Reliability Analysis of Binary-Imaged Generalized Multi-State k-out-of-n Systems

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Abstract

A celebrated reliability model is the binary k-out of-n system, which is a dichotomous system that is successful if and only if at least k out of its n components are successful. In contrast to general reliability systems whose handling entails exponential complexity, this system possesses an elegant quadratic-time algorithm for evaluating its reliability. The aim of this paper is to extend the utility of this algorithm to the reliability analysis of a homogeneous binary-imaged multi-state coherent generalized k-out-of-n system, which is still described as a non-repairable system with independent non-identical components. The paper characterizes such a system *via* switching-algebraic expressions of either system success or system failure at each non-zero level, or equivalently, via, minimal upper vectors or maximal lower vectors. We also adapt the afore-mentioned quadratic-time algorithm to compute the reliability and unreliability at each non-zero system level. We point to the inconvenience of using fixed-point reliability values for systems with good components, and recommend using floating-point unreliability values in this case.

Keywords— System reliability, Multi-state system, k-out-of-n system, Minimal upper vector, Maximal lower vector. Quadratic-time algorithm.

Introduction

A binary *k*-out-of-*n*:G (*k*-out-of-*n*:F) system is a dichotomous system that is successful (failed) if and only if at least k out of its n components are successful (failed) [1-15]. The *k*-out-of-*n*:G system is dual to the *k*-out-of-*n*:F system, and equivalent to the (n-k+1)-out-of-*n*:F system [3, 9, 16, 17]. We might omit the G and F designations, and refer collectively to these two dual systems as binary k-out-of-n systems or as partially-redundant systems [17]. The binary k-out-of-n system has many attractive features and applications, and a symmetric structure that has many convenient mathematical descriptions. Binary k-out-of-n systems play a central role (and constitute a basis) for the general class of binary coherent systems [18]. While virtually all nontrivial network reliability problems are known to be NP-hard for general networks, the regular structure of the k-out-of-n system allows the existence of efficient algorithms for its reliability analysis. Notable among these algorithms are the Belfore algorithm [12], which is of $O(n(\log_2 n)^2)$ complexity, and the AR algorithm [3, 7, 9, 16], which has $O(n^2)$ complexity.

Though the former algorithm is asymptotically more efficient, it experiences a lengthy overhead, since it is based on a recursive application of the Fast Fourier Transform (FFT) for computing generating function products. The AR algorithm is much simpler, and it has a nice interpretation in terms of a very regular signal flow graph, which turns out to be a reduced ordered binary decision diagram (ROBDD) [19, 20], representing a monotone symmetric switching function.

This paper shifts interest to the more-powerful modeling paradigm of a multi-state system (MSS), in which there are multiple levels of system capacity or performance and/or different component performance levels and multiple component failure modes having different impacts on the system performance [21, 22]. Specifically, we study a multi-state generalized k-out-of-n:G system [23-38], which is a multi-state system whose multi-valued success is greater than or equal to a certain value j (lying between 1 (the lowest non-zero output level) and M (the highest output level)) whenever at least k_m components are in state m or above for all m such that $1 \leq m \leq j$. We further assume that the k_m values are constant or increasing, i.e., we assume that $k_1 \leq k_2 \leq k_3 \leq \cdots \leq k_n$. This means that the system considered is binary-imaged [39] and hence it is possible to compute its probability of success at each non-zero level using the AR algorithm.

The organization of the remainder of this paper is as follows. Section 2 offers a mathematical description of a binary-imaged multi-state coherent k-out-of-n system, while Sec. 3 reviews the AR algorithm for computing the reliability of a binary k-out-of-n system, and visualizes it through a regular Mason signal flow graph (SFG) [40-43]. Section 4 introduces a generalized multi-state k-out-of-n system (with increasing or constant k) that is used as a running example for this paper. Section 5 presents a switching-algebraic characterization of the example system *via* minimal sum-of-products formulas of the success and failure at each non-zero system level, and then adds two other equivalent characterizations in terms of minimal upper vectors and maximal lower vectors. Section 6 applies the AR algorithm to compute the reliability and unreliability at each non-zero system level. Section 6 also points to the inconvenience of using fixed-point reliability values for systems with good components, and recommends using floating-point unreliability values in this case. Section 7 concludes the paper.

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Description of a binary-imaged multi-state coherent k-out-of-n system

The model considered herein is one of a multi-state system with multistate components, specified by the multi-valued structure or success function $S(\boldsymbol{X})$ [21, 22]

$$S: \{0,1,\cdots,m_1\} \times \{0,1,\cdots,m_2\} \times \ldots \times \{0,1,\cdots,m_n\} \rightarrow \{0,1,\cdots,M\}. \quad (1)$$

Though the number of system states $(M+1)$ and the numbers of component states $(m_1+1),(m_2+1),\cdots,(m_n+1)$ might differ, we consider herein a homogeneous system, in which these numbers have a common value. We use the symbol $X_k\{\geq j\}$ to denote the binary success of component k at level j [21, 22, 36]

$$X_k\{\geq j\} = X_k\{j, j+1, \ldots, m_k\} = \vee_{i=j}^{m_k} X_k\{i\} = X_k\{j\} \vee X_k\{j+1\} \vee \ldots \vee X_k\{m_k\}. \quad (2)$$

The complement of $X_k\{\geq j\}$ is called the binary failure of component k at level j, and is given by [21, 22, 36]

$$X_k\{< j\} = X_k\{0,1,\ldots,j-1\} = X_k\{0\} \vee X_k\{1\} \ldots \vee X_k\{j-1\} = X_k\{k \leq (j-1)\}. \quad (3)$$

The symbol $X_k\{j\}$ denotes a binary variable representing instant j of the multivalued variable X_k [21, 22, 36]

$$X_k\{j\} = X_k\{\geq j\}\, X_k\{<(j+1)\} = X_k\{\geq j\}\, \overline{X}_k\{\geq (j+1)\} = X_k\{\leq j\}\, X_k\{>(j-1)\} = X_k\{\leq j\}\, \overline{X}_k\{\leq (j-1)\}. \quad (4)$$

Similar definitions can be given to $S\{\geq j\}$ (the binary system success at level j), $S\{< j\}$ (the binary system failure at level j), and $S\{j\}$ (the binary variable representing instant j of the multivalued system structure function S). As usual, we designate the expectations $E\{S\{\geq j\}\}$ and $E\{\mathbf{X}\{\geq j\}\}$ by R_j and $\mathbf{p}\{\geq j\}$.

A binary-imaged multi-state system is a system whose success at level j is a function only of component successes at the same level (i.e., it is an MSS such that $S\{\geq j\}$ is a function of $\mathbf{X}\{\geq j\}$ only), or equivalently, it is a system whose failure at level j is a function only of component failures at the same level (i.e., it is an MSS such that $S\{\leq (j-1)\}$ is a function of $\mathbf{X}\{\leq (j-1)\}$ only) [21, 22]. For a binary-imaged system, elements of the set of MUVs $\theta(j)$ are vectors of j or 0 components only, and elements of the set of MLVs $\sigma(j)$ are vectors of j or M components only [21, 22].

A multi-state generalized k-out-of-n: G system is a multi-state system whose multi-valued success is greater than or equal to a certain value j (lying between 1 (the lowest non-zero output level) and M (the highest output level)) whenever at least k_m components are in state m or above for all m such that $1 \leq m \leq j$ [24-27, 36]. This system is binary imaged if the k_m values are constant or increasing, i.e., if $k_1 \leq k_2 \leq k_3 \leq \cdots \leq k_n$ [24-26].

The AR Algorithm for computing the Reliability of binary k-out-of-n systems

The AR algorithm for computing the reliability $R(k,n,\mathbf{p})$ of a k-out-of- n:G binary system is an iterative (non-recursive) algorithm of quadratic temporal complexity, that is governed by the two-dimensional recursive relation [3, 7, 9, 15-18]

$$R(k,n,\mathbf{p}) = (1-p_n) * R(k,n-1,\mathbf{p}/p_n) + p_n * R(k-1,n-1,\mathbf{p}/p_n), \ 1 \leq k \leq n, \quad (5a)$$

together with the boundary conditions

$$R(k,n,\mathbf{p}) = 1.0, \quad k = 0, n \geq 0, \quad (5b)$$
$$R(k,n,\mathbf{p}) = 0.0, \quad k = (n+1), n \geq 0, \quad (5c)$$

Note that $R(k,n,\mathbf{p})$ is the Complementary Cumulative Distribution Function (CCDF) of the generalized binomial distribution. The unreliability $U(k,n,\mathbf{p}) = 1.0 - R(k,n,\mathbf{p})$ is therefore the Cumulative Distribution Function (CDF) of the generalized binomial distribution [44-46]. It obeys a recursive relation that is similar to the recursive relation (5a), but with the values in the boundary conditions (5b) and (5c) complemented, i.e.

$$U(k,n,\mathbf{p}) = (1-p_n) * U(k,n-1\mathbf{p}/p_n) + p_n * U(k-1,n-1,\mathbf{p}/p_n), \ 1 \leq k \leq n, \quad (6a)$$
$$U(k,n,\mathbf{p}) = 0.0, \quad k = 0, n \geq 0, \quad (6b)$$
$$U(k,n,\mathbf{p}) = 1.0, \quad k = (n+1), n \geq 0, \quad (6c)$$

The set of relations (5) and (6) were first derived by Rushdi [3] by utilizing properties of monotone symmetric switching (two-valued Boolean) functions. Many authors (see, e.g., [13, 23, 25]) mysteriously credit (5) also to an extremely short (albeit fruitful) exchange by Barlow and Heidtmann [2], though this exchange is confined to the generating-function paradigm, with no mention whatsoever of recursion or boundary conditions. Recently, many authors [47-50] began to realize that (5) and (6) are both derived for the first time in, and only in, [3].

