Modeling Exponential Growth in Population using Logistic, Gompertz and ARIMA Model: An Application on New Cases of COVID-19 in Pakistan

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Summary
In the mid of the December 2019, the virus has been started to spread from China namely Corona virus. It causes fatalities globally and WHO has been declared as pandemic in the whole world. There are different methods which can fit such types of values which obtain peak and get flattened by the time. The main aim of the paper is to find the best or nearly appropriate modeling of such data. The three different models has been deployed for the fitting of the data of Coronavirus confirmed patients in Pakistan till the date of 20th November 2020. In this paper, we have conducted analysis based on data obtained from National Institute of Health (NIH) Islamabad and produced a forecast of COVID-19 confirmed cases as well as the number of deaths and recoveries in Pakistan using the Logistic model, Gompertz model and Auto-Regressive Integrated Moving Average Model (ARIMA) model. The fitted models revealed high exponential growth in the number of confirmed cases, deaths and recoveries in Pakistan.

Keywords: Logistics, Gompert, ARIMA , Pakistan , Covid-19, time series, NIH

1. Introduction

In December 2019, a viral infection named as coronavirus disease (COVID-19) was began in Wuhan city of China, which became a global pandemic. It caused thousands fatalities in the world, on these basis WHO declared it as pandemic in March 2020[1]. Corona virus causes illness of respiratory system and causes flu, cough fever, diarrhea and different symptoms in different age group was come in knowledge [2]. The disease is highly contagious and transmit with the physical contact of bodies and respiratory droplets of the infected patients which is the main source of the virus transmission. The virus lives around 12 hours or more than 24 hours on the surface or the things touched by the infected person or in the air [3].

In Pakistan, first case was registered on 27th February 2020 and after two days more cases was registered in different cities. Gradually the number of patients were increases and in the month of April the cases were more than 5000 in total (Punjab 2,826, 1452 cases, 800 cases, 233 cases, 231 cases, 131 cases, and 43 cases of Sindh, KPK, GB, Baluchistan, Islamabad and AJK respectively), in which most effected cases the recoveries were recorded to be 1,378 with 96 deaths [6]. In the last nine months, the pandemic has spread around the world and infected more than four million people with 300,000 fatalities in the world. The number of patients has been increased immensely in Pakistan as well. The pandemic became worst after the holy festival of Eid uz Zuha and holy month of Ramadan when people went for shopping and following of SOPs and social distancing was not careful done. The high numbers of cases also overburdened the health care system in country and causes a nightmare of bad condition in the country. The fatality rate is higher among the young children and elderly aged group ≥ 60 years [4]. It is expected that these statistics is increasing exponentially in the upcoming days[5].

The government took precautionary measure to fight this war in timely fashion, such as authorities took further measures of closing borders, suspending community services and schools, minimizing both domestic and international travels until further notice [7]. The purpose of these measures is to limit the chances of physical contacts among people, so that transmission of COVID-19 is controlled, as the incubation period for this virus is relatively longer than other viruses. Due to the novel nature of the virus, there is greater uncertainty around the decision on optimal time of disappearance of this disease. Therefore, short term forecasting is extremely important even in the slightest hint for predicting the upcoming month for the better management of the social, economical, cultural and public health issues [8].

In this paper, we will compare three model fitted on the data Logistic model, Gompertz model and ARIMA. Logistic model (GLM) indicates that the epidemic growth was exponential in china [9]. ARIMA model has been used to estimate the number of infected people in the months [10]. Predictions help to strengthen the strategies in order to prevent the pandemic from worsening.

