Convergence Analysis of Multigrid Method for Shifted Laplace at Various Levels Using Fourier Modes

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Abstract
In this paper, the convergence analysis of Multigrid solver is discussed for shifted Laplace equation. Multigrid is considered best choice for elliptic type partial differential equations, so is for Laplace equation. However inclusion of shift in Laplace equation disturbs spectral properties, which are usually not favorable for basic iterative methods. An analysis is inevitable to know the reasons for bad convergence of Multigrid for shifted Laplace equation. Multigrid components are separately analyzed and spectral expressions are derived. Their graphical interpretation is presented. The analysis of components of Multigrid is combined, in order to derive the closed-form of convergence factor of Multigrid method in two-grid fashion. The graphical interpretation of analysis is given, with recommendations of best and optimal parameters. This helps to recognize components of Multigrid causing slow convergence. Recommendation for fine tuning such components is given in order to obtain better convergence for shifted Laplace problem.

Key words:
Finite differences, Multigrid method, indefinite matrix, eigenvalues, Shifted Laplace equation
Mathematics Subject Classification:
65F05, 65F08, 65F10, 65N22, 65N06, 65N55

1. Introduction

This research work is aimed at determining composed form of convergence factor of Multigrid method in context of Solving Shifted Laplace equation. The efficient solution of linear system derived from discretization of Partial Differential Equation (PDE), denoted by,

$$A_h u_h = f_h \tag{1}$$

A problem considered is highly indefinite, however, Multigrid methods face difficulties in addressing the uncertainty [1, 2] , which is the subject of this research. This needs fine tuning of the components of the Multigrid method to optimize performance [3]. This is done using Fourier analysis for the problems under discussion. The pioneering analysis of Multigrid can be referred to in 1977 in [4]. A general Fourier analysis was presented in [5] for the iterative method itself. Multigrid as a preconditioner for undefined has been widely used and an analysis in its context is presented in [6-9]. The objective of this research is to obtain the convergence factor of the two networks method and its graphic interpretation.
The fundamental steps of Multigrid method are

1. Solving system $A_h u_h = f_h$ on fine grid Ω_h using BIM, which shall smooth error components
2. Transforming (restricting) error components on coarser grid Ω_{2h}
3. Smoothing error components on coarser grid Ω_{2h} and interpolating back to fine grid Ω_h
4. Updating solution on fine grid Ω_h with error components so interpolated.

The same is illustrated considering, considering two levels in the Figure 1

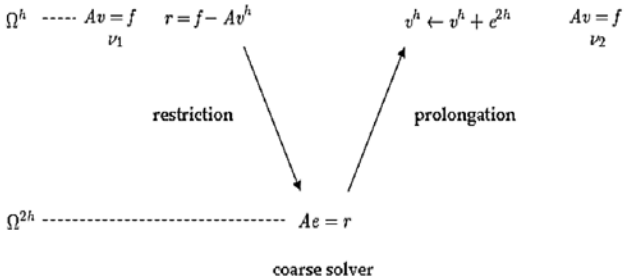

Fig. 1 Sketch of two-grid method

Further, the objective of research is to analyzing performance of Multigrid method for solving shifted Laplace equation as well as objectified to derive composed form of convergence factor of Multigrid method and the last objective is to make use of Fourier analysis at different levels of Multigrid methods.

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1.1 Outline of Paper

The rest of the paper is outlined as follows. Section 2 gives brief overview of problem statement with questions. Section 3 presents the methodology which covers the solution of one dimensional Laplace equation considering the eigenvalues and Eigen-functions along with derivations of the Multigrid elements are discussed. Section 4 is about the two grid method which subsequentially is used for result part. Section 5 is contains the results to analyse the efficiency of Multigrid method at various Fourier levels. In the end, Section 6 contains conclusion based on resulted are presented in Section 5.