Rushdi and Rushdi [16] provided a much simpler novel proof of (5) or (6) via the celebrated total probability theorem [51-53]. In this proof, the underlying mutually-exclusive and exhaustive events are the two complementary events: {component n is not working} and {component n is working}, with respective probabilities q_n and p_n. A k-out-of-n event becomes a k-out-of-(n−1) event under the condition that component n is not working, and becomes a (k−1)-out-of-(n−1) event under the condition that component n is working. This is still true when the concerned events are qualified by the adverb "at least". Recently, Efrem and Panagopoulos [50] prided themselves in exploring the power and beauty of recursion, as they (obviously independently) rediscovered the afore-mentioned simpler proof of (5) via the law/theorem of total probability.

Figure 1 shows a regular Mason signal flow graph (SFG) that illustrates the computation of $R(k, n, \mathbf{p})$. Note that in column I, each diagonal arrow has a transmittance equal to p_i, while each horizontal arrow carries a transmittance equal to its complement $q_i = (1.0 - p_i)$. There are two types of nodes: (a) Source nodes of known values which are either black or white. A black node has a value of 1.0 while a white node has a value of 0.0, and (b) Non-source nodes drawn as shaded ones, which include (at least) one sink node whose value is the final result sought. Figure 1 first appeared in Rushdi [3], and can be viewed in the Boolean domain as an SFG for a monotone symmetric switching function representing the success function of a k-out-of-n system. In such a graph, algebraic multiplication and addition are replaced by their logical counterparts (ANDing and ORing), and the graph can be identified to be an early precursor of a Reduced Ordered Binary Decision Diagram (ROBDD), which is currently known to be the state-of-the art data structure for encoding and manipulating switching functions. Moreover, Fig. 1 has certain similarities and minor dissimilarities with al-Karkhi's Triangle (Pascal's Triangle), which is constructed for the binomial (combinatorial) coefficient (n choose k) $c(k, n)$ via al-Karkhi's (Pascal's) identity [17]

$$c(k, n) = c(k, n-1) + c(k-1, n-1), \quad 0 < k < n, \tag{7a}$$

together with the boundary conditions

$$c(k, n) = 1,, \ (k = 0 \ \text{ or } \ k = n) \ \text{ and } \ n \geq 0. \tag{7b}$$

Subject to the condition of constant or increasing k ($k_1 \leq k_2 \leq \ k_3 \leq \cdots \leq \ k_n$), the reliability at level j of a multi-state generalized k-out-of-n:G system has a binary image given by

$$E\{S\{\geq j\}(k_j, n, \mathbf{X}\{\geq j\})\} = R_j(k_j, n, \mathbf{p}\{\geq j\}), \tag{8}$$

which depends solely on k_j and $\mathbf{p}\{\geq j\}$. The system success $S\{\geq j\}(k_j, n, \mathbf{X}\{\geq j\})$ at level j can be expressed by a single monotone symmetric switching function [3, 9, 36], of a characteristic set $\{m |\ k_j \leq m \leq n\} = \{k_j, k_j + 1, \ldots, n\}$ and arguments $\mathbf{X}\{\geq j\}$, and, hence, of the form

$$S\{\geq j\}(k_j, n, \mathbf{X}\{\geq j\}) = Sy(n;\ \{m |\ k_j \leq m \leq n\}; \mathbf{X}\{\geq j\}) = Sy(n;\ \{k_j, k_j + 1, \ldots, n\}; \mathbf{X}\{\geq j\}), \tag{9}$$

thanks to the fact that

$$\{k_j \leq k_{j+1}\} \ \Rightarrow \ \{Sy(n;\ \{m |\ k_j \leq m \leq n\}; \mathbf{X}\{\geq j\}) \ \geq Sy(n;\ \{m |\ k_{j+1} \leq m \leq n\}; \mathbf{X}\{\geq (j+1)\})\}, \tag{10}$$

which is true, since for all j and k

$$X_k\{\geq j\} = X_k\{j\} \ \vee \ X_k\{\geq (j+1)\} \ \geq \ X_k\{\geq (j+1)\}. \tag{11}$$

Therefore, the reliability $R_j(k_j, n, \mathbf{p}\{\geq j\})$ is directly computable by the iterative AR algorithm, since it is governed by the two-dimensional recursive relation in (5), i.e.,

$$R_j(k_j, n, \mathbf{p}\{\geq j\}) = \ p_n\{< j\} * R_j(k_j, n-1, \mathbf{p}/p_n\{\geq j\}) + p_n\{\geq j\} * R_j(k_j - 1, n-1, \mathbf{p}/p_n\{\geq j\}), \ 1 \leq k_j \leq n, \tag{12a}$$

together with the boundary conditions

$$R_j(k_j, n, \mathbf{p}\{\geq j\}) = 1.0, \quad k_j = 0, n \geq 0\ , \tag{12b}$$

$$R_j(k_j, n, \mathbf{p}\{\geq j\}) = 0.0, \quad k_j = (n+1), n \geq 0, \tag{12c}$$

A running multi-state example

We now present an example borrowed from Huang and Zuo [23]. This example deals with a production management problem of a plant having five production lines for a specific product. The plant has four different production levels: full scale for maximum or extensive customer demand (state 3), average scale for normal or usual customer demand (state 2), and low scale for low customer demand (state 1), and zero scale when the plant is shut down, and no customer demand whatsoever is met. All the five production lines have to work full scale (at state 3) for the system to be in state 3. At least three lines have to work at least at the average scale for the system to be at least in sate 2. At least 2 lines have to work at the low scale (state 1) for the system to be in state 1 or above. Such a system can be represented by an increasing multi-state k-out-of-n:G system model with k_1 = 2, k_2 = 3, and k_3 = 5. Table 1 summarizes the verbal description of system behavior at various levels. Table 2 lists the probabilities of the various states of the multistate components, as given in the original problem [23]. Table 3 transforms the information in Table 2 so as to suit our purposes by stating the success and failure probabilities at the various levels of the multistate components.

A switching-algebraic characterization

This section presents a switching-algebraic characterization of the example system via minimal sum-of-products formulas of the success and failure at each non-zero system level, and then adds two other equivalent characterizations in terms of minimal upper vectors and maximal lower vectors. Table 4 displays expressions of system success $S\{\geq j\}$ and system failure $\{S \leq (j-1)\} = \{S < j\}$ at every non-zero level j for $j = 1, 2,$ and 3. As expected, the number of minimal paths for level j (the number of prime implicants of the switching function $S\{\geq j\}$) is the binomial coefficient $c(k_j, n)$ [9, 17, 54]. Hence, for $j = 1, 2,$ and 3, we have the numbers of minimal paths given by $c(2, 5) = 10$, $c(3, 5) = 10$, and $c(5, 5) = 1$. On the other hand, the number of minimal cutsets for level j (the number of prime implicants of the switching function $S\{\leq (j-1)\}$) is the binomial coefficient $c(n - k_j + 1, n) = (k_j - 1, n)$ [9, 17, 54]. Hence, for $j = 1, 2,$ and 3, we have the numbers of minimal cutsets given by $c(1, 5) = 5$, $c(2, 5) = 10$, and $c(4, 5) = 5$.

We now clarify a subtle relation between a minimal upper vector (MUV) at a certain level and a prime implicant of success (minimal path) at that level, and a similarly subtle analogous or 'dual' relation between a maximal lower vector (MLV) at a certain level and a prime implicant of failure (minimal cutset) at that level [21, 22]. We stress that, contrarily to widespread belief, the MUVs and MLVs do not exactly play the role of (or might be considered synonymous to) minimal paths and minimal cutsets, respectively. In fact, a minimal path constitutes all the upper vectors extending (inclusively) from a particular MUV to the all-highest vector, while a minimal cutset comprises all the lower vectors extending (inclusively) from the all-0 vector to a particular MLV. Therefore, the mapping from a minimal upper vector (MUV) at a certain level and a prime implicant of success at that level is one-to-one and onto. Likewise, the mapping from a maximal lower vector (MLV) at a certain level and a prime implicant of failure at that level is also one-to-one and onto. This clarification might be conveniently visualized via a multi-valued Karnaugh map [55], as shown in [21, 22]. Based on this clarification, Table 5 displays the minimal upper vectors (MUVs) and the maximal lower vectors (MLVs) for our running example. For the current binary-imaged system, elements of the set of MUVs $\theta(j)$ are vectors of j or 0 components only, and elements of the set of MLVs $\sigma(j)$ are vectors of j or M components only [21, 22].

Application of the AR algorithm to the running example

Based upon the recursive relation (12a) and boundary conditions (12b) and (12c), a quadratic time non-recursive algorithm for computing $R_j(k_j, n, \mathbf{p}\{\geq j\})$ (and $U_j(k_j, n, \mathbf{p}\{\geq j\})$) can be immediately constructed. This algorithm has a pictorial interpretation in terms of an SFG generalizing the one in Fig. 1, by replacing the graph transmissions p_n and q_n preceding each column n by the qualified symbols $p_n\{\geq j\}$ and $p_n\{< j\}$, respectively, as shown in Figs. 2-4, which demonstrate the required k_j values of $2, 3$, and 5. The algorithm constructs an array of values inclusively bounded in the xy-plane by the four straight lines, $x = 1$, $x = k_j$, $x = y$, $x = (y - n + k_j)$, which are the edges of a parallelogram with corners (x, y) at $(1,1)$, (k_j, k_j), (k_j, n), and $(1, n - k_j + 1)$. The algorithm has three versions depending on the order of traversing or sweeping the aforementioned parallelogram elements, namely:

6.1. The vertical-sweep version

Nodes are visited column-wise, starting from the leftmost column $(y = 1)$ and ending at the rightmost column $(y = n)$. Within each column y, the bottom node $(x = \min(y, k_j))$ is visited first, and then followed by upper nodes till the top node $(x = \min(1, y - n + k_j))$ is reached.

