6-Point Non-Stationary Subdivision Scheme by Hyperbolic B-Spline

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Abstract
This paper proposes the construction of binary 6-point non stationary subdivision scheme using hyperbolic B-spline basis function. Hyperbolic B-spline basis functions are more consistent and have immense propensity to produce curves which are proximate to the control polygon unlike non-stationary subdivision scheme by trigonometric B-spline basis function in computational mathematics. Using the concept of asymptotically equivalence, smoothness and convergence are analyzed. Parabolas, hyperbolic polynomials and hyperbolic splines are constructed using the proposed scheme.

Key words:
Binary Schemes, Approximating, Non-Stationary Subdivision Schemes Hyperbolic B-Spline Smoothness.

1. Introduction

Subdivision defines a curve or surface from an initial control mesh by recursive refinement. Starting with initial control points $P^0 = \left\{P_i^0|_{i\in R^d}\right\}_{i=-1}^{n+1}$, application of the subdivision rule Sn defines a new set of points $P_n = \left\{P_i^n\right\}_{i=-1}^{2^n k+1}$,which can be written as:

Pn= Sn....S0P0,n∈Z+

If Sn does not depend on n, subdivision schemes are stationary otherwise these are non stationary schemes. In interpolating subdivision schemes refinement of control points is obtained by assigning different values corresponding to the intermediate points with the help of linear combination of neighboring points. Approximating subdivision schemes do not preserve the original control points while original points are preserved by interpolating subdivision schemes exactly. Stationary subdivision schemes produce curves and surfaces but are devoid of the construction of conic sections and spiral bands. Non stationary schemes paved the way for the production of analytical shapes. Non stationary subdivision schemes are constructed with the aid of trigonometric Lagrange, trigonometric B-spline and hyperbolic B-spline functions. Non-stationary subdivision schemes by using Lagrange and Trigonometric B-spline functions generate only cubic polynomials and specific families of conics. Reproduction of ellipses and circles using tension parameter are the salient features of non stationary schemes [1-3]. However, these schemes were devoid of generation of hyperbolas and parabolas. So, in order to overcome this deficiency Siddiqi et. al. [4-7] present non-stationary subdivision schemes using hyperbolic Lagrange interpolation and hyperbolic B-spline. Consistency level of hyperbolic non-stationary subdivision schemes is higher than that of non-stationary subdivision schemes using trigonometric B-spline. Also in case of large parametric values, the resultant curves are more smooth and flexible. The proposed scheme by hyperbolic B-spline is able to generate parabolas/hyperbolas and excluding the zeros of sine hyperbolic function, is convergent for positive real values of parameter. No approximating non stationary subdivision scheme has been found using hyperbolic functions. Conti and Romani [8] produce a new family of interpolating 6-point non-stationary subdivision scheme by hyperbolic B-spline. Cubic exponential B-spline symbol generating functions helped the 6-point non-stationary scheme to develop conics. Cm-1 Limiting curves are produced by using uniform trigonometric B-spline basis function of order m-1 from non stationary schemes [9-11]. Moreover, the generalized forms generated by [9-11] are the competent producer of trigonometric polynomials, trigonometric splines and conics. Mustafa and Bari [12] proposes a new family of odd point ternary non-stationary subdivision schemes by using Lagrange identities. The resultant ellipses have a contrast with the virtual ellipses. Deviation from actual

Manuscript received April 5, 2020
Manuscript revised April 20, 2020

ellipses is reduced up to great extent in these schemes. The generalized symbols of uni-variate stationary and non-stationary subdivision schemes proposed by Asghar [13] with base of Lane Riesenfeld generate B-spline schemes, Hormann and Sabin family schemes. As non-stationary schemes produce spiral curves but odd-point non-stationary schemes with fewer initial control points that are presented by Ghaffar et. al. [14] have less propensity to detach from tangent and osculating plane.
In organization of the remaining work, Section-2, holds preliminaries and asymptotic equivalence while Section-3, describes the definition of hyperbolic B-spline. Convergence analysis with some properties of scheme are also the part of Section-4. In Section-5, graphical representation of proposed hyperbolic scheme is done.

2. Preliminaries

Given a set of control points P0 = {Pi0 ∈ R : i ∈ Z} at level 0, a binary scheme for designing curves generates recursively a new set of control points $P^{n+1} = \{P^n_{i+1} : i \in Z\}$ at $(n+1)^{th}$ level by a subdivision rule:

$$P^{n+1} = \sum_{j\in Z} a_n^{i-2j} P_i^n, i \in Z$$

The set of coefficients an := {αin: i ∈ Z} determines the subdivision rule at level n and is termed as the mask at nth level. If the mask an is independent of n, the subdivision scheme S n corresponding to the mask an is called stationary otherwise it is called non-stationary. In case of binary scheme if up to order m the limit-function has continuous derivatives, for any initial data then convergence exists. A subdivision scheme Sa with symbol $\alpha(z) = \left(\frac{1+z}{2}\right)^m b(z)$ is said to be Cm continuous if b(z) is convergent and c(z) is contractive. Where c(z) is a difference scheme of b(z).

3. Construction of hyperbolic B-spline scheme

Theorem-2.1: Two binary schemes S_a^n and S_b^n are asymptotically equivalent if

$$\sum_{n=1}^{\infty} \left\|S_a^n - S_b^n\right\|_\infty < \infty$$

Where

$$\left\|S_a^n\right\|_\infty = \max\left\{\sum_{i\in Z}\left|a_{2i}^n\right|, \sum_{i\in Z}\left|a_{2i+1}\right|\right\}$$

The proof is exactly similar to the proof given in (Theorem 2.2).