The aim of the study is to use the available data for the forecasting of the number of infected, dead and recovered people in Pakistan in the second wave of the pandemic. This forecasting is aimed to help government institutions and

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policy makers as well as public in adopting new strategies and strengthening the existing preventive measures against COVID-19 pandemic. In addition, this study may help in reliving current socioeconomic and psychosocial distress caused by COVID-19 among public in Pakistan

2. Method

The data has been obtained from official reports of National Institute of Health(NIH)- Islamabad, Pakistan[11]. The number of confirmed cases of COVID-19, deaths and recovered people from February 26th to November 20th ,2020. NIH is an autonomous health research institute of the Ministry of National Health Services of Pakistan, situated in Islamabad; its primary responsibilities include biomedical and health related research and vaccine manufacturing15. Since the 1st case of COVID-19 observed in Pakistan, NIH collects and publishes the daily reports on COVID2019 regularly [11]. The published data includes total number of confirmed cases in Pakistan (Province wise), deaths and recovery from COVID-2019. The analysis is based numbers of confirmed cases published online by NIH; therefore, ethical approval is not required for this study. In this paper, the Logistic model, Gompetrz and ARIMA model has been fitted on the confirmed cases of corona virus in Pakistan. Logistic models give similar results as the Gompertz model and ARIMA model.

2.1 Logistic Model

The logistic growth curve, which is also called the Verhulst model as it was first proposed as a model of population growth Pierre Verhulst 1845, 1847) is one of the simplest of the S-shaped growth curves. A generalization of the logistic, which is no longer symmetrical about the point of inflection, was developed by Richard (1959) and terms as generalization of logistic. Assume that the rate of growth of an organism declines with size so that the rate of change in size may be described by:

$$\frac{dl}{dt} = l(k - \delta)$$

where t is time, l is length (size), K is the growth rate and delta a term which expresses the rate at which growth declines with size. After integration and some rearrangement, we arrive at the 3 parameter logistic growth curve:

$$l_t = \frac{L_\infty}{1 + e^{-k(t-I)}}$$

where I is the age at the inflection point and L_∞ is the upper asymptote (maximum size reach after infinite growing time).The 3 parameter logistic has a lower asymptote of 0. The point of inflection on the y-axis occurs at

$$I_y = \frac{L_\infty}{2}$$

This last formula states that the point of inflection is always at at 50 % of the asymptotic size (L_∞).

2.2. Gompers growth Curve:

Actuary B. Gompertz, presenting his law of human mortality, proposed using the exponential function to describe the relationship between increasing death rate and age[14]. Gompertz presented only the probability density function, whilst Makeham stated the well-known cumulative form of this model [15]. In the literature, numerous forms of the Gompertz model can be found, and a detailed review of the Gompertz model was presented by Tjørve [16]. Assume that the rate of growth of an organism declines with size so that the rate of change in length*, l,* (or any other measure of size of weight) may be described by:

$$\frac{dlogl}{dt} = k(\log L_\infty - \log l)$$

where K is the growth rate and L_∞, termed 'L infinity', is the asymptotic length at which growth is zero.Integrating this becomes:

$$l_t = L_\infty e^{e^{-k(t-I)}}$$

where t is age and I is the age at the inflection point. The equation above is the 3 parameter version of the Gompertz growth curve (see below for an example plot). Growth II can also fit the 4 parameter version:

$$l_t = A = Be^{e^{-k(t-I)}}$$

in which A is the lower asymptote (see below for an example plot) and B is the upper asymptote minus A.The point of inflection on the y-axis occurs at

$$I_y = \frac{B}{e}$$

This last formula states that the point of inflection is always at at about 36.8 % of the asymptotic size (L_∞). This does not hold true for all growth processes.

2.3 ARIMA Model:

ARIMA is the most common model in the statistic model. It regards the data sequence formed by the prediction object over time as a random sequence. It analyzes a portion of the data in the sequence to obtain specific parameters that describe the mathematical model of the sequence to achieve time series modeling. And use the remaining data in the sequence to validate the validity of the model. The validated model can be used to predict the subsequent values of the data series.

