Comparing Three Different Estimators of fuzzy hazard Rate Function for Mixed Failure to Time Model

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Abstract
This paper deals with constructing a new mixed probability failure model distribution, from mixing f1(x,β) and f2(x,β) were f1(x,β)is exponential and f2(x,β) is Gamma with (2,β), the mixing proportion are ($p = \frac{\alpha}{\alpha+1}$) and $(1-p) = \frac{1}{\alpha+1}$, and after the p.d.f is constructing, we derive cumulative distribution function and reliability function, and also derive moments formula about origin.
And then the two parameters (α,β) are estimated by maximum likelihood, and moments and proposed method, then we use $(\hat{\alpha}, \hat{\beta})$ with fuzzy parameter $(\tilde{k}_i)$ to compare different estimators of fuzzy hazard rate function, using different sets of initial values, and also of sample size (n), and the comparison is done through simulation. All results are explained in tables.
Key words:
fuzzy hazard rate ; Mixed Failure ; maximum likelihood estimated.

1. Introduction

Many researchers work on constructing mixed distribution like Lindley (1958), were he introduce mixed distribution from exponential and Gamma, and in (1970), Sankaran introduce discrete mixed (Lindely-Poisson) and estimate its parameters, while in (1990), Hoskins indicates "L-moments" method foe estimations. Cupta R. D. and Kundu, D. (2001) gives different methods for estimating parameters of Generalized exponential. And Mahmoudi, E. and Zakerzadeh, H. (2010) work on the generalized Poisson Lindely distribution.
And in (2012) the researchers (H. S. Bakouch et al.) obtained expanded Lindely, which is mixed from Lomax distribution and Weibull distribution. They studied it's different properties like Reliability Function and Risk function and moments, as well as estimating by maximum likelihood and application on real data.
Also in (2013), Rama Shnker and A. Mishra, studied properties that is related with moments and risk function, and explain methods of estimation like, maximum likelihood and moments, with application.
In (2014) Erisoglu and Erisoglu indicates L-moments method of estimations for mixture of Weibull distribution.

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2. Definition of Distribution

Mixed exponential (β) and Gamma (2,β)

$$f(x,\alpha,\beta) = \frac{\alpha}{\alpha+1}\mathrm{f}_1(\mathrm{x}) + \frac{1}{(\alpha+1)}\mathrm{f}_2(\mathrm{x}) \quad (1)$$

$$= \frac{\alpha}{(\alpha+1)}\beta \mathrm{e}^{-\beta \mathrm{x}} + \frac{1}{(\alpha+1)}\beta^2 e^{-\beta x} \quad (2)$$

Then $f(x,\alpha,\beta) = \frac{\alpha\beta e^{-\beta x} + \beta^2 x e^{-\beta x}}{(\alpha+1)} = \frac{\beta(\alpha+\beta x)\beta e^{-\beta x}}{(\alpha+1)}$

x>0, β>0, α>-1 (3)

And the cumulative distribution function is

$$F(x) = \int_0^x f(u,\alpha,\beta)du = 1 - \frac{(1+\alpha+\beta x)}{(\alpha+1)}e^{-\beta x}$$

x>0, β>0, α>-1 (4)

While the survival function or reliability function is

$$R_x(x) = 1 - F_x(x) = \frac{(1+\alpha+\alpha\beta x)e^{-\beta x}}{1+\alpha}$$

x>0, β>0, α>-1 (5)

And the Hazard Rate function is

$$h(x) = \frac{f(x)}{R_x(x)} = \frac{\beta(\alpha+\beta x)}{1+\alpha+\beta x} \quad x>0,\ \beta>0,\ \alpha>-1 \quad (6)$$

and when x1<x2
h(x1)<h(x2)
h(x) is monotone increasing function
and when α ≥ 1, f(x) is decreasing function, and the mode of f(x) is

$$Mode = \begin{cases} \frac{(1-\alpha)}{\beta} & |\alpha| < 1 \\ 0 & olw \end{cases}$$

Also we can prove that the rth moments formula about origin

$$\mu'_r = E(x^r) = \int_0^\infty x^r f(x,\alpha,\beta)dx$$

This yield

$$\frac{\Gamma(r+1)}{\beta^r}\left[\frac{\alpha}{\alpha+1} + \frac{(r+1)}{(\alpha+1)}\right] \quad (7)$$