***Theorem-2.2**: The binary non-stationary six point scheme S_α^n is C^1.*

Proof: Let S n and Sα be two asymptotically equivalent subdivision schemes having finite mask of the same support. Suppose S n is non-stationary and Sα is stationary. As the binary 6-point stationary scheme Sα is C1 continuous, so in order to prove the proposed non-stationary scheme to be C1 continuous [7], it is sufficient to show that the scheme Sαn corresponding to αn(z) is C1 continuous. Therefore it is necessary to show

$$\sum_{n=1}^{\infty} 2^n \left\|S_\alpha^n - S_\alpha\right\| < \infty$$

Where

$$\left\|S_\alpha^n - S_\alpha\right\| = \max\left\{\sum_{j\in Z}\left|b_{i+2j}^n - b_{i+2j}\right| : i = 0,1\right\}$$

$$\left\|S_\alpha^n - S_\alpha\right\| = \max\left\{\sum_{j\in Z}\left|b_{2j}^n - b_{2j}\right|, \sum_{j\in Z}\left|b_{1+2j}^n - b_{1+2j}\right|\right\}$$

$$2\sum_{j\in Z}\left|b_{2j}^n - b_{2j}\right| = 10\left|\beta_0 - \frac{243}{122880}\right| + 6\left|\beta_1 - \frac{15349}{122880}\right| + 2\left|\beta_2 - \frac{63858}{122880}\right| + 2\left|\beta_3 - \frac{40314}{122880}\right| + 6\left|\beta_4 - \frac{3119}{122880}\right| + 10\left|\beta_5 - \frac{1}{122880}\right|.$$

Moreover,

$$2\sum_{j\in Z}\left|b_{1+2j}^n - b_{1+2j}\right| = 8\left|\beta_0 - \frac{243}{122880}\right| + 4\left|\beta_1 - \frac{15349}{122880}\right| + 4\left|\beta_3 - \frac{40314}{122880}\right| + 8\left|\beta_4 - \frac{3119}{122880}\right| + 10\left|\beta_5 - \frac{1}{122880}\right|.$$

By using Lemma 3.2 we have,

$$\sum_{n=0}^{\infty} 2^n \left|\beta_5 - \frac{1}{122880}\right| < \infty$$

$$\sum_{n=0}^{\infty} 2^n \left|\beta_4 - \frac{3119}{122880}\right| < \infty$$

$$\sum_{n=0}^{\infty} 2^n \left|\beta_3 - \frac{40314}{122880}\right| < \infty$$

$$\sum_{n=0}^{\infty} 2^n \left|\beta_2 - \frac{63858}{122880}\right| < \infty$$

$$\sum_{n=0}^{\infty} 2^n \left|\beta_1 - \frac{15349}{122880}\right| < \infty$$

$$\sum_{n=0}^{\infty} 2^n \left|\beta_0 - \frac{243}{122880}\right| < \infty$$

we can write it as:

$$\sum_{n=0}^{\infty} 2^n \left\| S_\alpha^n - S_\alpha \right\| < \infty$$

Hence S_α^n is C^1 continuous.

4. Hyperbolic B-spline

Uniform hyperbolic B-spline is denoted by T_i^n of order n knot sequence

$X := \{t_i := i\alpha : i = 0,1,....m + n\}$ are defined by the following Recurrence relation

$$T_i^1(t;\alpha) = \begin{cases} 1 & \text{if} \quad t \in (t_i, t_{i+1}) \\ 0 & \text{Otherwise} \end{cases}$$

For n >1

$$T_i^n(t;\alpha) = \frac{S(t-t_i)}{S(t_{n+i-1}-t_i)} T_i^{n-1}(t;\alpha) + \frac{S(t_{i+n}-t)}{S(t_{n+i-1}-t_{i+1})} T_{i+1}^{n+1}(t;\alpha)$$

Where
S(t) = sin h(t).

The mesh size is α and $T_i^n = T_0^n(t - i\alpha : \alpha)$. The support of hyperbolic B-spline
Tin(t;α) is [ti, ti+n] and in the interior of its support it remains positive. Hence each uniform hyperbolic B-spline f(t) of order "n" knot sequence X has a unique representation

$$f(t) = \sum_i P_i T_i^n(t;\alpha)$$

where P_i are real numbers.

4.1 Definition of Binary 6-Point Non-Stationary Subdivision Scheme by Hyperbolic B-Spline

Now mask of binary 6-point approximating non-stationary subdivision scheme by hyperbolic B-spline, structure and its Laurant polynomial is presented here.

Mask of proposed scheme

To obtain the mask

$\beta_i^n(\alpha) = \beta_i^n, i = 0,1,2,......n-1$

the following "Recreance relation" for any value of n is used.

$$\beta_i^n(\alpha) = T_0^n\left(\left(n-i-1\right)\frac{\alpha}{2^n} + \frac{\alpha}{2^{n+2}} : \frac{\alpha}{2^n}\right)$$

where

$$T_0^n\left(t; \frac{\alpha}{2^n}\right)$$

with mesh size,

$$\left(\frac{\alpha}{2^n}\right)$$

is a trigonometric B-spline basis function of order n-1.

The mask of binary 6-point scheme for any "n" can be obtained as:

$$\beta_i^n = T_0^6\left(\left(5-i\right)\frac{\alpha}{2^n} + \frac{\alpha}{2^{n+2}} : \frac{\alpha}{2^n}\right) \text{ for } i = 0,1,2,3,4,5$$

Where $T_0^6\left(t; \frac{\alpha}{2^n}\right)$ is the trigonometric B-spline basis function having mesh size $\frac{\alpha}{2^n}$.