According to whether the data show evidence of stationary in the different parts of regression, the ARIMA model has three basic types: moving average (MA) model, autoregressive (AR) model, and autoregressive integral moving average model (ARIMA). A non-seasonal ARIMA model is generally denoted ARIMA (*p*,*d*,*q*). If the series is the stationary series, ARIMA model can be expressed as:

$$x_t = \sum_{i=1}^{p} a_i x_{t-i} + \sum_{j=1}^{q} \beta_j \epsilon_{t-j}$$

In the formula, p means the autoregressive parameter, q means the moving average order, x_t means the model parameter to be estimated. The autocorrelation functions (ACF) and partial autocorrelation functions (PACF) and their graphs has been used to determine the parameter values of the model. If the PACF is truncated and the ACF is trailing, then the stationary sequence fits the autoregressive model AR. If the PACF is trailing and the ACF is truncated, the stationary sequence fits the moving average model RA. In other cases, the ARIMA model is recommended.

3. Data Analysis:

The data has been collected from NCOC the logistic growth model, Gompert Growth curve and classic susceptible-infected-recovered dynamic model are used to estimate the final size of the second phase of coronavirus pandemic.In the mid of December 2019 corona virus has started and became a global pandemic. It has spread to the countries and causes fatalities. In Pakistan, the total number of infected cases till the month of September 2020 are more than 300,000.

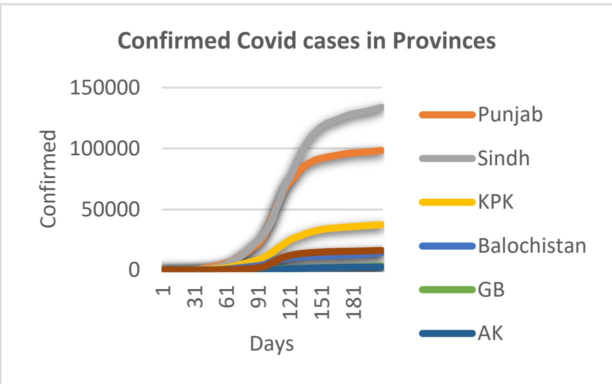

Figure 1 Graphical representation of the confirmed cases in Pakistan province vise.

Graph in Figure 1 shows that the number of confirmed cases in Pakistan provinces wise. This figure presents the actual values for new reported cases nationally Punjab, Sindh, KPK, Balochistan, Gilgit Baltistan, Ajaz Kashmir, Islamabad.

Graph in fig 2 shows that the number of confirmed cases, number of deaths and recovered cases in Pakistan.

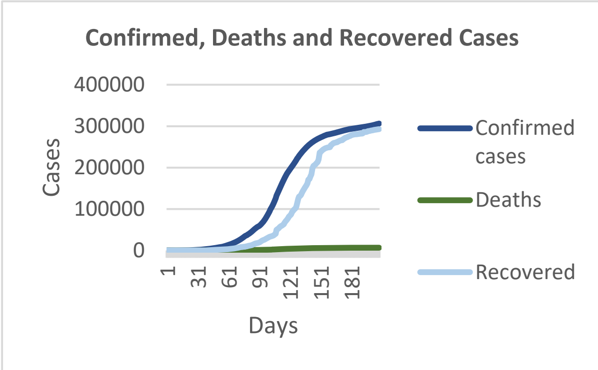

Figure 2 Graphical representation of the confirmed, deaths and recovered cases in Pakistan

3.1 Logistic Model:

The logistic model is a data-driven model. The data which grows exponentially can be fitted by logistic growth model, such as covid-19 data. The numeric data used for fitting the models. The dependent variable y is the number of confirmed cases and the independent variable x is the number of days since the first case appeared. The standard error of the estimated parameters has been calculated .In reality, the parameters are correlated, as well as the data points, so we should work with the entire covariance, but we will leave that for another article in the future. As a final step, we store the parameters and the errors.