6.2. The horizontal-sweep version

Nodes are visited row-wise, starting from the topmost row $(x = 1)$ and ending at the bottom row $(x = k_j)$. Within each row x, the algorithm proceeds from the left diagonal $(y = x)$ to the right diagonal $(y = x + n - k_j)$.

6.3. The diagonal-sweep version

Nodes are visited diagonal-wise, starting from the leftmost diagonal $(y - x = 0)$ and ending at the rightmost one $(y - x = n - k_j)$. Within each diagonal, the algorithm proceeds downwards from the top row $(x = 1)$ to the bottom row $(x = k_j)$.

There exists a "dual" version of the AR algorithm that computes the unreliability $U_j(k_j, n, \mathbf{p}\{\geq j\})$ instead of the reliability $R_j(k_j, n, \mathbf{p}\{\geq j\})$ with no change whatsoever in complexity [3, 9, 16, 17]. All we need in any of Figs. 2-4 is to switch every black node (of value 1.0) to a white one (of value 0.0), and vice verse, thereby switching the value of every node in the entire SFG to the complementary (to one) value. Examples of the resulting SFG are shown in Figs. 5-7, which are "node-wise" complementary to Figs. 2-4.

The parallelogram of nodes in each of Figs. 2-7 has a height of k_j, a width of $(n - k_j + 1)$, and an area of $k_j(n - k_j + 1)$. For $k_j = n$ (a series system at level j), the area of the parallelogram diminishes as it degenerates into a straight-line segment of n nodes on the diagonal $(x = y)$, for which it is far better to use the reliability version of the algorithm than the unreliability one. Dually, for $k_j = 1$ (a parallel system at level j), the parallelogram again degenerates, but this time into a straight-line segment of n nodes on the horizontal line $(x = 1)$, for which it is far better to use the unreliability version of the algorithm than the reliability one.

We now utilize graphical means to discuss the use of the two versions of the AR algorithm for analyzing our running example. Figure 8 demonstrates the actual computations implemented on the SFG of Fig. 2 to obtain $E\{s\{\geq 1\}\}(2, 5, \mathbf{p}\{\geq 1\})$ of system success at level 1. The figure manifests the inconvenience of representing ultra-reliabilities in fixed-point format. Figure 9 (in a totally complementary fashion) illustrates the actual computations implemented on the SFG of Fig. 5 to obtain the expectation $E\{s\{< 1\}\}(2, 5, \mathbf{p}\{\geq 1\})$ of system failure at level 1. The figure demonstrates the numerical convenience of dealing with probability of failure rather than that of success when dealing with ultra-reliable systems. Every node value in the figure is complementary to the corresponding node in Fig. 8. The tasks achieved by Fig. 8 and 9 for level 1 are attained by Figs. 10 and 11, respectively, for level 2, and Figs. 12 and 13, respectively, for level 3. Figure 10 indicates the actual computations implemented on the SFG of Fig. 3 to obtain $E\{s\{\geq 2\}\}(3, 5, \mathbf{p}\{\geq 2\})$. Again, the figure manifests the inconvenience of representing ultra-reliabilities in fixed-point format. Figure 11 clarifies the actual computations implemented on the SFG of Fig. 6 to obtain the expectation $E\{s\{< 2\}\}(2, 5, \mathbf{p}\{\geq 2\})$ of system failure at level 2. Again, the figure demonstrates the numerical convenience of dealing with probability of failure rather than that of success when dealing with ultra-reliable systems. Every node value in the figure is complementary to the corresponding node in Fig. 10. Now, Fig. 12. Demonstrates the actual computations implemented on the SFG of Fig. 4 to obtain $E\{s\{\geq 3\}\}(5, 5, \mathbf{p}\{\geq 3\})$. This reliability figure neatly recovers the celebrated reliability-product formula of a series system. Dually, the unreliability version of the AR algorithm is equally efficient in obtaining the dual (albeit much less known) unreliability-product formula of a parallel system. Figure 13 shows the actual computations implemented on the SFG of Fig.

7 to obtain the expectation $E\{s\{<3\}\}(5,5,\mathbf{p}\{\geq 3\})$ of system failure at level 3. Every node value in the figure is complementary to the corresponding node in Fig. 12. This kind of unreliability computation is not recommended for the present series system (but is preferable for a parallel system).

For further clarification, we give a more detailed explanation for our analysis of level 3. Figure 4 shows the SFG used by the AR algorithm to compute the reliability at level 3 for our running example, namely $R_3(5,5,\ \mathbf{p}\{\geq 3\})$. The algorithm recovers the celebrated product formula for a series (n-out-of-n:G) system, viz.

$$R_3(5,5,\mathbf{p}\{\geq 3\}) = p_1\{\geq 3\} * p_2\{\geq 3\} * p_3\{\geq 3\} * p_4\{\geq 3\} * p_5\{\geq 3\}. \quad (13)$$

Figure 12 demonstrates the actual numerical computation performed in (13), i.e.,

$$R_3(5,5,\mathbf{p}\{\geq 3\}) = (0.80) * (0.81) * (0.82) * (0.83) * (0.84) = 0.370464192. \quad (14)$$

Figure 7 shows the SFG used by the AR algorithm to compute the unreliability at level 3 for our running example, namely $U_3(5,5,\ \mathbf{p}\{\geq 3\})$. The algorithm produces the much involved formula

$$U_3(5,5,\mathbf{p}\{\geq 3\}) = p_5\{<3\} + p_5\{\geq 3\}(p_4\{<3\} + p_4\{\geq 3\}(p_3\{<3\} + p_3\{\geq 3\}(p_2\{<3\} + p_2\{\geq 3\}(p_1\{<3\} + p_1\{\geq 3\}(0))))). \quad (15)$$

Figure 13 demonstrates the actual numerical computation performed in (15), i.e.,

$$U_3(5,5,\mathbf{p}\{\geq 3\}) = (0.16) + (0.84)\Big((0.17) + (0.83)\Big((0.18) + (0.82)\big((0.19) + (0.81)(0.20 + 0)\big)\Big)\Big) = 0.62953580. \quad (16)$$

Finally, we have two options to obtain the probabilities of various system states via

$$E\{S\{0\}\} = 1.0 - E\{S\{\geq 1\}\} = E\{S\{<1\}\}, \quad (17)$$
$$E\{S\{1\}\} = E\{S\{\geq 1\}\} - E\{S\{\geq 2\}\} = E\{S\{<2\}\} - E\{S\{<1\}\}, \quad (18)$$
$$E\{S\{2\}\} = E\{S\{\geq 2\}\} - E\{S\{\geq 3\}\} = E\{S\{<3\}\} - E\{S\{<2\}\}, \quad (19)$$
$$E\{S\{3\}\} = E\{S\{\geq 3\}\} = 1.0 - E\{S\{<3\}\}. \quad (20)$$

Table 6 reports the numerical values obtained by our method (twice) compared to that in [23]. It is important to note the very small probability of $E\{S\{0\}\}$ cannot be approximated as zero (as done in [23]). Such a catastrophic approximation amounts to a relative error of 100% [51, 52]. In fact, in rare-event assessment, one should be concerned with relative error, not absolute error (i.e., one needs to know whether the probability is of order 10^{-4} or 10^{-6}, not that it is just close to 0 [56-60].

Conflict of interest

The authors assert that no conflict of interest exists

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تحليل معولية النظم الو افرة جزئيا ذات الحالات المتعددة والمعممة وذات الصور الثنائية

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المستخلص

إن أحد نماذج المعولية المشهورة هو النظام الثنائي للوفرة الجزئية، وهو نظام ثنائي ينجح إذا وفقط إذا نجح على الأقل ك من بين ن من مكوناته. وعلى النقيض من أنظمة المعولية العامة التي تتطلب معالجتها تعقيدًا أسيًا، يمتلك هذا النظام خوارزمية أنيقة لا تحتاج لأكثر من وقت تربيعي. الهدف من هذه الورقة هو توسيع فائدة هذه الخوارزمية لتقوم بتحليل معولية نظام وافر جزئيا معمم متسق متعدد الحالات فضلا عن كونه متجانسا ذا صور ثنائية، والذي لا يزال متصفا بأنه نظام غير قابل للإصلاح ذو مكونات غير متطابقة ولكنها مستقلة. تميز هذه الورقة مثل هذا النظام من خلال استخدام صيغ تبديل جبرية لنجاح النظام أو فشله عند كل مستوى غير صفري، أو بصورة مكافئة، باستعمال المتجهات العليا الأصغرية أو المتجهات الدنيا الأعظمية. نقوم أيضًا بتطبيق خوارزمية تربيعية الوقت لحساب المعولية وعدم المعولية عند كل مستوى غير صفري للنظام. نشير إلى عدم الملاءمة في استخدام القيم ذات العلامة العشرية الثابتة لمعولية الأنظمة ذات المكونات الجيدة، ونوصي باستخدام قيم عدم المعولية ذات العلامة العشرية العائمة في هذه الحالة

الكلمات الدالة

معولية النظم، النظام متعدد الحالات، النظام الوافر جزئيا، المتجه الأعلى الأصغري، المتجه الأدنى الأعظمي، خوارزمية تربيعية الزمن.