Laurant polynomial

$$\alpha^n(z) = \{\beta_5^n z^{-6} + \beta_0^n z^{-5} + \beta_4^n z^{-4} + \beta_1^n z^{-3} + \beta_3^n z^{-2} + \beta_2^n z^{-1} + \beta_2^n z^0 + \beta_3^n z^1 + \beta_1^n z^2 + \beta_4^n z^3 + \beta_0^n z^4 + \beta_5^n z^5\}$$

$$\beta_0^n = \frac{S^5\left(\frac{3\alpha}{2^n}\right)}{S\left(\frac{\alpha}{2^n}\right)S\left(\frac{2\alpha}{2^n}\right)S\left(\frac{3\alpha}{2^n}\right)S\left(\frac{4\alpha}{2^n}\right)S\left(\frac{5\alpha}{2^n}\right)}$$

$$S(t) = \sinh(t)$$

$$\begin{aligned}\beta_1^n = \frac{1}{S\left(\frac{\alpha}{2^n}\right)S\left(\frac{2\alpha}{2^n}\right)S\left(\frac{3\alpha}{2^n}\right)S\left(\frac{4\alpha}{2^n}\right)S\left(\frac{5\alpha}{2^n}\right)} & \left\{S\left(\frac{17\alpha}{2^{n+2}}\right)S^4\left(\frac{3\alpha}{2^{n+2}}\right)\right. \\ & + S^3\left(\frac{3\alpha}{2^{n+2}}\right)S\left(\frac{7\alpha}{2^{n+2}}\right)S\left(\frac{13\alpha}{2^{n+2}}\right) + S\left(\frac{9\alpha}{2^{n+2}}\right)S^2\left(\frac{3\alpha}{2^{n+2}}\right)S^2\left(\frac{7\alpha}{2^{n+2}}\right) \\ & \left. + S^3\left(\frac{7\alpha}{2^{n+2}}\right)S\left(\frac{5\alpha}{2^{n+2}}\right)S\left(\frac{3\alpha}{2^{n+2}}\right) + S^4\left(\frac{7\alpha}{2^{n+2}}\right)S\left(\frac{\alpha}{2^{n+2}}\right)\right\}\end{aligned}$$

$$\begin{aligned}\beta_2^n = \frac{1}{S\left(\frac{\alpha}{2^n}\right)S\left(\frac{2\alpha}{2^n}\right)S\left(\frac{3\alpha}{2^n}\right)S\left(\frac{4\alpha}{2^n}\right)S\left(\frac{5\alpha}{2^n}\right)} & \left\{S^2\left(\frac{13\alpha}{2^{n+2}}\right)S^3\left(\frac{3\alpha}{2^{n+2}}\right)\right. \\ & + S\left(\frac{13\alpha}{2^{n+2}}\right)S\left(\frac{3\alpha}{2^{n+2}}\right)S\left(\frac{5\alpha}{2^{n+2}}\right)S^2\left(\frac{7\alpha}{2^{n+2}}\right) + S\left(\frac{13\alpha}{2^{n+2}}\right)S^2\left(\frac{7\alpha}{2^{n+2}}\right) \\ & S\left(\frac{5\alpha}{2^{n+2}}\right)S\left(\frac{3\alpha}{2^{n+2}}\right) + S\left(\frac{13\alpha}{2^{n+2}}\right)S^3\left(\frac{7\alpha}{2^{n+2}}\right)S\left(\frac{\alpha}{2^{n+2}}\right) + S\left(\frac{11\alpha}{2^{n+2}}\right) \\ & S^2\left(\frac{9\alpha}{2^{n+2}}\right)S^2\left(\frac{3\alpha}{2^{n+2}}\right)S\left(\frac{11\alpha}{2^{n+2}}\right)S\left(\frac{7\alpha}{2^{n+2}}\right)S\left(\frac{5\alpha}{2^{n+2}}\right)S\left(\frac{9\alpha}{2^{n+2}}\right) \\ & S\left(\frac{3\alpha}{2^{n+2}}\right) + S\left(\frac{9\alpha}{2^{n+2}}\right)S\left(\frac{\alpha}{2^{n+2}}\right)S^2\left(\frac{7\alpha}{2^{n+2}}\right)S\left(\frac{11\alpha}{2^{n+2}}\right) + S^2\left(\frac{5\alpha}{2^{n+2}}\right) \\ & S\left(\frac{3\alpha}{2^{n+2}}\right)S\left(\frac{11\alpha}{2^{n+2}}\right) + S\left(\frac{5\alpha}{2^{n+2}}\right)S\left(\frac{7\alpha}{2^{n+2}}\right)S\left(\frac{\alpha}{2^{n+2}}\right)S^2\left(\frac{11\alpha}{2^{n+2}}\right) \\ & \left. + S^3\left(\frac{11\alpha}{2^{n+2}}\right)S^2\left(\frac{\alpha}{2^{n+2}}\right)\right\}\end{aligned}$$