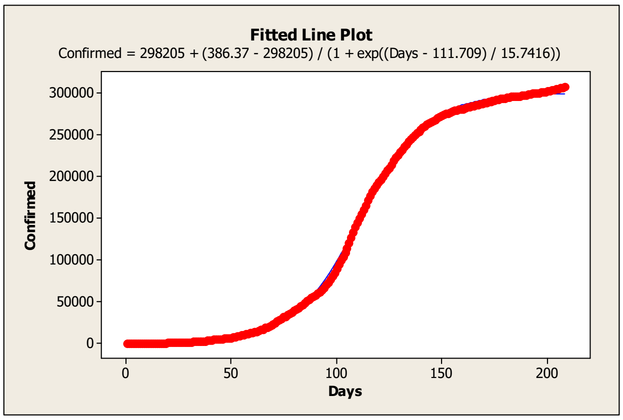

Figure 3 Fitted model graphical representation of the confirmed cases in Pakistan

Graphical representation of the fitted model. The estimated logistic growth function and actual values for new reported cases nationally. Estimated and actual values for cumulative cases nationally are represented. The evolution of the model with these parameters is depicted in figure 3. The first case was reported on February 26, 2020, and spans for 210 days. Moreover, figure 3 displays the number of active cases per day obtained from the model along with the reported number of cases obtained from data. The evolution of the number of active cases per day can be equivalently described by the time-varying logistic equation. It is concluded from figure 3 that both systems are equivalent so that the time-varying logistic equation is a single equation that appropriately describes the number of active cases per day.

$$\mathbf{Confirmed} = \mathbf{298205} + (\mathbf{386.37} - \mathbf{298205}) / (\mathbf{1} + \mathbf{exp}((\mathbf{Days} - \mathbf{111.709}) / \mathbf{15.7416}))$$

Equation above using estimated parameters from logistic model. Confirmed cases analysis due to pandemic in Pakistan up to 20th November 2020. Red curve demonstrate the predicted more infected cases via relating the model and blue curve demonstrate the authentic figure of confirmed cases of Covid-19.

Parameter	Estimate	SE
Theta1	298205	518.704
Theta2	386	448.504
Theta3	112	0.143
Theta4	16	0.13

Table 1. Estimated logistic model parameters for confirmed cases.

The estimation of parameters of the logistic model by using the MLE method depending on Wald statistic for the final model is showed in table 1 which gives the results of fitting the logistic model to Covid-19 data and showing coefficients, which are used in the equation for making the classifications. The standard error of the estimated values are also given.

R^2	MSLE
0.715	1.036

Table 2. R^2 and Mean Square Log Error of model

To test the significance of model and parameters, we are using MSLE and R^2. A model summary of the logistic model is presented in Table 2. It could be observed that the model has a relatively larger R^2 of 0.715 and 1.036 for MSLE. That is, the fitted model can explain or account for 71.5% of the variation in the dependent variable. This is an indication of a good model.The MSLE (mean square logarithmic error) is better to use in terms of assessment of epidemics. The MSLE gives us a rough idea of the actual errors. The R^2 also known as coefficient of determination, is a common metric to test the goodness of fit of a model. A value very close to 1, means that the model can predict very well the data, while a value close to zero or even negative, means that the model is a very bad fit. The numeric data used for fitting the models.

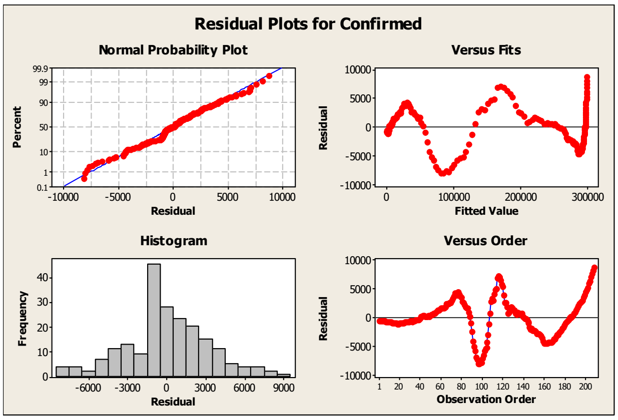

Figure 4 Residual Analysis of fitted model

Residual analysis of fitted model. The estimated logistic growth function and actual values for new reported cases nationally. When integration occurs, one needs to add some constant C, and therefore this study added a constant that was the difference between the mean of all observed data points and that of all predicted data points.