2. Problem Statement

Multigrid Method is considered the best for solving elliptic partial differential equation in general and the Poisson problem i.e. $\Delta u = f$, in particular. However, with adding shift, problem becomes difficult to solve with Multigrid, also with other solvers. Reason is apparent i.e. there are negative eigenvalues of A. This causes Multigrid method to stagnates, means slow or bad convergence. Few questions are remarked in this research are "why Multigrid convergence is not as good as in case of Poisson equation"? "What is effect of shift in deriving convergence"? How parameters in components in Multigrid influence convergence? These posed questions are problem statement.

A two–dimensional shifted Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-k^2u = f(x,y) \tag{2}$$

is considered in unit square domain (geometry) $\Omega = (0,1)\times(0,1)$ along-with Homogeneous Dirichlet's boundary conditions $u(x,0)=u(0,y)=0=(x,1)=(1,y)$

Where $u(x,y)$ is pressure and k is the speed of wave

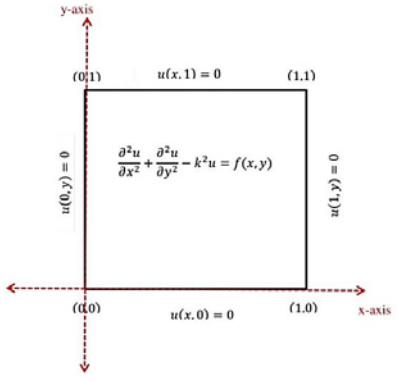


Fig. 2 Problem description along-with domain

3. Methodology

One-dimensional model problem (MP) is considered to explain methodology. The methodology for MP is divided into three different steps, first one is to obtain eigenvalues of Jacobi (smoother) iteration matrix $I - D^{-1}A$, where A is discrete Laplace operator $\Delta := A$.

The second step is to derive spectral expression of coarse grid correction operator (CGC), and last step is to join smoothing analysis and coarse grid correction analysis, to obtain closed form eigenvalues and Eigen-functions, which will be in block from of size 2×2, because of orthogonally of Eigen-function. We start the analysis by considering the one-dimensional Laplace differential equations is

$$\Delta u = \frac{\partial^2 u}{\partial x^2} = 0 \tag{3}$$

on domain $0 \le x \le 1$ with homogenous Dirichlet's boundary conditions

$$u(0) = 0$$
$$u(1) = 0 \tag{4}$$

Now the corresponding eigenvalue problem to the differential equation is

$$\Delta u = \frac{\partial^2 u}{\partial x^2} = \lambda u \tag{5}$$

The problem in Eq. (5) can be solved using Auxiliary equation

$$D^2 - \lambda = 0 \tag{6}$$

Solution(s) now depends upon value of "λ" this arises three difference cases, depending upon value of λ , that is $\lambda = 0, \lambda <$ or $\lambda > 0$

Case-1 When $\lambda = 0$ so Eq. (6) becomes

$$D^2 = 0 \tag{7}$$

This gives two repeated roots i.e. $D = 0, 0$. Thus solution will be

$$u(x) = Ae^{o.x} + Bx.\mathrm{e}^{0.x} = A + Bx \tag{8}$$

Using boundary conditions given in Eq. (4), the both constant *A* and *B* become zero, which means the particular solution is the trivial solution and is obtained from Eq. (7) using values of *A* and *B* which is given by

$$u(x) = 0 \tag{9}$$

This is trivial solution of Eigen-value and eigen-function problem, but as the interest lies in non-trivial Eigen function, this case is ignored

Case-2 When $\lambda > 0$

With this choice, Eq. (5) has two distinct positive roots, which are

$$D = \pm\sqrt{\lambda} \tag{10}$$

Two different real roots are there and solution will be formed as,

$$u(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} \tag{11}$$

Again, using boundary conditions from Eq. (4) , both the coefficients are zero i.e. $A = 0$ and $B = 0$. Therefore, again trivial solution which is of no interest, as Eigen-functions cannot be zero.