$$\beta_3^n=\frac{1}{S\left(\frac{\alpha}{2^n}\right)S\left(\frac{2\alpha}{2^n}\right)S\left(\frac{3\alpha}{2^n}\right)S\left(\frac{4\alpha}{2^n}\right)S\left(\frac{5\alpha}{2^n}\right)}\left\{S^3\left(\frac{9\alpha}{2^{n+2}}\right)S^2\left(\frac{3\alpha}{2^{n+2}}\right)\right.$$
$$+S^2\left(\frac{9\alpha}{2^{n+2}}\right)S\left(\frac{5\alpha}{2^{n+2}}\right)S\left(\frac{3\alpha}{2^{n+2}}\right)S\left(\frac{7\alpha}{2^{n+2}}\right)+S\left(\frac{9\alpha}{2^{n+2}}\right)S\left(\frac{3\alpha}{2^{n+2}}\right)$$
$$S\left(\frac{11\alpha}{2^{n+2}}\right)S^2\left(\frac{5\alpha}{2^{n+2}}\right)+S^2\left(\frac{9\alpha}{2^{n+2}}\right)S\left(\frac{\alpha}{2^{n+2}}\right)S^2\left(\frac{7\alpha}{2^{n+2}}\right)+S\left(\frac{9\alpha}{2^{n+2}}\right)$$
$$S\left(\frac{5\alpha}{2^{n+2}}\right)S\left(\frac{\alpha}{2^{n+2}}\right)S\left(\frac{7\alpha}{2^{n+2}}\right)S\left(\frac{11\alpha}{2^{n+2}}\right)+S\left(\frac{9\alpha}{2^{n+2}}\right)S^2\left(\frac{\alpha}{2^{n+2}}\right)$$
$$S^2\left(\frac{11\alpha}{2^{n+2}}\right)+S\left(\frac{15\alpha}{2^{n+2}}\right)S^3\left(\frac{5\alpha}{2^{n+2}}\right)S\left(\frac{3\alpha}{2^{n+2}}\right)+S\left(\frac{15\alpha}{2^{n+2}}\right)S^2\left(\frac{5\alpha}{2^{n+2}}\right)$$
$$S\left(\frac{7\alpha}{2^{n+2}}\right)S\left(\frac{\alpha}{2^{n+2}}\right)+S\left(\frac{15\alpha}{2^{n+2}}\right)S\left(\frac{11\alpha}{2^{n+2}}\right)S^2\left(\frac{\alpha}{2^{n+2}}\right)S\left(\frac{5\alpha}{2^{n+2}}\right)$$
$$\left.+S^2\left(\frac{15\alpha}{2^{n+2}}\right)S^3\left(\frac{\alpha}{2^{n+2}}\right)\right\}$$

$$\beta_4^n=\frac{1}{S\left(\frac{\alpha}{2^n}\right)S\left(\frac{2\alpha}{2^n}\right)S\left(\frac{3\alpha}{2^n}\right)S\left(\frac{4\alpha}{2^n}\right)S\left(\frac{5\alpha}{2^n}\right)}\left\{S^4\left(\frac{5\alpha}{2^{n+2}}\right)S\left(\frac{3\alpha}{2^{n+2}}\right)\right.$$
$$+S^3\left(\frac{5\alpha}{2^{n+2}}\right)S\left(\frac{\alpha}{2^{n+2}}\right)S\left(\frac{7\alpha}{2^{n+2}}\right)+S\left(\frac{19\alpha}{2^{n+2}}\right)S\left(\frac{\alpha}{2^{n+2}}\right)+S\left(\frac{11\alpha}{2^{n+2}}\right)$$
$$\left.S^2\left(\frac{5\alpha}{2^{n+2}}\right)S^2\left(\frac{\alpha}{2^{n+2}}\right)+S\left(\frac{15\alpha}{2^{n+2}}\right)S^3\left(\frac{\alpha}{2^{n+2}}\right)S\left(\frac{5\alpha}{2^{n+2}}\right)\right\}$$

$$\beta_5^n=\frac{S^5\left(\frac{\alpha}{2^{n+2}}\right)}{S\left(\frac{\alpha}{2^n}\right)S\left(\frac{2\alpha}{2^n}\right)S\left(\frac{3\alpha}{2^n}\right)S\left(\frac{4\alpha}{2^n}\right)S\left(\frac{5\alpha}{2^n}\right)}$$

The mask of the proposed scheme converges to the mask of stationary scheme. So

$$P_{2i}^{n+i}=\frac{243}{122880}P_{i-2}^n+\frac{15349}{122880}P_{i-1}^n+\frac{63858}{122880}P_i^n+\frac{40314}{122880}P_{i+1}^n+\frac{3119}{122880}P_{i+2}^n+\frac{1}{122880}P_{i+3}^n$$

$$P_{2i+1}^{n+i}=\frac{1}{122880}P_{i-2}^n+\frac{3119}{122880}P_{i-1}^n+\frac{40314}{122880}P_i^n+\frac{63858}{122880}P_{i+1}^n+\frac{15349}{122880}P_{i+2}^n+\frac{243}{122880}P_{i+3}^n$$

Remark-3.3: Sum of the weights of the proposed scheme tends to one as $n\to\infty$. Let

$\beta_0+\beta_1+\beta_2+\beta_3+\beta_4+\beta_5=\lambda^n$.

Following inequalities are used,

$$\frac{\sinh(a)}{\sinh(b)}\le\frac{a}{b},0<a\le b,a,b\in R^+-\{m'\pi\}$$
$$\frac{\theta}{\sinh(\theta)}<\frac{t}{\sinh(t)},0<\theta\le t,\theta,t\in R^+-\{m'\pi\}$$
$$\cosh(x)>\frac{\sinh(x)}{x},x\in R^+-\{m'\pi\}$$