TABLE 1. BEHAVIOR AT VARIOUS LEVLES

Level	Description
1	2-out-of-5:G (4-out-of-5:F)
2	3-out-of-5:G (3-out-of-5:F)
3	Series (5-out-of-5:G) (1-out-of-5:F)

TABLE 2. PROBABILITIES OF THE VARIOUS STATES OF THE MULTISTATE COMPONENTS

i	1	2	3	4	5
$p_i\{3\}$	0.80	0.81	0.82	0.83	0.84
$p_i\{2\}$	0.10	0.11	0.11	0.11	0.10
$p_i\{1\}$	0.05	0.04	0.05	0.03	0.02
$p_i\{0\}$	0.05	0.04	0.02	0.03	0.04
$\sum_{j=0}^{3} p_i\{j\}$	1.00	1.00	1.00	1.00	1.00

TABLE 3. SUCCESS AND FAILURE PROBABILITIES AT THE VARIOUS LEVELS OF THE MULTISTATE COMPONENTS

Level	i	1	2	3	4	5
3	$p_i\{\geq 3\}$	0.80	0.81	0.82	0.83	0.84
	$p_i\{< 3\}$	0.20	0.19	0.18	0.17	0.16
2	$p_i\{\geq 2\}$	0.90	0.92	0.93	0.94	0.94
	$p_i\{< 2\}$	0.10	0.08	0.07	0.06	0.06
1	$p_i\{\geq 1\}$	0.95	0.96	0.98	0.97	0.96
	$p_i\{< 1\}$	0.05	0.04	0.02	0.03	0.04

TABLE 4. EXPRESSIONS OF SYSTEM SUCCESS AND SYSTEM FAILURE AT EVERY NON-ZERO LEVEL J {J = 1,2,3}

Level j	**System success at level j**	**System failure at level j**
1 ***(2-out-of-5:G)*** ***(4-out-of-5:F)***	$S\{\geq 1\} = X_1\{\geq 1\}X_2\{\geq 1\}$ $\vee X_1\{\geq 1\}X_3\{\geq 1\}$ $\vee X_1\{\geq 1\}X_4\{\geq 1\}$ $\vee X_1\{\geq 1\}X_5\{\geq 1\}$ $\vee X_2\{\geq 1\}X_3\{\geq 1\}$ $\vee X_2\{\geq 1\}X_4\{\geq 1\}$ $\vee X_2\{\geq 1\}X_5\{\geq 1\}$ $\vee X_3\{\geq 1\}X_4\{\geq 1\}$ $\vee X_3\{\geq 1\}X_5\{\geq 1\}$ $\vee X_4\{\geq 1\}X_5\{\geq 1\}$	$S\{\leq 0\}$ $= X_1\{\leq 0\}X_2\{\leq 0\}X_3\{\leq 0\}X_4\{\leq 0\}$ $\vee X_1\{\leq 0\}X_2\{\leq 0\}X_3\{\leq 0\}X_5\{\leq 0\}$ $\vee X_1\{\leq 0\}X_2\{\leq 0\}X_4\{\leq 0\}X_5\{\leq 0\}$ $\vee X_1\{\leq 0\}X_3\{\leq 0\}X_4\{\leq 0\}X_5\{\leq 0\}$ $\vee X_2\{\leq 0\}X_3\{\leq 0\}X_4\{\leq 0\}X_5\{\leq 0\}$
2 ***(3-out-of-5:G)*** ***(3-out-of-5:F)***	$S\{\geq 2\}$ $= X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 2\}$ $\vee X_1\{\geq 2\}X_2\{\geq 2\}X_4\{\geq 2\}$ $\vee X_1\{\geq 2\}X_2\{\geq 2\}X_5\{\geq 2\}$ $\vee X_1\{\geq 2\}X_3\{\geq 2\}X_4\{\geq 2\}$ $\vee X_1\{\geq 2\}X_3\{\geq 2\}X_5\{\geq 2\}$ $\vee X_1\{\geq 2\}X_4\{\geq 2\}X_5\{\geq 2\}$ $\vee X_2\{\geq 2\}X_3\{\geq 2\}X_4\{\geq 2\}$ $\vee X_2\{\geq 2\}X_3\{\geq 2\}X_5\{\geq 2\}$ $\vee X_2\{\geq 2\}X_4\{\geq 2\}X_5\{\geq 2\}$ $\vee X_3\{\geq 2\}X_4\{\geq 2\}X_5\{\geq 2\}$	$S\{\leq 1\} = X_1\{\leq 1\}X_2\{\leq 1\}X_3\{\leq 1\}$ $\vee X_1\{\leq 1\}X_2\{\leq 1\}X_4\{\leq 1\}$ $\vee X_1\{\leq 1\}X_2\{\leq 1\}X_5\{\leq 1\}$ $\vee X_1\{\leq 1\}X_3\{\leq 1\}X_4\{\leq 1\}$ $\vee X_1\{\leq 1\}X_3\{\leq 1\}X_5\{\leq 1\}$ $\vee X_1\{\leq 1\}X_4\{\leq 1\}X_5\{\leq 1\}$ $\vee X_2\{\leq 1\}X_3\{\leq 1\}X_4\{\leq 1\}$ $\vee X_2\{\leq 1\}X_3\{\leq 1\}X_5\{\leq 1\}$ $\vee X_2\{\leq 1\}X_4\{\leq 1\}X_5\{\leq 1\}$ $\vee X_3\{\leq 1\}X_4\{\leq 1\}X_5\{\leq 1\}$
3 ***Series*** ***(5-out-of-5:G)*** ***(1-out-of-5:F)***	$S\{\geq 3\} = X_1\{\geq 3\}X_2\{\geq 3\}$ $X_3\{\geq 3\}X_4\{\geq 3\}X_5\{\geq 3\}$	$S\{\leq 2\} = X_1\{\leq 2\} \vee X_2\{\leq 2\} \vee X_3\{\leq 2\}$ $\vee X_4\{\leq 2\} \vee X_5\{\leq 2\}$

TABLE 5. THE MINIMAL UPPER VECTORS (MUVS) AND THE MAXIMAL LOWER VECTORS (MLVS)

MUVs	**MLVs**
$\theta(1) = \left\{\begin{matrix}(1,1,0,0,0),\\(1,0,1,0,0),\\(1,0,0,1,0),\\(1,0,0,0,1),\\(0,1,1,0,0),\\(0,1,0,1,0),\\(0,1,0,0,1),\\(0,0,1,1,0),\\(0,0,1,0,1),\\(0,0,0,1,1),\end{matrix}\right\}$	$\sigma(0) = \left\{\begin{matrix}(0,0,0,0,3),\\(0,0,0,3,0),\\(0,0,3,0,0),\\(0,3,0,0,0),\\(3,0,0,0,0),\end{matrix}\right\}$
$\theta(2) = \left\{\begin{matrix}(2,2,2,0,0),\\(2,2,0,2,0),\\(2,2,0,0,2),\\(2,0,2,2,0),\\(2,0,2,0,2),\\(2,0,0,2,2),\\(0,2,2,2,0),\\(0,2,2,0,2),\\(0,2,0,2,2),\\(0,0,2,2,2),\end{matrix}\right\}$	$\sigma(1) = \left\{\begin{matrix}(1,1,1,3,3),\\(1,1,3,1,3),\\(1,1,3,3,1),\\(1,3,1,1,3),\\(1,3,1,3,1),\\(1,3,3,1,1),\\(3,1,1,1,3),\\(3,1,1,3,1),\\(3,1,3,1,1),\\(3,3,1,1,1),\end{matrix}\right\}$
$\theta(3) = \{(3,3,3,3,3)\}$	$\sigma(2) = \left\{\begin{matrix}(2,3,3,3,3,3),\\(3,2,3,3,3,3),\\(3,3,2,3,3),\\(3,3,3,2,3),\\(3,3,3,3,2),\end{matrix}\right\}$

TABLE 6. THE PROBABLITY IN EACH STATE ACCORDING TO TWO METHODS

System state	**3**	**2**	**1**	**0**	**Total**
Probability [23]	0.3705	0.6260	0.0035	0	1.0000
Probability, present results	0.370464192	0.626015792	0.003512848	0.000007168	1.000000000

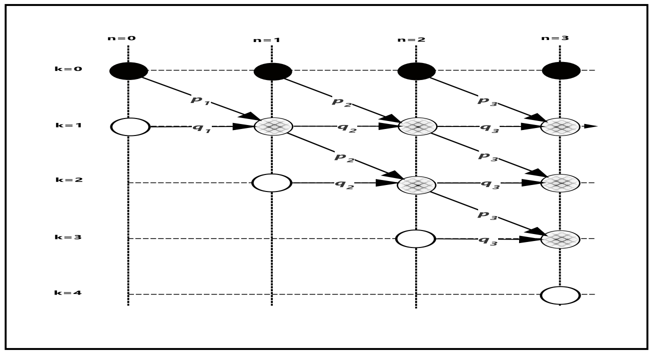

Fig. 1. A Mason signal flow graph that illustrates the computation of the reliability $R(k, n, \mathbf{p})$ of a binary k-out-of-n system.