Case-3 When $\lambda < 1$

Taking $\lambda < 1$,the roots to Eq. (6) will be distinct but a pair of complex conjugate and are given as

$$D = \pm i\sqrt{\lambda} \tag{12}$$

These two complex conjugate roots will form solution to Eq. (5) as

$$u(x) = A\cos\left(\sqrt{\lambda}x\right) + B\left(\sqrt{\lambda}x\right) \tag{13}$$

Using boundary conditions from Eq. (6) , the coefficient $A = 0$ and also

$$B\sin(\sqrt{\lambda}) = 0 \tag{14}$$

If $B = 0$,then only possibility of having solution is trivial. Hence choosing $B \neq 0$, it shall give

$$\sin(\sqrt{\lambda}) = 0 \tag{15}$$

Solving Eq. (15), yields

$$\sqrt{\lambda} = n\pi \qquad \text{for} \qquad n = 0,1,2,3,4,... \tag{16}$$

Now using $A = 0$ and $\sqrt{\lambda} = n\lambda$ in eq. (5)

$$u(x) = B\sin(n\pi x) \tag{17}$$

Now in order to transform in discrete phenomena, the following notations are introduced [10]. $u(x)$ at node x_i on 1D grid is denoted by $u_i = u(x_i)$.

$u_i = u(x_i)$ The operator $\Delta = \frac{\partial^2}{\partial x^2}$ is discretized using Finite Difference method. ,

$\frac{\partial u_i}{\partial x} = \frac{u_{i+1} - u_i}{h}$ Again using backward difference,

$$\frac{\partial}{\partial x}\left(\frac{\partial u_i}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{u_{i+1} - u_i}{h}\right]$$

$$\Delta u_i = \frac{\partial^2 u_i}{\partial x^2} = \frac{u_{i+1} - 2u_i + u_{i+1}}{h^2}$$

Now discrete eigenvalue problem is given by

$$\Delta u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = \lambda_h u_i$$

Since,

$$u(x) = \sin(n\pi x_i)$$

$$\frac{\sin\left(n\pi x_{i+1}\right) - 2\sin\left(n\pi x_i\right) + \sin\left(n\pi x_{i-1}\right)}{h^2} = \lambda_h \sin(n\pi x_i)$$

Few steps of simplification will be

$$\lambda_h = \frac{2\cos(n\pi h) - 2}{h^2} = \lambda_h(n) \qquad n = 0,1,2,3,4,.....$$

3.1 Smoothing Analysis

For linear system, $A_h u_h = f_h$, the Jacobi iterations are obtained by splitting

$$A = D + L + U$$

Where D , L and U are is diagonal, lower and upper triangular matrices respectively.

$$Au = f$$

$$(L + U + D)u = f$$

$$u^{(i+1)} = D^{(-1)} f - D^{(-1)}(L + U)u^i$$

Since $L + U = A - D$, therefore

$$u^{i+1} = u^i + D^{-1}\left(f - Au^i\right)$$

And damped version will be

$$u^{i+1} = u^i + \omega D^{-1}\left(f - Au^i\right)$$

From above, iteration matrix of Jacobi and convergence depend upon spectral radius of this matrix

$$S_h = I_h - D_h^{-1} A_h \tag{18}$$

which is easily computed and is given by

$$\lambda_h(n)(s_h) = 1 - \lambda_h(D_h^{-1}).\lambda_h(A_h) = \frac{4}{h}\sin^2(n\pi h) \quad (19)$$

And "D" is diagonal of Laplace operator therefore and Eigen function are same for A and

$$\phi_h(n) = \sqrt{2h}\sin(n\pi x_i), \quad (20)$$

where $i = 1, 2, ..., N-1$ and $n = 1, 2, ...N-1$

3.2 Coarse Grid Correction

After smoothing analysis, second component of Multigrid is to be analyzed which is Coarse Grid Correction (CGC). Mainly first it involves the computation of residual $r_h = f_h - A_h u_h$ then get error equation, which is $A_h e_h = r_h$ this is equivalent to original system $A_h u_h = f_h$.