$$\lambda^n=\frac{1}{\left(\frac{\alpha}{2^n}\right)\cosh\left(\frac{\alpha}{2^n}\right)\left(\frac{2\alpha}{2^n}\right)\cosh\left(\frac{2\alpha}{2^n}\right)\left(\frac{3\alpha}{2^n}\right)\operatorname{sech}\left(\frac{3\alpha}{2^n}\right)\left(\frac{4\alpha}{2^n}\right)\cosh\left(\frac{4\alpha}{2^n}\right)\left(\frac{5\alpha}{2^n}\right)\cosh\left(\frac{5\alpha}{2^n}\right)}$$
$$\left\{\left(\frac{243\alpha^5}{2^{5n+10}}\right)+\left(\frac{17\alpha}{2^{n+2}}\right)\left(\frac{81\alpha^4}{2^{4n+8}}\right)+\left(\frac{27\alpha^3}{2^{3n+6}}\right)\left(\frac{7\alpha}{2^{n+2}}\right)\left(\frac{13\alpha}{2^{n+2}}\right)+\left(\frac{9\alpha}{2^{n+2}}\right)\left(\frac{9\alpha^2}{2^{2n+4}}\right)\left(\frac{49\alpha^2}{2^{2n+4}}\right)\right.$$
$$+\left(\frac{343\alpha^3}{2^{3n+6}}\right)\left(\frac{5\alpha}{2^{n+2}}\right)\left(\frac{3\alpha}{2^{n+2}}\right)+\left(\frac{2401\alpha^4}{2^{4n+8}}\right)\left(\frac{\alpha}{2^{n+2}}\right)+\left(\frac{169\alpha^2}{2^{2n+4}}\right)\left(\frac{27\alpha^3}{2^{3n+6}}\right)+\left(\frac{13\alpha}{2^{n+2}}\right)\left(\frac{9\alpha}{2^{n+2}}\right)$$
$$\left(\frac{7\alpha}{2^{n+2}}\right)\left(\frac{9\alpha^2}{2^{2n+2}}\right)+\left(\frac{13\alpha}{2^{n+2}}\right)\left(\frac{49\alpha^2}{2^{2n+4}}\right)\left(\frac{5\alpha}{2^{n+2}}\right)\left(\frac{3\alpha}{2^{n+4}}\right)+\left(\frac{13\alpha}{2^{n+2}}\right)\left(\frac{343\alpha^3}{2^{3n+6}}\right)\left(\frac{\alpha}{2^{n+2}}\right)+\left(\frac{11\alpha}{2^{n+2}}\right)$$
$$\left(\frac{81\alpha^2}{2^{2n+4}}\right)\left(\frac{9\alpha^2}{2^{2n+4}}\right)+\left(\frac{11\alpha}{2^{n+2}}\right)\left(\frac{7\alpha}{2^{n+4}}\right)\left(\frac{5\alpha}{2^{n+2}}\right)\left(\frac{9\alpha}{2^{n+2}}\right)\left(\frac{3\alpha}{2^{n+2}}\right)+\left(\frac{9\alpha}{2^{n+2}}\right)\left(\frac{\alpha}{2^{n+2}}\right)\left(\frac{49\alpha^2}{2^{2n+4}}\right)$$
$$\left(\frac{11\alpha}{2^{n+4}}\right)+\left(\frac{25\alpha^2}{2^{2n+4}}\right)\left(\frac{3\alpha}{2^{n+2}}\right)\left(\frac{121\alpha^2}{2^{2n+4}}\right)+\left(\frac{5\alpha}{2^{n+2}}\right)\left(\frac{7\alpha}{2^{n+2}}\right)\left(\frac{\alpha}{2^{n+2}}\right)\left(\frac{121\alpha^2}{2^{2n+4}}\right)+\left(\frac{133\alpha^3}{2^{3n+4}}\right)$$
$$\left(\frac{\alpha^2}{2^{2n+4}}\right)+\left(\frac{729\alpha^3}{2^{3n+4}}\right)\left(\frac{9\alpha^2}{2^{2n+4}}\right)+\left(\frac{81\alpha^2}{2^{2n+4}}\right)+\left(\frac{5\alpha}{2^{n+2}}\right)\left(\frac{3\alpha}{2^{n+2}}\right)\left(\frac{7\alpha}{2^{n+2}}\right)+\left(\frac{81\alpha^2}{2^{2n+4}}\right)\left(\frac{\alpha}{2^{n+2}}\right)$$
$$\left(\frac{49\alpha^2}{2^{2n+4}}\right)+\left(\frac{27\alpha}{2^{n+2}}\right)\left(\frac{11\alpha}{2^{n+2}}\right)\left(\frac{25\alpha^2}{2^{2n++4}}\right)+\left(\frac{9\alpha}{2^{n+2}}\right)\left(\frac{5\alpha}{2^{n+2}}\right)\left(\frac{\alpha}{2^{n+2}}\right)\left(\frac{7\alpha}{2^{n+2}}\right)\left(\frac{11\alpha}{2^{n+2}}\right)+\left(\frac{9\alpha}{2^{n+2}}\right)$$
$$\left(\frac{\alpha^2}{2^{2n+4}}\right)\left(\frac{121\alpha^2}{2^{2n+4}}\right)+\left(\frac{15\alpha}{2^{n+2}}\right)\left(\frac{125\alpha^3}{2^{3n+6}}\right)\left(\frac{3\alpha}{2^{n+2}}\right)+\left(\frac{15\alpha}{2^{n+2}}\right)\left(\frac{25\alpha^2}{2^{2n+4}}\right)\left(\frac{7\alpha}{2^{n+2}}\right)+\left(\frac{3\alpha}{2^{n+2}}\right)$$
$$\left(\frac{15\alpha}{2^{n+2}}\right)\left(\frac{11\alpha}{2^{n+2}}\right)\left(\frac{\alpha^2}{2^{2n+4}}\right)\left(\frac{5\alpha}{2^{n+6}}\right)+\left(\frac{225\alpha^2}{2^{2n+4}}\right)\left(\frac{\alpha^3}{2^{3n+6}}\right)+\left(\frac{625\alpha^4}{2^{4n+8}}\right)\left(\frac{3\alpha}{2^{n+2}}\right)+\left(\frac{125\alpha^3}{2^{3n+6}}\right)$$
$$\left.\left(\frac{\alpha}{2^{n+2}}\right)\left(\frac{7\alpha}{2^{n+2}}\right)+\left(\frac{19\alpha^2}{2^{n+2}}\right)\left(\frac{\alpha^4}{2^{4n+5}}\right)+\left(\frac{11\alpha}{2^{n+2}}\right)\left(\frac{25\alpha^2}{2^{2n+4}}\right)\left(\frac{\alpha^2}{2^{2n+4}}\right)+\left(\frac{15\alpha^2}{2^{2n+4}}\right)\left(\frac{\alpha^3}{2^{3n+6}}\right)+\left(\frac{\alpha^5}{2^{5n+10}}\right)\right\}$$