From equation (7) we prove that the variance of this mixed distribution is

$$\sigma^2 = E(x-\mu)^2 = \frac{\alpha^2+4\alpha+2}{\beta^2(\alpha+1)^2} \quad (8)$$

And coefficient of variation (C.V)

$$C.V = \frac{\sigma}{\mu_1} = \frac{\sqrt{\alpha^2+4\alpha+2}}{\alpha+2} \quad (9)$$

The value of (C.V) is increased when α is also increased.
We can obtain another measure like skewness and Kurtosis but it can found in the references [6].
After the derivation of (rth) moments formula about origin (equation 7) we can find moment estimator of α, β from solving

$$\mu'_r = \frac{\sum_{i=1}^{n} x_i^r}{n}$$

And since we have two parameters we need $E(x) = \frac{\alpha+2}{\beta(\alpha+1)}$ and

$$E(x^2) = \frac{2(\alpha+3)}{\beta^2(\alpha+1)}$$

Then the moment estimator obtained from $\mu_1 = \bar{x}$

$$\frac{\Gamma(2)}{\beta} = \left[\frac{\alpha}{\alpha+1} + \frac{2}{(\alpha+1)}\right] = \bar{x} \quad (10)$$

And

$$\mu_2 = \frac{\sum x_i^2}{n}$$

$$\frac{\Gamma(3)}{\beta^2} = \left[\frac{\alpha}{\alpha+1} + \frac{3}{(\alpha+1)}\right] = \frac{\sum_{i=1}^{n} x_i^2}{n} \quad (11)$$

$$\frac{1}{\hat{\beta}}\left(\frac{\hat{\alpha}+2}{\hat{\alpha}+1}\right) = \bar{x}$$

And

$$\frac{2}{\hat{\beta}^2}\left[\frac{\hat{\alpha}+3}{\hat{\alpha}+1}\right] = \frac{\sum_{i=1}^{n} x_i^2}{n}$$

$$\therefore \hat{\beta}_{MOM} = \frac{1}{\bar{x}} \quad \frac{(\hat{\alpha}+2)}{(\hat{\alpha}+1)}$$

And since

$$\hat{\beta}_{MOM}^2 = \frac{2n}{\sum_{i=1}^{r} x_i^2}\left(\frac{\hat{\alpha}+3}{\hat{\alpha}+1}\right) \quad (12)$$

Equation (12) gives ($\hat{\beta}_{MOM}$) which is an implicate function of ($\hat{\alpha}$), and can solved numerically by fixed point method according to given of ($\hat{\alpha}$>-1) until ($\hat{\alpha}_{MOM}$) for two successive interactive make the absolute differences less than tolerance.

$$\left|\hat{\beta}_{newmom} - \hat{\beta}_{oldmom}\right| < \text{tolerance}$$

3. Estimation by Maximum likelihood

Let x1, x2, …., xn be a r.s. from p.d. f. given in equation (2), then

$$L = \prod_{i=1}^{n} f(x_i, \alpha, \beta)$$

$$= \frac{\beta^n \prod_{i=1}^{n}(\alpha+\beta x_i) e^{-\beta(\sum x_i)}}{(\alpha+1)^2}$$

Taking logarithm for L

$$logL = nlog\beta + \sum_{i=1}^{n} log(\alpha+\beta x_i) - \beta\sum x_i - nlog(\alpha+1) \quad (13)$$

$$\frac{\partial logL}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\frac{x_i}{(\alpha+\beta x_i)} - \sum_{i=1}^{n} x_i$$

and

$$\frac{\partial logL}{\partial\alpha} = \sum_{i=1}^{n}\frac{x_i}{(\alpha+\beta x_i)} - \frac{n}{\alpha+1} \quad (14)$$

$$\frac{\partial logL}{\partial\beta} = 0$$

$$\frac{n}{\hat{\beta}} = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i(\hat{\alpha}+\hat{\beta}x_i)^{-1}$$

$$\hat{\beta}_{MLE} = \frac{n}{\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i(\hat{\alpha}+\hat{\beta}x_i)^{-1}} \quad (15)$$

From $\frac{\partial logL}{\partial\alpha} = 0$

$$\frac{n}{\hat{\alpha}} = \sum_{i=1}^{n}\left(\hat{\alpha}+\hat{\beta}x_i\right)^{-1} \quad (16)$$

Equation (16) solved numerically to find ($\hat{\alpha}_{MLE}$) .