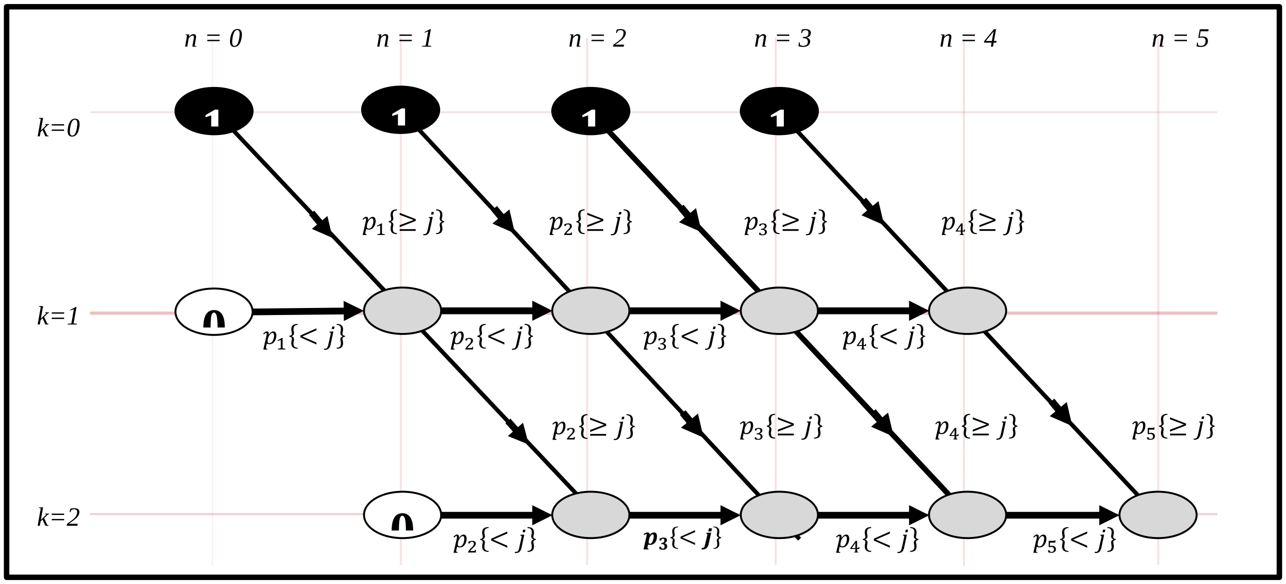

Fig. 2. A signal flow graph implementing the reliability version of the AR algorithm for computing the expectation $E\{s\{\geq j\}\}(2, 5, \boldsymbol{p}\{\geq j\})$ of system success at level j when $k_j = 2$ and $n = 5$.

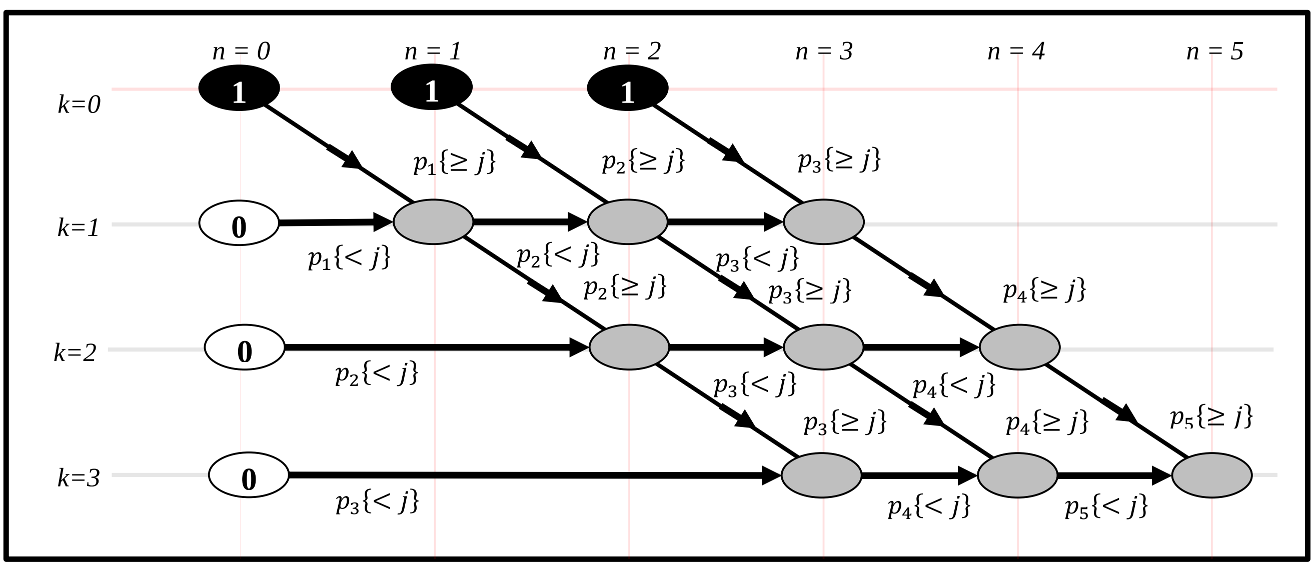

Fig. 3. A signal flow graph implementing the reliability version of the AR algorithm for computing the expectation $E\{s\{\geq j\}\}(3, 5, \boldsymbol{p}\{\geq j\})$ of system success at level j when $k_j = 3$ and $n = 5$.

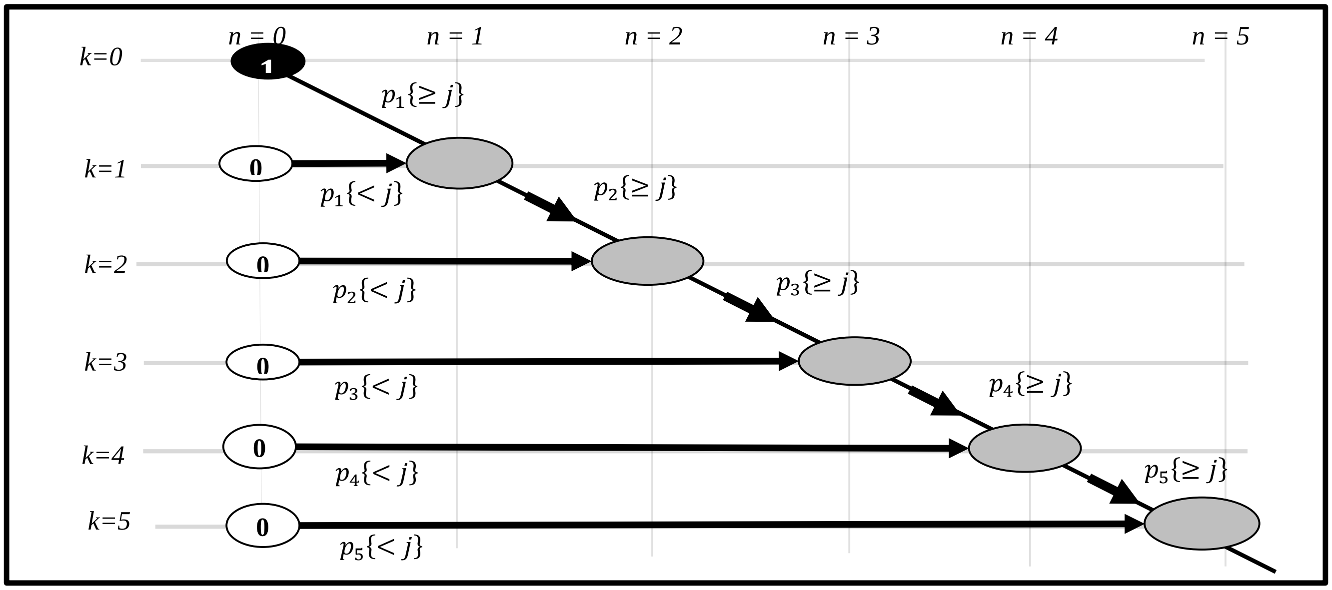

Fig. 4. A signal flow graph implementing the reliability version of the AR algorithm for computing the expectation $E\{s\{\geq j\}\}(5, 5, \boldsymbol{p}\{\geq j\})$ of system success at level j when $k_j = 5$ and $n = 5$.

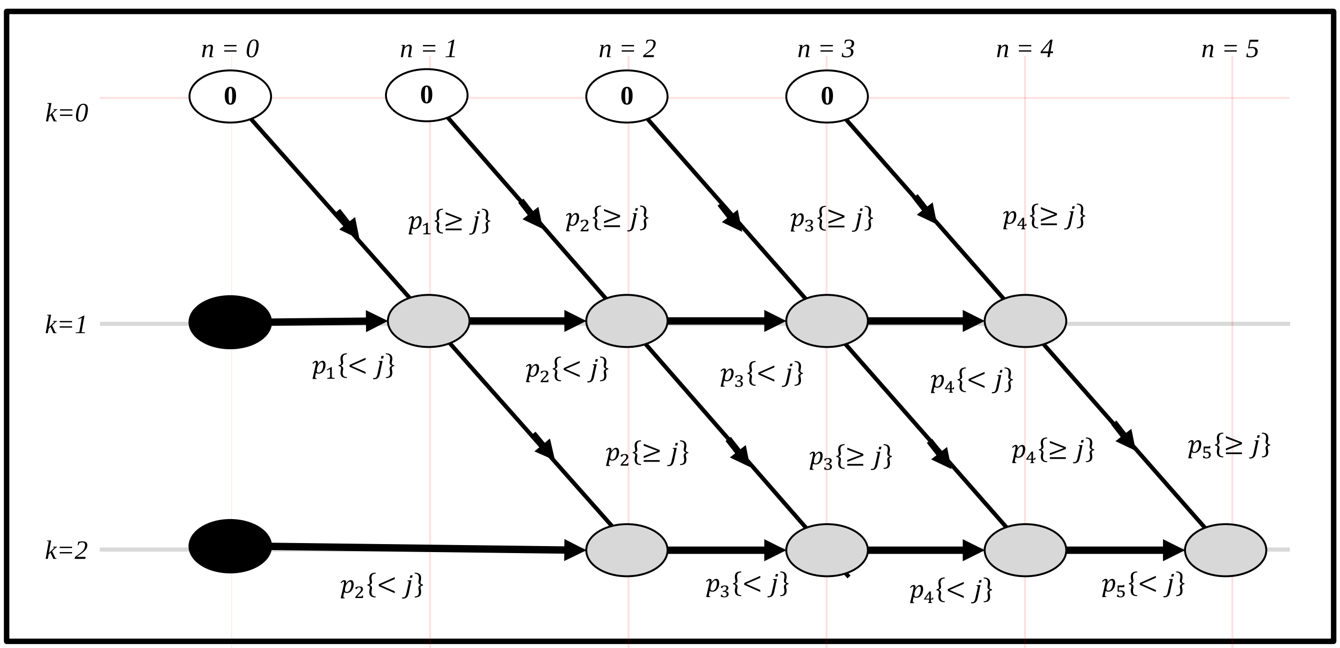

FIG. 5. A SIGNAL FLOW GRAPH IMPLEMENTING THE UNRELIABILITY VERSION OF THE AR ALGORITHM FOR COMPUTING THE EXPECTATION $E\{s\{< j\}\}(2, 5, \boldsymbol{p}\{\geq j\})$ OF SYSTEM FAILURE AT LEVEL j WHEN $k_j = 2$ AND $n = 5$. EVERY NODE VALUE IN THE FIGURE IS COMPLEMENTARY TO THE CORRESPONDING NODE VALUE IN FIG. 2.