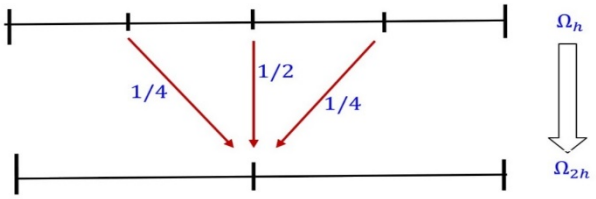


Fig. 3 Prolongation (linear interpolation) scheme for a one-dimensional grid.

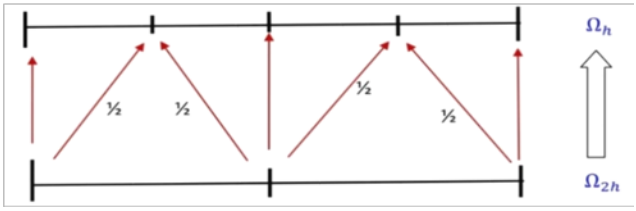


Fig. 4 Restriction scheme for a one-dimensional grid.

This problem takes same effort of solving original problem. Idea is to solve this problem on coarse grid Ω_{2h} to transfer residual to coarser grid Ω_{2h}, the restriction operator is need which explained in Figure 3. Same Problem on grid Ω_{2h} can immediately be solved by the framework of two-grid method, the equation,

$$A_{2h}e_{2h} = r_{2h}$$

Where A_{2h} is obtained by re-discretization on grid Ω_{2h} and r_{2h} is obtained by restricting original residual r_h to coarser grid using *restriction matrix*

$$r_{2h} = I_h^{2h} r_h$$

Where $\boldsymbol{I_h^{2h}}$ is restriction matrix, for any function f(x), defined as

$$\mathrm{I}_{\mathrm{h}}^{2\mathrm{h}} f(x) = \tfrac{1}{2} f(x-h) + f(x) + \tfrac{1}{2} f(x+h)$$

Next is to transform error e_{2h} from grid Ω_{2h} to Ω_h this is typically done using prolongation, which linear interpolation in one-dimensional as illustrated in Figure 3. In case of two-dimensional problem, this becomes bi-linear interpolation. Equation will be then

$$I_{2h}^h e_{2h} = e_{2h},$$

Where I_{2h}^h is prolongation operator and mathematically it is designed as

$$I_{2h}^h f(x) = \begin{cases} f(x) & \text{if } x \in \Omega_{2h} \\ \frac{1}{2}\left[f(x+h) + f(x-h) \right] & \text{otherwise} \end{cases}$$

This process of coarse grid correction can be written algebraically in equation as

$$T_h^{CGC} = I_h - I_{2h}^h A_{2h}^{(-1)} I_h^{2h} A_h,$$

Where it is important to note that eigenvalues of CGC operator are 0 and1, which leads to say that CGC is not useful on its own. However when it is combined with smoothing process, it produced effective method.

4. Two Grid Method

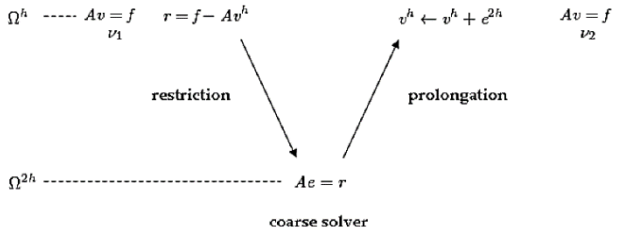


Fig. 5 Sketch of two-grid method

First of all, the sketch of two-grid cycle is illustrated in Figure 5, which explains how residual travels from fine to coarse grid.