4. Proposed Method

Here we have (β) scale parameter and (α shape) we can estimate (β) by

$$\hat{\beta}_{prop} = x_{(1)} = Min(x_i)$$

or

$$\hat{\beta} = \frac{\bar{y}(\hat{\alpha}-1)}{\hat{\alpha}} \quad (17)$$

and then

$$\hat{\alpha}_{prop} = Ln2/Ln\left(\frac{me}{\hat{\beta}}\right) \quad (18)$$

or

$$\hat{\alpha} = \frac{\widehat{M}_{1,0,1} - \widehat{M}_{1,0,0}}{2\widehat{M}_{1,0,1} - \widehat{M}_{1,0,0}} \quad (19)$$

$$\widehat{M}_{1,0,0} = \bar{y}$$

$$\widehat{M}_{1,0,1} = \frac{1}{n(n-1)} \sum_{i=1}^{n} (i-1) y_i \quad (20)$$

y(1)<y(2)< …… <y(n)

5. Simulation Procedure

The comparison of fuzzy hazard rate function $\hat{h}(\breve{k}_i\, x_i)$ are obtained from simulation, using inverse transformation were

$$u_i = 1 - \frac{(1+\hat{\alpha}+\hat{\beta}x_i)}{(\hat{\alpha}+1)} e^{-\hat{\beta}x_i} (21)$$

$$\frac{(1+\hat{\alpha}+\hat{\beta}x_i)}{(\hat{\alpha}+1)} e^{-\hat{\beta}x_i} = (1-u_i)$$

Let $v_i = (1-u_i)$ then $\quad 0 \le v_i \le 1$

$$(1 + \frac{\hat{\beta}}{(\hat{\alpha}+1)} x_i) e^{-\hat{\beta}x_i} = v_i$$

$$1 + \frac{\hat{\beta}}{\hat{\alpha}+1} x_i = v_i e^{\hat{\beta}x_i}$$

$$\frac{\hat{\beta}}{\hat{\alpha}+1} x_i = (v_i e^{\hat{\beta}x_i} - 1)$$

Were $\hat{\alpha} > -1$, $\hat{\beta} > 0$, $x_i > 0$, $0 \le v_i \le 1$

$$x_i = \frac{(\hat{\alpha}+1)}{(\hat{\beta})} (v_i e^{\hat{\beta}x_i} - 1)$$

According to given values of

$\hat{\alpha} > -1$, $\hat{\beta} > 0$ (n=30,60,90)

We generate values of xi and then the estimated values of $(\hat{\alpha}, \hat{\beta})$ are obtained and then the estimated values of $\hat{h}(\breve{k}_i\, x_i)$ fuzzy hazard rate are obtained.

6. Simulation Results for Fuzzy Hazard Rate $\hat{h}(\breve{k}_i\, x_i)$

Were α=0.5 β=0.8

α=0.3 β=1.2

$\widetilde{K} = 0.3 \quad 0.6$

n=25, 50,75

The comparison depend on the value smallest hazard rate function is the best one

Table 1: $\tilde{h}_i(k_i x_i)$

	α=0.25 θ=0.8 k_i=0.3					
N	ti	Real	MLE $\hat{h}_i(t_i)$	MOM $\hat{h}_i(t_i)$	Proposed $\hat{h}_i(t_i)$	Best
25	1.6	0.3898	0.3763	0.4336	0.3824	MLE
	2.6	0.4526	0.4467	0.402	0.4587	MOM
	3.6	0.5082	0.5329	0.4828	0.5041	MOM
	4.6	0.5362	0.5536	0.5302	0.5344	MOM
	5.6	0.5674	0.5620	0.5622	0.5268	MLE
	6.6	0.5826	0.5820	0.5872	0.5849	Prop
	7.6	0.6202	0.5933	0.6396	0.5872	Prop
	8.6	0.6136	0.6029	0.6361	0.6063	MLE
	9.6	0.6126	0.5992	0.6472	0.6136	MLE
	10.6	0.6172	0.5997	0.6532	0.6602	MLE
50	1.6	0.3879	0.3817	0.4005	0.3885	MLE
	2.6	0.4526	0.4548	0.4714	0.4612	MLE
	3.6	0.5072	0.5098	0.5067	0.5064	Prop
	4.6	0.5328	0.5406	0.5468	0.5367	Prop
	5.6	0.5612	0.5636	0.5622	0.5589	Prop
	6.6	0.5783	0.5793	0.5853	0.5599	Prop
	7.6	0.5914	0.5991	0.5972	0.5974	MOM
	8.6	0.6032	0.6024	0.6084	0.5993	Prop
	9.6	0.6104	0.6115	0.6143	0.5997	Prop
	10.6	0.6182	0.6108	0.6244	0.6152	MLE
75	1.6	0.3879	0.3888	0.4226	0.3994	MLE
	2.6	0.4526	0.4626	0.4902	0.4732	MLE
	3.6	0.5072	0.5081	0.5328	0.5268	MLE
	4.6	0.5328	0.5392	0.5532	0.5472	MLE
	5.6	0.5612	0.5642	0.5706	0.5602	Prop
	6.6	0.5783	0.5774	0.5741	0.6274	MOM
	7.6	0.5914	0.5913	0.5832	0.6218	MOM
	8.6	0.6032	0.6022	0.5844	0.6341	MOM
	9.6	0.6104	0.6108	0.6029	0.6442	MOM
	10.6	0.6182	0.6181	0.6348	0.6501	MLE

