{component, } i\le \frac{n}{2} \\ i+2 & \text{,for } 3^{th}\text{component,} i\le \left(\frac{n}{2}\right)-1 \\ n-i & \text{, for } 3^{th}\text{component, } \frac{n}{2}\le i\le n-1 \\ & \text{for } 5^{th}\text{component, } i>\frac{n}{2} \\ n-i+1 & \text{, for } 4^{th}\text{component,} i>\frac{n}{2} \\ 0 & \text{, for } 1^{st}\text{component, } i \text{ even, } i\le n-2 \\ & \text{for } 2^{nd}\text{component, } i\text{odd, } i\le n-1 \\ 2 & \text{, for } 1^{st}\text{component and } 3^{th}\text{, } i=n \\ 1 & \text{, otherwise.} \end{cases}$$

$$c_\Pi(v_i)=\begin{cases} i+1 & \text{, for } 1^{st}\text{component, } i\le \frac{s}{2} \\ i+2 & \text{, for } 2^{nd}\text{component,} i\le \frac{s}{2} \\ & \text{for } 3^{th}\text{component, } i\le \frac{s}{2} \\ s-i+2 & \text{, for } 1^{st}\text{component,} i>\frac{s}{2} \\ & \text{for } 3^{th}\text{component, } i>\frac{s}{2} \\ s-i+3 & \text{, for } 2^{nd}\text{component, } i>\frac{s}{2} \\ 0 & \text{, for } 4^{th}\text{component, } i \text{ odd, } 1\le i\le s\text{-1} \\ & \text{for } 5^{th}\text{component, } i \text{ even, } 1\le i\le s \\ 1 & \text{, otherwise.} \end{cases}$$

Since all vertices in $B^{*s}_{P_{n,1}}$ for even n have distinct color codes, then c is the locating coloring using 4 colors. Therefore, the locating chromatic number of the subdividing barbell graph containing generalized Petersen graph, $\chi_L\left(B^{*s}_{P_{n,1}}\right)\le 5$.This completes the proof.

Acknowledgement

We are thankful to DRPM Dikti for the Fundamental Reseach Grant 2019.

References

[1] Chartrand, G., Erwin, D., Henning, M.A., Slater, P.J., and Zhang, P., The Locating-Chromatic Number of a Graph, Bull. Inst. Combin. Appl., vol. 36, pp. 89-101, 2002.
[2] Chartrand, G., Salehi, E., and Zhang, P., On the partition dimension of graph. Congr. Numer., 130, pp. 157-168, 1998.
[3] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater, P. Zhang, Graph of order n with locating-chromatic number n−1, Discrete Math. vol 269, pp. 65-79, 2003.
[4] Asmiati, H. Assiyatun, E.T. Baskoro, Locating-Chromatic Number of Amalgamation of Stars, ITB J. Sci., vol. 43, no. 1, pp. 1-8, 2011.
[5] A. Behtoei, B. Omoomi, On the locating chromatic number Kneser Graphs, Discrete Applied Mathematics, 159, pp. 2214-2221, 2011.
[6] Asmiati, Wamiliana, Devriyadi, L. Yulianti, On Some Petersen Graphs Having Locating-Chromatic Number Four or Five, Far East Journal of Mathematical Sciences, vol. 102, no. 4, pp. 769-778, 2017.
[7] D.K. Syofyan, E.T. Baskoro, H. Assiyatun, Trees with Certain Locating-Chromatic Number, J. Math. Fund. Sci., 48(1), pp. 39 – 47, 2016.
[8] M.E. Watkins, A theorem on Tait coloring with an application to the generalized Petersen graphs, J. Combin. Theory. 6., pp. 152-164, 1969.
[9] Asmiati, K.G. Yana, L. Yulianti, On the locating chromatic number of certain barbell graphs, International Journal of Mathematics and Mathematical Sciences, pp. 1-5, 2018.
[10] I.A. Purwasih, E.T. Baskoro, H. Assiyatun, D. Suprijanto, The Bounds on The Locating-Chromatic Number for a Subdivision of a Graph on One Edge, Procedia Computer Science., 74, pp. 84 – 88, 2015.