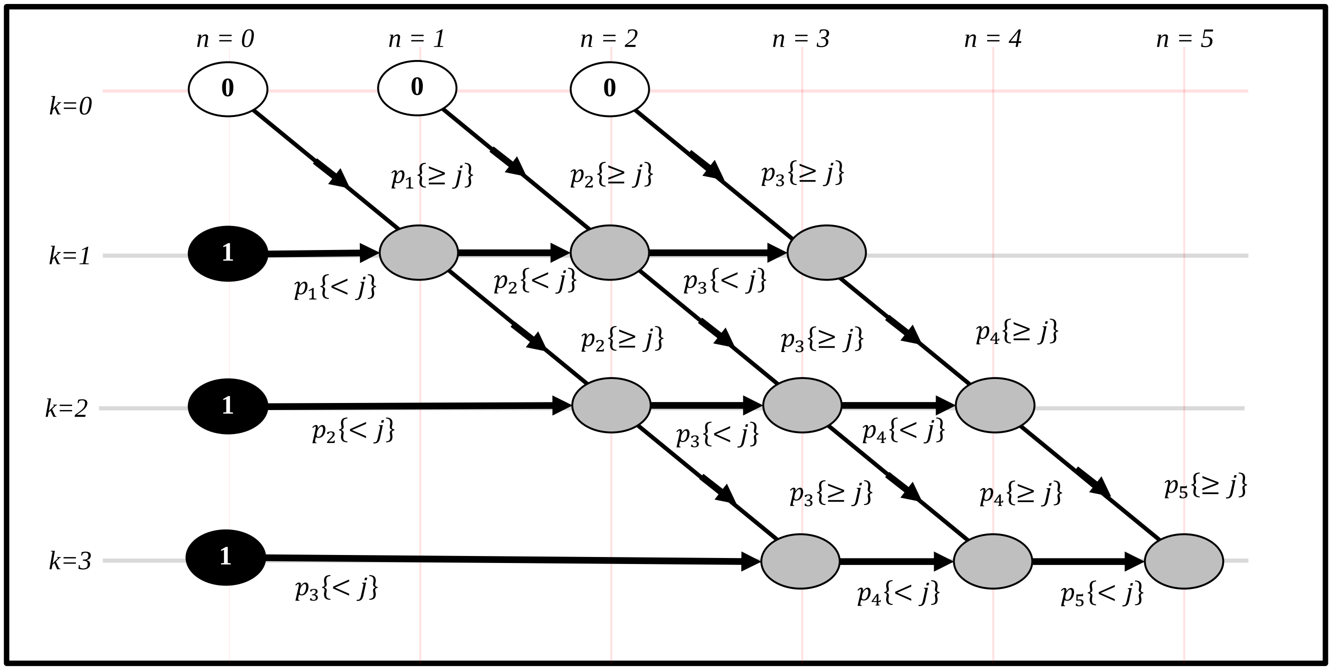

Fig. 6. A signal flow graph implementing the unreliability version of the AR algorithm for computing the expectation $E\{s\{< j\}\}(3, 5, \boldsymbol{p}\{\geq j\})$ of system failure at level j when $k_j = 3$ and $n = 5$. Every node value in the figure is complementary to the corresponding node value in Fig. 3.

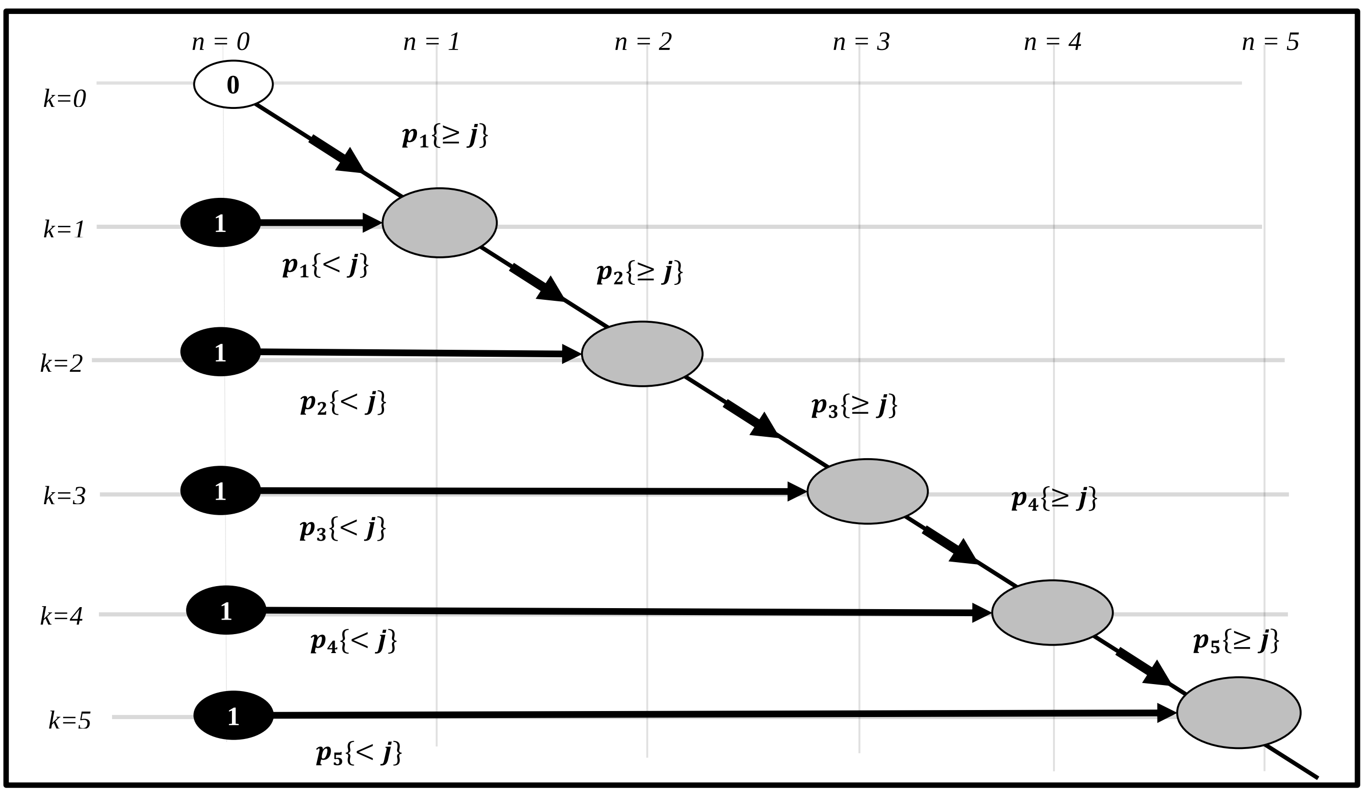

Fig. 7. A signal flow graph implementing the unreliability version of the AR algorithm for computing the expectation $E\{S\{< j\}\}$ $(5, 5, \boldsymbol{p}\{\geq j\})$ of system failure at level j when $k_j = 5$ and $n = 5$. Every node value in the figure is complementary to the corresponding node value in Fig. 4.

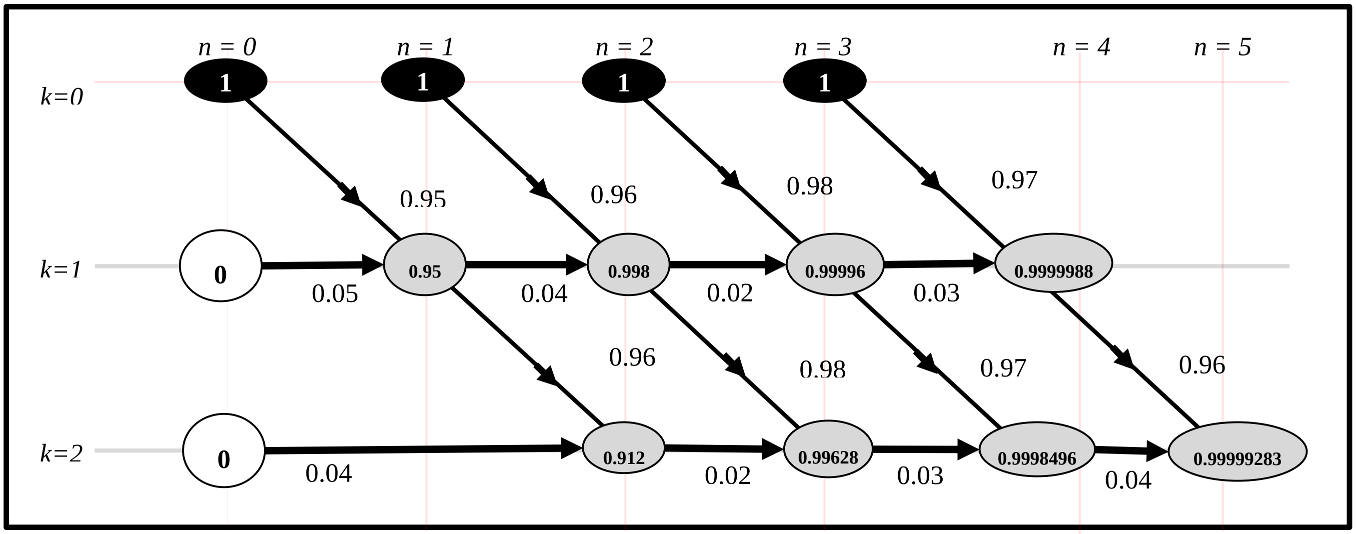

Fig. 8. Actual computations implemented on the SFG of Fig. 2 to obtain $E\{s\{\geq 1\}\}(2, 5, \boldsymbol{p}\{\geq 1\})$ of system success at level 1. The figure manifests the inconvenience of representing ultra-reliabilities in fixed-point format.