Now we construct table (2) for α=0.5, β=0.8 $\widetilde{K} = 0.6$, n=25, 50,75

Table 2: values of fuzzy hazard rate function $\tilde{h}_i(\tilde{k}_i x_i)$ for $\tilde{k} = 0.6$

	α=0.25 β=0.8					
n	Ti	Real	MLE $\hat{h}_i(t_i)$	MOM $\hat{h}_i(t_i)$	Proposed $\hat{h}_i(t_i)$	Best
25	1.6	0.5066	0.6064	0.6104	0.6114	MLE
	2.6	0.50358	0.6688	0.6642	0.676	MOM
	3.6	0.50793	0.7064	0.6653	0.7168	MOM
	4.6	0.50882	0.7372	0.7145	0.7425	MOM
	5.6	0.51154	0.7564	0.7437	0.7688	MOM
	6.6	0.514583	0.7714	0.7641	0.7913	MOM
	7.6	0.51708	0.7854	0.7789	0.7966	MOM
	8.6	0.51808	0.7943	0.8721	0.8014	MLE
	9.6	0.52367	0.8022	0.8094	0.8093	MLE
	10.6	0.5342	0.8098	0.8164	0.8163	MLE
50	1.6	0.6115	0.6115	0.6222	0.6264	MLE
	2.6	0.674	0.676	0.6851	0.6884	MLE
	3.6	0.6168	0.7164	0.7254	0.7271	MLE
	4.6	0.645	0.744	0.7534	0.7544	MLE
	5.6	0.7656	0.7656	0.774	0.7653	MLE
	6.6	0.7814	0.7821	0.7882	0.7892	MLE
	7.6	0.7837	0.7846	0.7885	0.8012	MLE
	8.6	0.8037	0.8036	0.8011	0.8113	MOM
	9.6	0.812	0.8121	0.8019	0.8192	MOM
	10.6	0.8189	0.8364	0.8264	0.8266	MOM
75	1.6	0.5066	0.61151	0.6394	0.6049	Prop
	2.6	0.50358	0.6850	0.6972	0.6489	Prop
	3.6	0.50793	0.7253	0.7352	0.6099	Prop
	4.6	0.50882	0.7531	0.7611	0.6321	Prop
	5.6	0.51154	0.7732	0.7802	0.6334	Prop
	6.6	0.514583	0.7846	0.7954	0.7804	Prop
	7.6	0.51708	0.7884	0.8009	0.7961	MLE
	8.6	0.51808	0.7893	0.8162	0.8003	MLE

	9.6	0.52367	0.8013	0.8242	0.8402	MLE
	10.6	0.5342	0.8015	0.8351	0.8098	MLE

7. Conclusion

In this work, we have attended both results. The first concern the results of simulation (which comparing three different estimator's of fuzzy hazard rate function $\tilde{h}_i(\tilde{k_\iota t_i})$, when we find the first best estimator one is MLE (with $\left(\frac{13}{30}\right) * 100\% = 43.33\%$ and then Proposed is best with percentage $\left(\frac{9}{30}\right) * 100\% = 30\%$ while the third category is for moment estimator which is best with percentage $\left(\frac{8}{30}\right) * 100\% = 26.66\%$. The second result is the possibility of the extension the concept of comparison by taking another sets, but we see that the given results is complete.

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