By using the above mentioned inequalities, we have

$$\lambda^n\ge\frac{\alpha^5 32^{-n}}{\left(\frac{\alpha}{2^n}\right)\left(\frac{2\alpha}{2^n}\right)\left(\frac{3\alpha}{2^n}\right)\left(\frac{4\alpha}{2^n}\right)\left(\frac{5\alpha}{2^n}\right)\cosh^5\left(\frac{5\alpha}{2^n}\right)1024}(13191+28349+36708+40080+3285)$$

$$\lambda^n\ge\frac{1}{\left(\frac{\alpha}{2^n}\right)\left(\frac{2\alpha}{2^n}\right)\left(\frac{3\alpha}{2^n}\right)\left(\frac{4\alpha}{2^n}\right)\left(\frac{5\alpha}{2^n}\right)\cosh^5\left(\frac{5\alpha}{2^n}\right)}\left(\frac{121622\alpha^5}{1024(32^n)}\right)$$

This implies $\lambda n\ge 1$.When $n\to\infty$.

Again, using above inequalities, we get

$$\lambda^n\le\frac{\alpha^5 32^{-n}}{\left(\frac{\alpha}{2^n}\right)\left(\frac{2\alpha}{2^n}\right)\left(\frac{3\alpha}{2^n}\right)\left(\frac{4\alpha}{2^n}\right)\left(\frac{5\alpha}{2^n}\right)}\operatorname{sech}\left(\frac{\alpha}{2^n}\right)\operatorname{sech}\left(\frac{2\alpha}{2^n}\right)\operatorname{sech}\left(\frac{3\alpha}{2^n}\right)\operatorname{sech}\left(\frac{4\alpha}{2^n}\right)\operatorname{sech}\left(\frac{5\alpha}{2^n}\right)$$

$$\lambda^n\le\frac{1}{\left(\frac{\alpha}{2^n}\right)\left(\frac{2\alpha}{2^n}\right)\left(\frac{3\alpha}{2^n}\right)\left(\frac{4\alpha}{2^n}\right)\left(\frac{5\alpha}{2^n}\right)}\operatorname{sech}^5\left(\frac{5\alpha}{2^n}\right)\left(\frac{121622\alpha^5}{1024(32^n)}\right)$$

This implies $n\le 1$. When $n\to\infty$, so $\lambda n = 1$ for $n\to\infty$.

5. Convergence Analysis

Convergence and smoothness of the proposed scheme are attained through the method of asymptotic equivalence. We denote the stationary scheme by $S\alpha$ and proposed non-stationary scheme by $S_\alpha{}^n$,

where

$$\alpha^n=\sum_{i\in Z}\alpha_i^n z^i$$

Some estimations of β_i^n, i = 0,1,...5 are needed in the following Lemmas.

Lemma-1: Since the proposed scheme is:

$$P_{2i}^{n+i} = \beta_0^n P_{i-2}^n + \beta_1^n P_{i-1}^n + \beta_2^n P_i^n + \beta_3^n P_{i+1}^n + \beta_4^n P_{i+2}^n + \beta_5^n P_{i+3}^n$$
$$P_{2i+1}^{n+i} = \beta_5^1 P_{i-2}^n + \beta_4^n P_{i-1}^n + \beta_3^n P_i^n + \beta_2^n P_{i+1}^n + \beta_1^n P_{i+2}^n + \beta_0^n P_{i+3}^n$$

and the stationary scheme is denoted by:

$$P_{2i}^{n+i} = \frac{243}{122880} P_{i-2}^n + \frac{15349}{122880} P_{i-1}^n + \frac{63858}{122880} P_i^n + \frac{40314}{122880} P_{i+1}^n + \frac{3119}{122880} P_{i+2}^n + \frac{1}{122880} P_{i+3}^n$$

$$P_{2i+1}^{n+i} = \frac{1}{122880} P_{i-2}^n + \frac{3119}{122880} P_{i-1}^n + \frac{40314}{122880} P_i^n + \frac{63858}{122880} P_{i+1}^n + \frac{15349}{122880} P_{i+2}^n + \frac{243}{122880} P_{i+3}^n$$

For

$$m' > 0, \alpha \in R^+ - \{m'\pi\}$$
$$\frac{243}{122880} > \beta_0^n > \frac{243}{122880}\frac{1}{\cosh^5\left(\frac{5\alpha}{2^n}\right)}, \frac{15349}{122880} > \beta_1^n > \frac{1}{122880\cosh^5\left(\frac{5\alpha}{2^n}\right)},$$
$$\frac{63858}{122880} > \beta_2^n > \frac{63858}{122880}\frac{1}{\cosh^5\left(\frac{5\alpha}{2^n}\right)}, \frac{40314}{122880} > \beta_3^n > \frac{40314}{122880\cosh^5\left(\frac{5\alpha}{2^n}\right)}$$
$$\frac{3119}{122880} > \beta_4^n > \frac{243}{122880}\frac{1}{\cosh^5\left(\frac{5\alpha}{2^n}\right)}, \frac{1}{122880} > \beta_5^n > \frac{1}{122880\cosh^5\left(\frac{5\alpha}{2^n}\right)}$$

Proof. By using the following inequalities, Lemmas can be proved

$$\frac{\sinh(a)}{\sinh(b)} \le \frac{a}{b}, 0 < a \le b, a, b \in R^+ - \{m'\pi\},$$
$$\frac{\theta}{\sinh(\theta)} < \frac{t}{\sinh(t)}, 0 < \theta \le t, \theta, t \in R^+ - \{m'\pi\},$$
$$\cosh(x) > \frac{\sinh(x)}{x}, x \in R^+ - \{m'\pi\}.$$