Combining smoothing process and coarse grid correction gives two grid iterative processes [11], in equation form it is written as

$$M_h^{TGM} = S_h^{post} ? T_h^{CGC})\mathfrak{B}_h^{pre}$$

$$M_h^{TGM} = S_h^{post}\left(I_h - I_{2h}^h A_{2h}^{-1} I_h^{2h} A_h\right) S_h^{pre}$$

Now using orthogonal property of eigenvectors/functions, we can write eigenvectors $\phi_h(n)$, $n = 1, 2, ?..N - 1$

$$Q_h = [\phi_h(1), \phi_h(N-1), \phi_h(2), \phi_h(N-2) \ldots \phi_h(1), \phi_h(N/2)$$

By orthogonalizing eigen-function, we have obtained eigenvalues of M_h^{TGM} in 2×2 blocks as

$$\lambda_h\left(M_h^{TGM}\right) = \begin{bmatrix} 1 - \omega \sin^2\left(\frac{n\pi h}{2}\right) & 0 \\ 0 & 1 - \omega \cos^2\left(\frac{n\pi h}{2}\right) \end{bmatrix}$$

for $n = 1, 2, \ldots N/2 - 1$

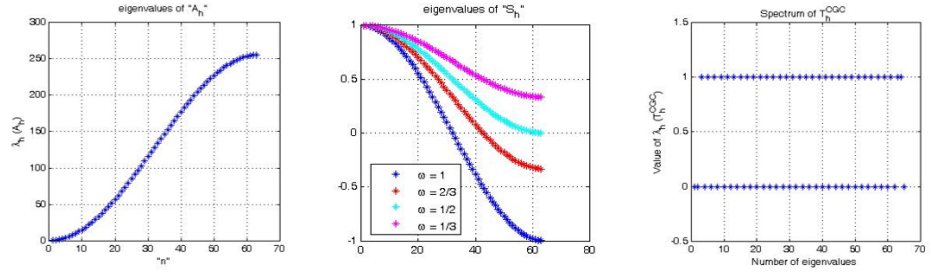
And $\lambda_h\left(M_h^{TGM}\right) = 1 - \omega$ for n

5. Results and Discussion

The closed form expressions of eigenvalues in all three parts of analysis [12] i.e. Smoothing, Coarse grid correction and two-grid method are graphically interpreted.

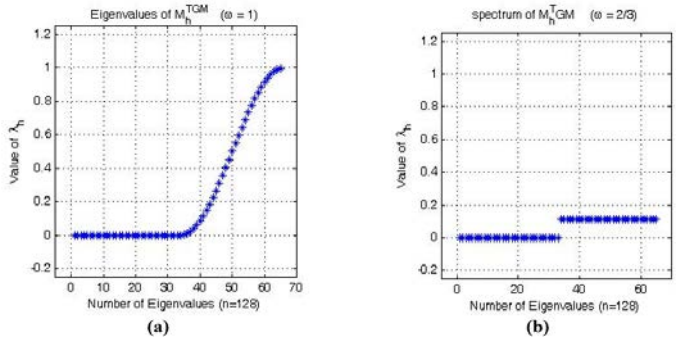
First of all spectrum of original matrix A_h is plotted in Figure 6 (a). It is noticeable that eigenvalues are increasing with respect index and all eigenvalues of positive unless shift is not involved.

Next, the spectrum of smoother matrix (Jacobi) S_h is plotted in Figure 6 (b). Four different choices of relaxation parameter ω are chosen which are $\omega = \frac{1}{2}, \frac{1}{3}, \frac{2}{3} and\ 1$. With parameter $\omega = 1/3$, spectrum is highly favorable is away from zero, however with other three choices, one can find that eigenvalues occur near to zero. Note that this does not remain same, when CGC is applied. It pushes to change behavior of smoother and same effective choice of ω is no more favorable. In Figure 6 (c), the eigenvalues of only coarse grid operator T_h^{CGC} are plotted. Spectrum consists of 0 and 1 moreover, CGC is not useful when applied alone. It is effective to project low frequency nodes hence after smoothing iterations, its application bring out favorable results. As compared to Figure 6(a), spectrum of two-grid method is presented with two different choices of relaxation parameter in smoother ω. It is interesting to notice that contrary to smoother only, the choice of $\omega = 2/3$ shows elegant spectrum. Half of eigenvalues becomes zero, not contributing in solver. Later half is very much clustered.