3.2 Gompertz Model:

The gompertz model is a data-driven model. The data which grows exponentially can be fitted by logistic growth model, such as covid-19 data. The numeric data used for fitting the models. The dependent variable y is the number of confirmed cases and the independent variable x is the number of days since the first case appeared. The standard error of the estimated parameters has been calculated .In reality, the parameters are correlated, as well as the data points, so we should work with the entire covariance, but we will leave that for another article in the future. As a final step, we store the parameters and the errors.

Graphical representation of the fitted model. The estimated gompertz function and actual values for new reported cases nationally. Estimated and actual values for cumulative cases nationally are represented. The evolution of the model with these parameters is depicted in figure 5. The first case was reported on February 26, 2020, and spans for 210 days. Moreover, figure 5 displays the number of active cases per day obtained from the model along with the reported number of cases obtained from data.It is concluded from figure 5 that gompertz equation is a single equation that not appropriately describes the number of active cases per day.

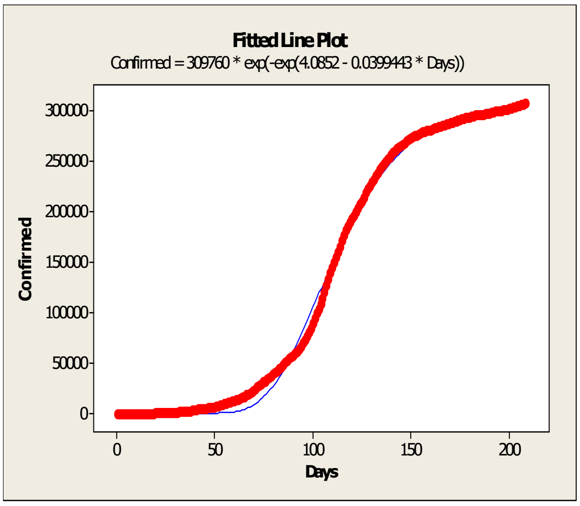

Figure 5 Fitted model graphical representation of the confirmed cases in Pakistan

$$\textbf{Confirmed} = 309760 * \exp(-\exp(4.0852 - 0.0399443 * \textbf{Days}))$$

Equation above using estimated parameters from logistic model. Confirmed cases analysis due to pandemic in Pakistan up to 20th November 2020. Red curve demonstrate the predicted more infected cases via relating the model and blue curve demonstrate the authentic figure of confirmed cases of Covid-19.

Parameter	Estimate	SE
Theta1	309760	1441.4
Theta2	4	0.07
Theta3	0.0399	0

Table 3. Estimated logistic model parameters for confirmed cases.

The estimation of parameters of the gompertz model by using the MLE method depending on Wald statistic for the final model is showed in table 3 which gives the results of fitting the logistic model to Covid-19 data and showing coefficients, which are used in the equation for making the classifications. The standard error of the estimated values are also given.

R^2	MSLE
0.640	0.634

Table 4. R^2 and Mean Square Log Error of model

To test the significance of model and parameters, we are using MSLE and R^2. A model summary of the gompertz model is presented in table 4. It could be observed that the model has a relatively larger R^2 of 0.640 and 0.634 for MSLE. That is, the fitted model can explain or account for 64.0% of the variation in the dependent variable. The MSLE (mean square logarithmic error) is better to use in terms of assessment of epidemics. The MSLE gives us a rough idea of the actual errors. The R^2 also known as coefficient of determination, is a common metric to test the goodness of fit of a model. A value very close to 1, means that the model can predict very well the data, while a value close to zero or even negative, means that the model is a very bad fit. It is above average type of model.