The proof of first inequality is provided and similarly the remaining ones can be solved. Since

$$\beta_0^n = \frac{S^5\left(\frac{3\alpha}{2^{n+2}}\right)}{S\left(\frac{\alpha}{2^n}\right)S\left(\frac{2\alpha}{2^n}\right)S\left(\frac{3\alpha}{2^n}\right)S\left(\frac{4\alpha}{2^n}\right)S\left(\frac{5\alpha}{2^n}\right)}$$
$$\beta_0^n < \frac{\left(\frac{3\alpha}{2^{n+2}}\right)^5}{\left(\frac{\alpha}{2^n}\right)\left(\frac{2\alpha}{2^n}\right)\left(\frac{3\alpha}{2^n}\right)\left(\frac{4\alpha}{2^n}\right)\left(\frac{5\alpha}{2^n}\right)}$$
$$\beta_0^n < \frac{243}{122880}$$

and

$$\beta_0^n > \frac{\left(\frac{3\alpha}{2^{n+2}}\right)^5}{\left(\frac{\alpha}{2^n}\right)\left(\frac{2\alpha}{2^n}\right)\left(\frac{3\alpha}{2^n}\right)\left(\frac{4\alpha}{2^n}\right)\left(\frac{5\alpha}{2^n}\right)}\operatorname{sech}\left(\frac{\alpha}{2^n}\right)$$
$$\operatorname{sech}\left(\frac{2\alpha}{2^n}\right)\operatorname{sech}\left(\frac{3\alpha}{2^n}\right)\operatorname{sech}\left(\frac{4\alpha}{2^n}\right)\operatorname{sech}\left(\frac{5\alpha}{2^n}\right)$$
$$\beta_0^n > \frac{243}{122880\cosh^5\left(\frac{5\alpha}{2^n}\right)}$$

This proof was missing so, please change it as I used the image

Proof of 1st inequality is missing

Hence first inequality is proved.

Lemma-2: For the constants di, i = 0,1,2,3,4,5 ,independent of n, the following inequalities hold.

$$\left|\beta_0^n - \frac{243}{122880}\right| < \frac{d_0}{2^n}$$
$$\left|\beta_1^n - \frac{15349}{122880}\right| < \frac{d_1}{2^n}$$
$$\left|\beta_2^n - \frac{63858}{122880}\right| < \frac{d_2}{2^n}$$
$$\left|\beta_3^n - \frac{40314}{122880}\right| < \frac{d_3}{2^n}$$
$$\left|\beta_4^n - \frac{3119}{122880}\right| < \frac{d_4}{2^n}$$
$$\left|\beta_5^n - \frac{1}{122880}\right| < \frac{d_5}{2^n}$$

Proof: Now we prove the first inequality by using the above mentioned lemma. We take

$$\beta_0^n > \frac{243}{122880\cosh^5\left(\frac{5\alpha}{2^n}\right)}$$
$$\beta_0^n - \frac{243}{122880} > -\frac{243}{122880}\left(1 - \frac{1}{\cosh^5\left(\frac{5\alpha}{2^n}\right)}\right),$$

Using the Maclaurin series [see Computer Algebra Systems],

$$\cosh\alpha = 1 + \frac{\alpha^2}{2} + \frac{\alpha^4}{4} + \frac{\alpha^6}{6}.....$$

Also

$$\cosh^5\left(\frac{5\alpha}{2^n}\right) \le \cosh^5(5\alpha)$$
$$-\beta_0^n + \frac{243}{122880} < \frac{2373046875\alpha^{10}}{(125829120)\cosh^5(5\alpha)2^n}$$

$$\left| -\beta_0^n + \frac{243}{122880} \right| < \frac{d_0}{2^n}$$

so,

$$d_0 = \frac{2373046875\alpha^{10}}{(125829120)\cosh^5(5\alpha)}$$

Proof of lemma-2 is also incorrect and incomplete, please type the whole proof again. Hence inequality (1) is proved. By adopting same method the remaining inequalities can be proved.

Lemma-3: The symbol αn(z) corresponding to the nth level of the non stationary scheme S n is presented in the form

$$\alpha^n(z) = \left(\frac{1+z}{2z}\right) b^n(z)$$

where,

$$\begin{aligned} b^n(z) = 2\big(&\beta_0^n z^{-5} + (\beta_0^n - \beta_5^n)z^{-4} + (\beta_4^n + \beta_5^n - \beta_0^n)z^{-3} + (\beta_0^n + \beta_1^n - \beta_4^n - \beta_5^n)z^{-2} \\ &+ (\beta_3^n + \beta_4^n + \beta_5^n - \beta_1^n - \beta_0^n)z^{-1} + (\beta_0^n + \beta_1^n + \beta_2^n - \beta_3^n - \beta_4^n - \beta_5^n)z^0 \\ &+ (\beta_3^n + \beta_4^n + \beta_5^n - \beta_1^n - \beta_0^n)z^1 + (\beta_0^n + \beta_1^n - \beta_4^n - \beta_5^n)z^2 \\ &+ (\beta_4^n + \beta_5^n - \beta_0^n)Z^3 + (\beta_0^n - \beta_5^n)z^4 + \beta_5^n z^5\big) \end{aligned}$$