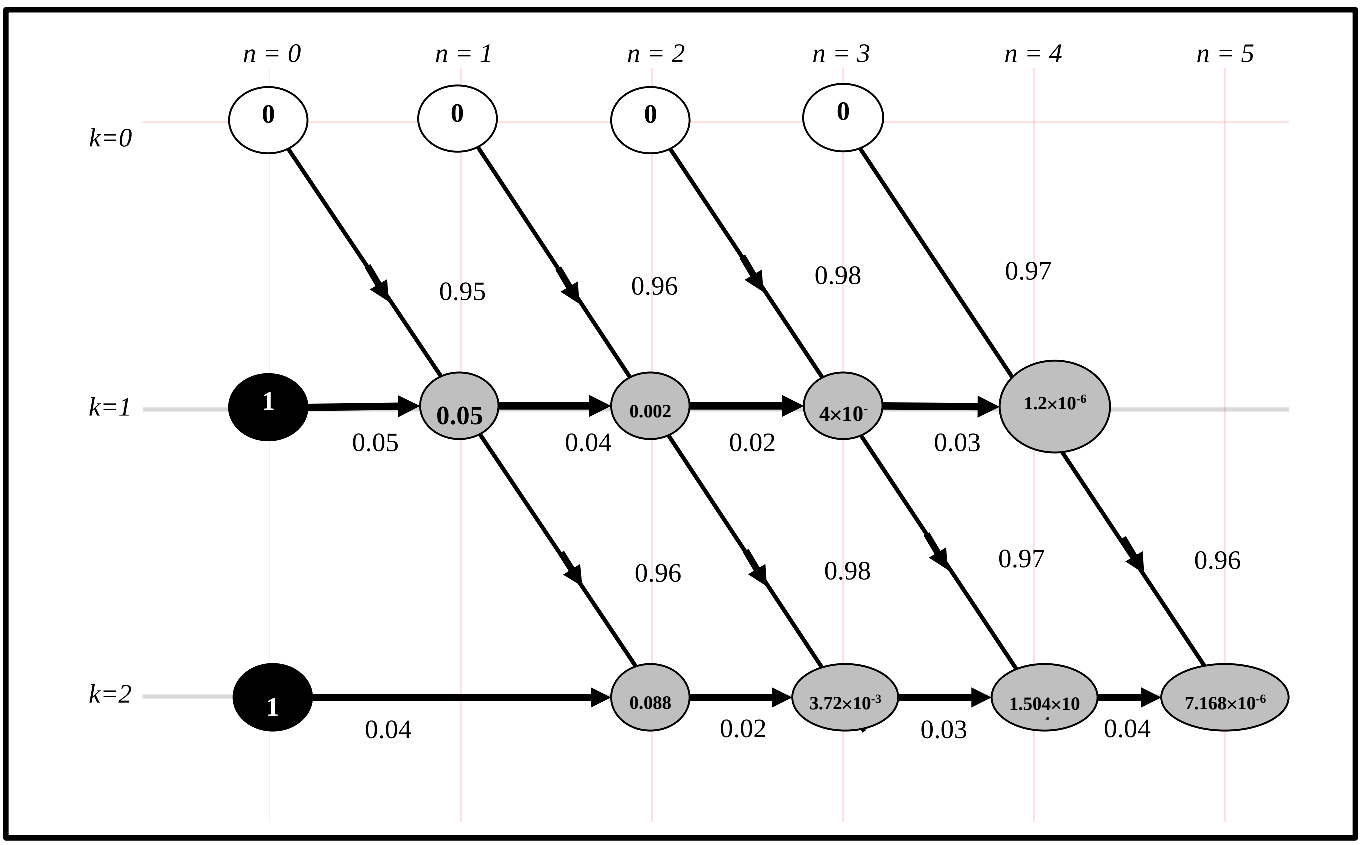

Fig. 9. Actual computations implemented on the SFG of Fig. 5 to obtain the expectation $E\{s\{< 1\}\}(2, 5, \boldsymbol{p}\{\geq 1\})$ of system failure at level 1 . The figure demonstrates the numerical convenience of dealing with probability of failure rather than that of success when dealing with ultra-reliable systems. Every node value in the figure is complementary to the corresponding node value in Fig. 8.

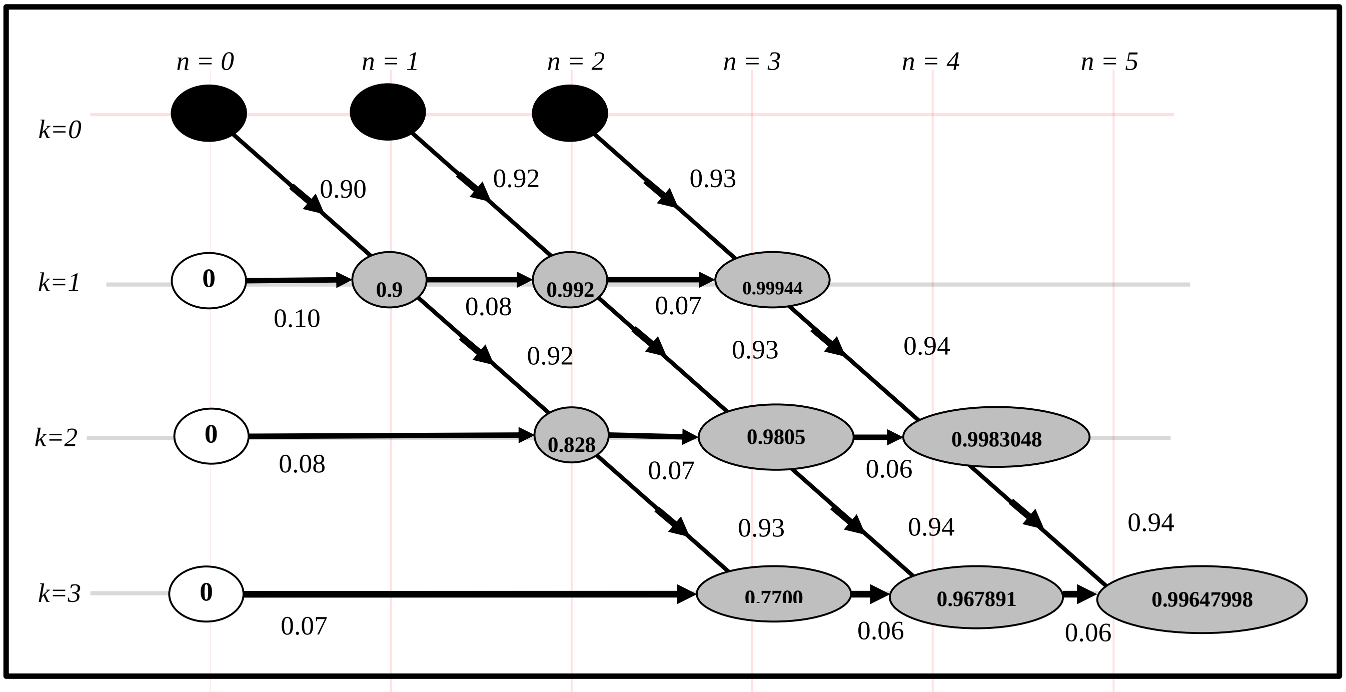

Fig. 10. Actual computations implemented on the SFG of Fig. 3 to obtain $E\{s\{\geq 2\}\}(3, 5, \boldsymbol{p}\{\geq 2\})$. Again, the figure manifests the inconvenience of representing ultra-reliabilities in fixed-point format.