(a) (b) (c)

Fig. 6 (a) A plot of eigenvalue with grid size n=64 (b) Eigen-values of smoother with different choice of $\boldsymbol{\omega}$ with grid size n=64 (c) Eigenvalues of coarse grid operator



(a) (b)

Fig. 7 Eigenvalue of two grid with different relaxation parameter (a) $\omega = 1$ (b) $\omega = 2/3$

The weightage of two distinct damping values is depicted clearly in Figure 7 of left at $\omega = 1$ and right at $\omega = 2/3$. It is observant that the change in relaxation parameter affects the value of Eigen values with respect to grid size. However, $\omega = 2/3$ is more efficient than the $\omega = 1$.

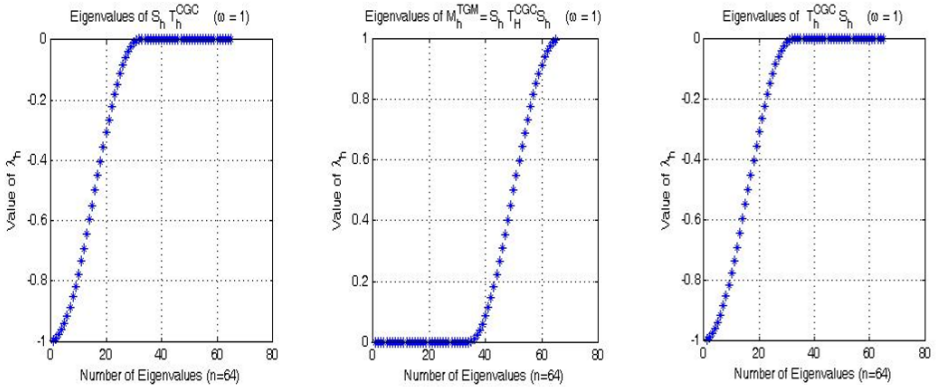


Fig. 8 Eigenvalue of two grid operator with different value of smoothing two grids at $\omega = 1$

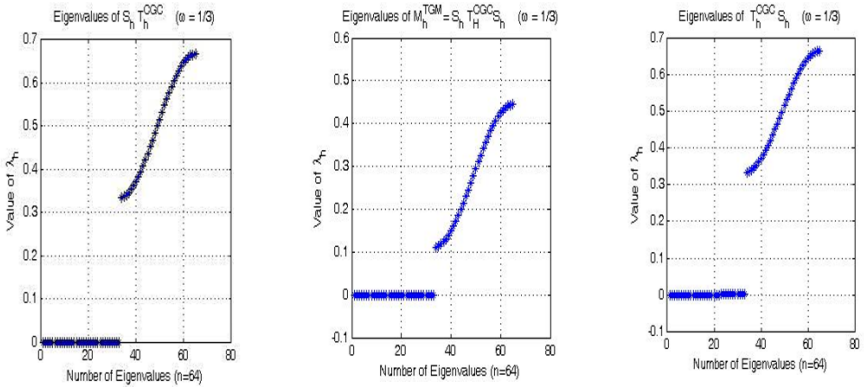


Fig. 9 Eigenvalue of with different value of smoothing two grids at $\omega = \frac{1}{3}$