Now we will prove

$$\begin{aligned} b^n(z) = 2\big(&\beta_0^n z^{-5} + (\beta_0^n - \beta_5^n)z^{-4} + (\beta_4^n + \beta_5^n - \beta_0^n)z^{-3} + (\beta_0^n + \beta_1^n - \beta_4^n - \beta_5^n)z^{-2} \\ &+ (\beta_3^n + \beta_4^n + \beta_5^n - \beta_1^n - \beta_0^n)z^{-1} + (\beta_0^n + \beta_1^n + \beta_2^n - \beta_3^n - \beta_4^n - \beta_5^n)z^0 \\ &+ (\beta_3^n + \beta_4^n + \beta_5^n - \beta_1^n - \beta_0^n)z^1 + (\beta_0^n + \beta_1^n - \beta_4^n - \beta_5^n)z^2 \\ &+ (\beta_4^n + \beta_5^n - \beta_0^n)Z^3 + (\beta_0^n - \beta_5^n)z^4 + \beta_5^n z^5\big) \end{aligned}$$

Proof: Since,

$$\begin{aligned} \alpha^n(z) = \big(&\beta_5^n z^{-6} + \beta_0^n z^{-5} + \beta_4^n z^{-4} + \beta_1^n z^{-3} + \beta_3^n z^{-2} + \beta_2^n z^{-1} \\ &+ \beta_2^n z^0 + \beta_3^n z^1 + \beta_1^n z^2 + \beta_4^n z^3 + \beta_0^n z^4 + \beta_5^n z^5\big) \end{aligned}$$

so "Affine Invariance" property,

$\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 = 1$,

helps to get the required result[see Remark 3.3].

Lemma-4: The stationary scheme Sα corresponding to the symbol is c1.

$$\alpha(z) = \left(\frac{1+z}{2}\right) g(z)$$

Where,

$$\begin{aligned} \alpha(z) = \frac{1}{122880}\big(&1 + 243z^1 + 3119z^2 + 15349z^3 + 40314z^4 + 63858z^5 \\ &+ 63858z^6 + 40314z^7 + 15349z^8 + 3119z^9 + 243z^{10} + z^{11}\big\} \end{aligned}$$

Proof: To check the "continuity" of the scheme {Sα} we know that if w(z) is contractive and g(z) is convergent, α(z) will be C1 continuous. Where,

$$\alpha(z) = \left(\frac{1+z}{2}\right)^m g(z)$$

$$\alpha(z) = \left(\frac{1+z}{2}\right)^1 \frac{1}{61440}\{1,242,2877,12472,27842,36016,27842,12472,2877,242,1\}$$

$$g(z) = \frac{1}{61440}\{1,242,2877,12472,27842,36016,27842,12472,2877,242,1\}$$

$$g(z) = \left(\frac{1+z}{2}\right)^2 w(z)$$

$$w(z) = \frac{1}{30720}\{1,241.2636.9836,1800,18010,9832,2640,237,5,-4\}$$

The norm of the scheme $w(z)$ is $\{S_w'\}$

$$\left\|S_w'\right\|_\infty = \max\left\{\sum_{j\in z}\left|w_{2j}^w\right|, \sum_{j\in z}\left|w_{1+2j}^n\right|\right\}$$

$$\left\|S_w'\right\|_\infty = \max\left\{\frac{30732}{61440}, \frac{30716}{61440}\right\} < 1$$

Hence α(z) is C^1 continuous.

Properties of Scheme
Basis Limit Function

The limit function of the data is the basis limit function of the proposed scheme.

$$P_i^0 = \begin{cases} 1 & \text{if} \quad i = 0 \\ 0 & \text{Otherwise} \end{cases}$$

Lemma-5: Symmetry of the basic limit function holds about Y-axis.

Proof: Let B be the basic limit function and define

$$G_n = \left\{\frac{i}{2^n} : i \in Z\right\}$$

also B must satisfy

$$B\left(\frac{i}{2^n}\right) = P_i^n, i \in Z$$

By using induction method on n we can prove the symmetry of B about Y-axis. It is easily seen that

$$B(i) = B(-i), i \in Z$$

$$B\left(\frac{i}{2^n}\right) = B\left(1\frac{i}{2^n}\right), i \in Z, n = 0$$

Now if we assume

$$B\left(\frac{2i+1}{2^{n+1}}\right) = B\left(-\frac{2i+1}{2^n+1}\right), i \in Z$$

then

$$P_{2i+1}^{n+1} = P_{-2i-1}^{n+1}, i \in Z$$

$$B\left(\frac{2i+1}{2^{n+1}}\right) = P_{2i+1}^{n+1}$$

$$B\left(\frac{2i+1}{2^{n+1}}\right) = P_{2i+1}^{n+1}$$

$$P_{2i+1}^{n+1} = \beta_0^n P_{i-2}^n + \beta_1^n P_{i-1}^n + \beta_2^n P_i^n + \beta_3^n P_{i+1}^n + \beta_4^n P_{i+2}^n + \beta_5^n P_{i+3}^n$$

$$= \beta_5^n P_{-i-2}^n + \beta_4^n P_{-i-1}^n + \beta_3^n P_{-i}^n + \beta_2^n P_{-i+1}^n + \beta_1^n P_{-i+2}^n + \beta_0^n P_{-i+3}^n$$

$$= P_{-2i-1}^{n+1}$$

Similarly,

$$B\left(\frac{2i}{2^{n+1}}\right) = P_{2i}^{n+1} = P_{-2i}^{n+1} = B\left(\frac{-2i}{2^{n+1}}\right)$$

Hence, $B\left(\frac{i}{2^n}\right) = B\left(-\frac{i}{2^n}\right), n \in Z^+$

and also from the continuity of basic limit function we can say

$B(x) = B(-x), x \in R$.

6. Conclusion

A 6-point binary approximating non stationary hyperbolic subdivision scheme has been produced which is able to construct C1 limiting curve. Hyperbolic B-spline basis function is used to regenerate parabolas/hyperbolas. Asymptotic equivalence relation has been used to analyze the smoothness. Numerical results of the proposed scheme authenticates the generation of parabolas and hyperbolic spline. It gives a new era in hyperbolic field.

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