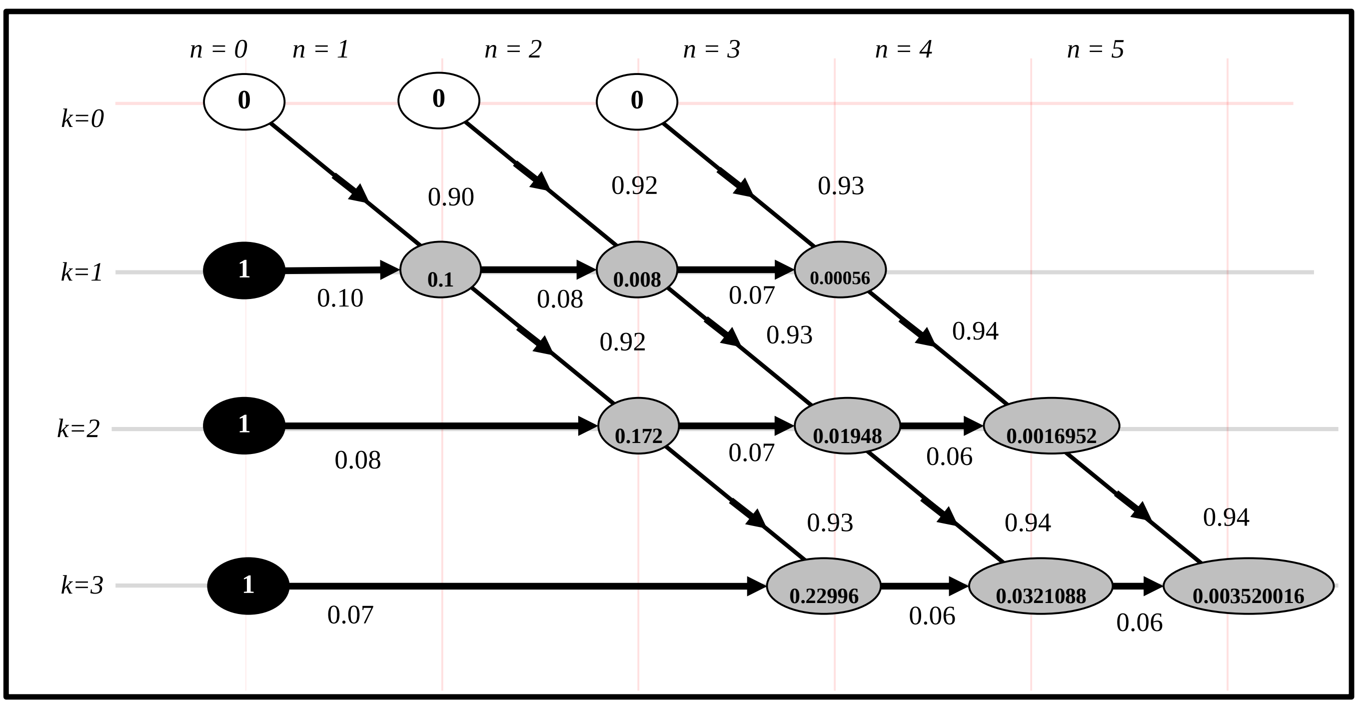

Fig. 11. Actual computations implemented on the SFG of Fig. 6 to obtain the expectation $E\{s\{< 2\}\}(2, 5, \boldsymbol{p}\{\geq 2\})$ of system failure at level 2 . Again, the figure demonstrates the numerical convenience of dealing with probability of failure rather than that of success when dealing with ultra-reliable systems. Every node value in the figure is complementary to the corresponding node value in Fig. 10.

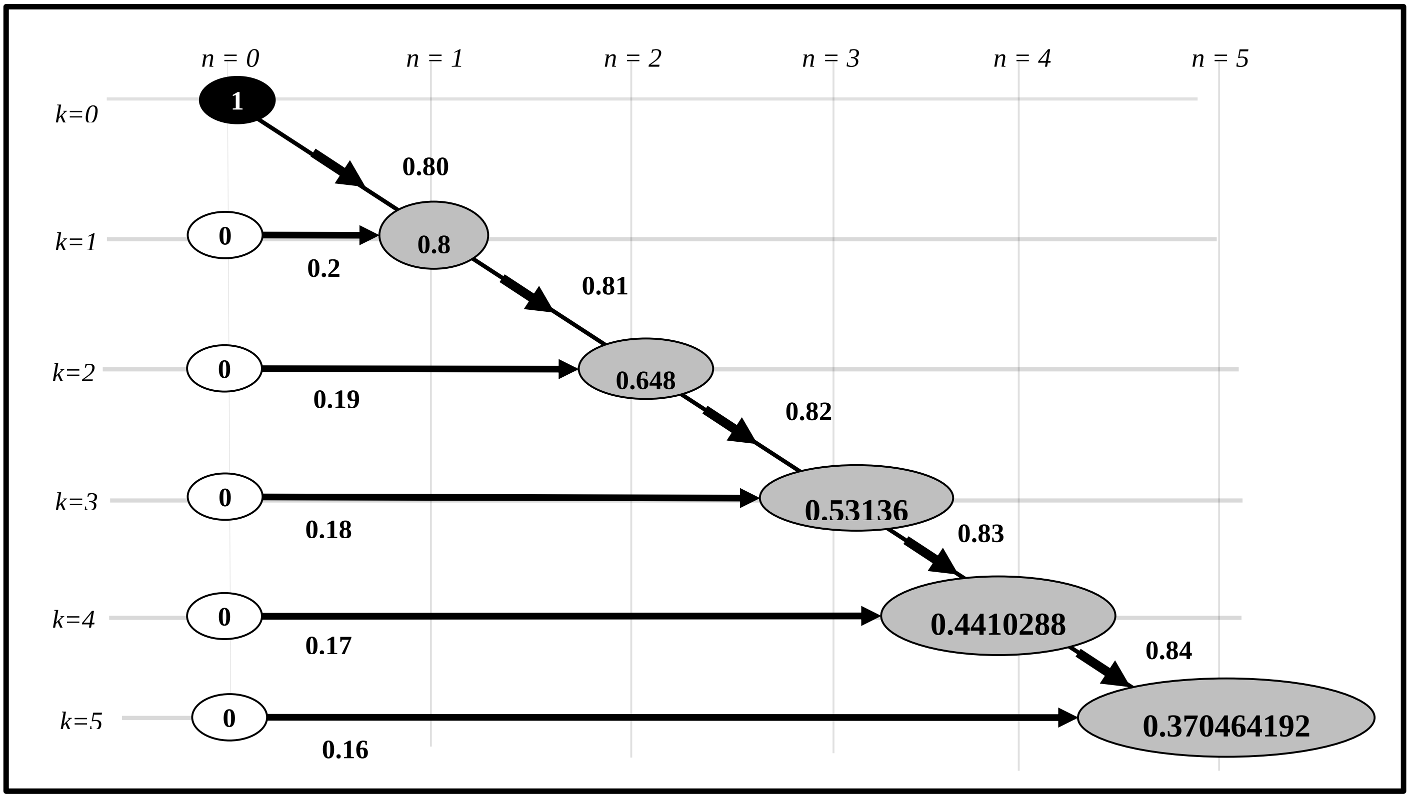

Fig. 12. Actual computations implemented on the SFG of Fig. 4 to obtain $E\{s\{\geq 3\}\}(5, 5, \boldsymbol{p}\{\geq 3\})$. This reliability figure neatly recovers the celebrated reliability-product formula of a series system. Dually, the unreliability version of the AR algorithm is equally efficient in obtaining the dual (albeit much less known) unreliability-product formula of a parallel system.

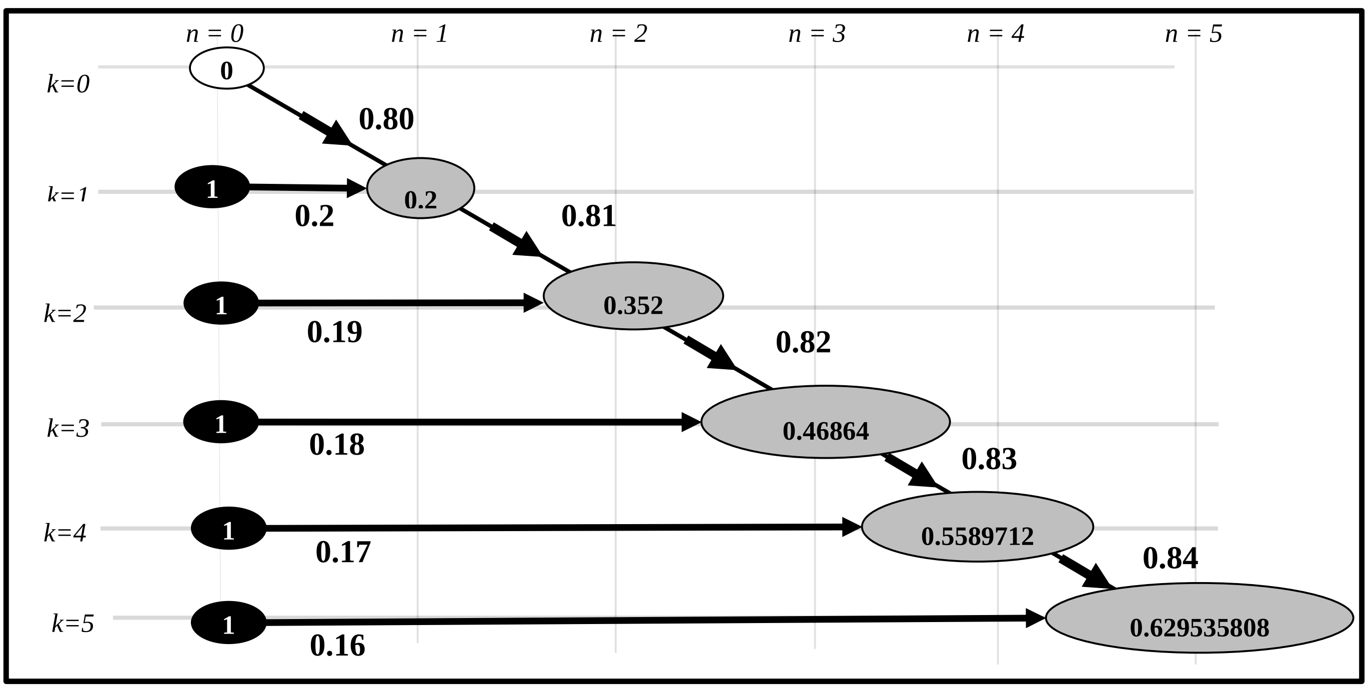

Fig. 13. Actual computations implemented on the SFG of Fig. 7 to obtain the expectation $E\{s\{< 3\}\}(5, 5, \boldsymbol{p}\{\geq 3\})$ of system failure at level 3 . Every node value in the figure is complementary to the corresponding node in Fig. 12. This kind of unreliability computation is not recommended for the present series system (but is preferable for a parallel system).