Table 1: Number of Pre- and Post-smoothing steps brings out same spectral radius

Smoothing	n	$\omega = 1$	$\omega = 2/3$	$\omega = 1/2$	$\omega = 1/3$
$\rho(S_h^{(n)} \cdot T_h^{CGC})$	1	0.9976	0.3333	0.5000	0.6667
	2	0.9976	0.1111	0.2500	0.4444
	3	0.9952	0.0786	0.1250	0.2963
	4	0.9952	0.0617	0.8320	0.2975
	5	0.9928	0.0501	0.0671	0.1317

Keeping in view the following relation, above results in Table 1 is framed.

$$\rho(S_h T_h^{CGC}) = \rho(T_h^{CGC} S_h)$$

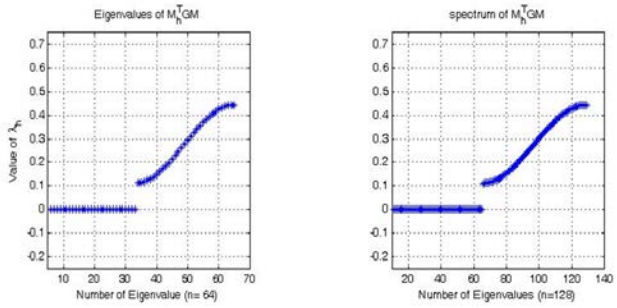


Fig. 10 Eigenvalue of two grid operator with pre and post smoothing

Change of grid size does not make change eigenvalues. Eigenvalues of only Two grid operator, M_h^{TGM}, with one pre- and post-smoothing $\omega = \frac{1}{3}$.

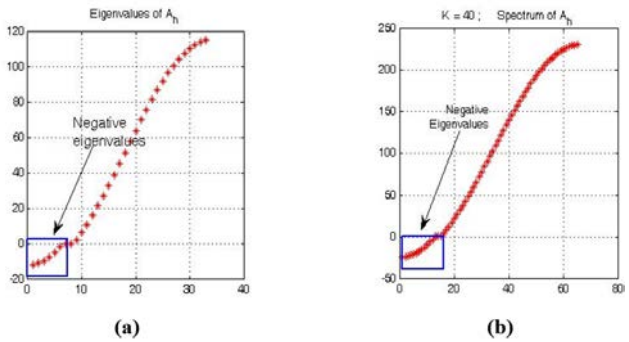


Fig. 11 Eigenvalue of original matrix for nonzero, (a) $k = 20$ (b) $k = 40$

Eigenvalues S_h when K is non-zero, in comparison of both graphs (a) and (b) we can see negative eigenvalues that leads to negative definite sparse matrix.

6. Conclusion

As the basic iterative methods are not solvers but smoothers. In an effort to complete the research three questions were framed. Why Multigrid has limitations for the case of Poisson equation? What are the effective consequences in deriving the convergence shifitng?and How Multigrid core parts will impact the parameters in convergence analysis? To answert these questions this work set out to investigate the convergence factor for one-dimensional Helmholtz equation in composite form that is derived, not only for smoother but for two grid method too. Moreover, the numerical experiments tell about optimal choice when CGC is not used. It is found to be $\omega = 1/3$ and as plotted, showed that better choice of relaxation parameter is $\omega = 2$ without using CGC. Further it is concluded that more iterations of smoothing does not reduce convergence factor for $\omega = 1$ more iterations reduces convergence factor more faster with choice $\omega = 2/3$. Smoothing either on left or right (Pre – or Post-) produces same spectrum as well as convergence factor and the spectrum of CGC contains eigenvalues 1 and 0.
The future analysis can be extended to be used for other problems which are being solved with Multigrid. We have used standard components of Multigrid, but using customized components of Multigrid will reshape and the same analysis can be extended for that problem. Analysis for Multigrid-based preconditioner (not solver) can be prospectus future work.

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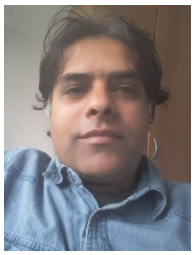

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