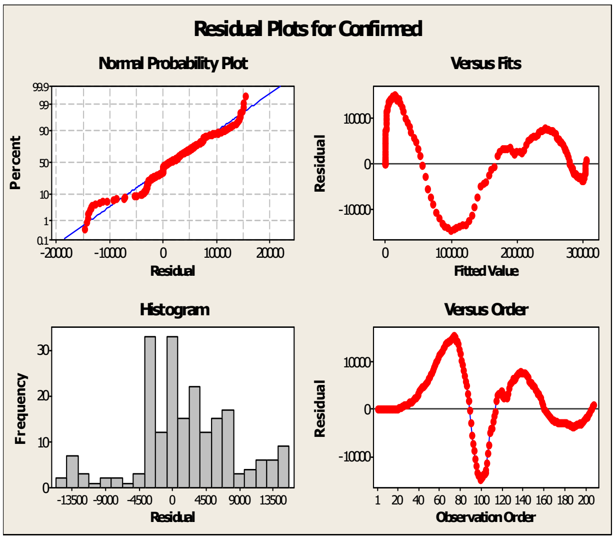

Figure 6 Residual analysis of model fitted.

Residual analysis of fitted model. The estimated gompertz function and actual values for new reported cases nationally. When integration occurs, one needs to add some constant C, and therefore this study added a constant that was the difference between the mean of all observed data points and that of all predicted data points.

3.3 ARIMA Model:

The data has ups and downs due to uncertainty, as cases were reported day by day. The confirmed number of cases per day data has been used to fit by the model. The cumulative number of COVID-19 confirmed cases are expected to show exponential growth over time. Therefore, we used the simple time series methods of Auto-Regressive Integrated Moving Average (ARIMA) Model [16] to model the number of cases every day. The ARIMA model has higher fitting and forecasting accuracy than exponential smoothing [17].It captures both the seasonal and non-seasonal forecasting trends. Due to the limited available data, we simply focus on non-seasonal models to describes the pattern (growth) over time. Hence, we assumed that the pattern of current cases will continue in the near future (20 observations). We believe that the ARIMA model, which is the combination of Autoregressive (AR) and Moving Average (MA) fits well to the nature of the available data and provide good forecasting for the short time series data. The forecasting and prediction intervals until the end of November 2020 is produced from the fitted model. In order to assess the model fit, parameters (p, d, q) are identified by Autocorrelation function (ACF) and Partial Autocorrelation function (PACF); whereas, p is the autoregressive term, d is the differencing order and q is the moving averages term. Furthermore, ARIMA (p, d, q) results are based upon Akaike information criterion (AIC) which is a goodness of fit test such that model with minimum AIC is considered best. All statistical analyses were conducted using the Exploratory software [18] for fitting ARIMA model.

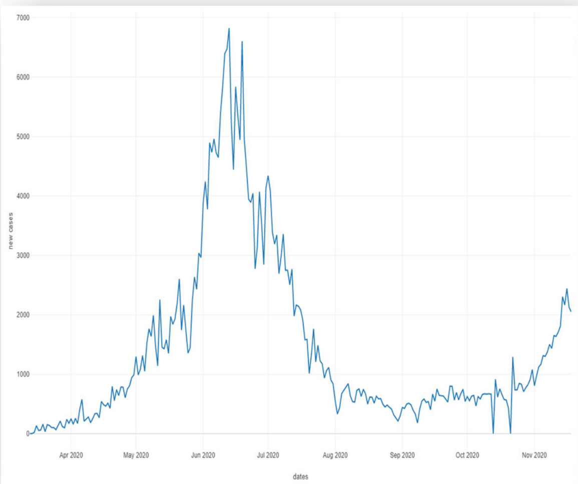

Figure 7 Confirmed cases in Pakistan

Using 210 days data from 26 February 2020 – 20^{h} November 2020 and ARIMA model, we forecasted the data up to 20 days ahead. As, we were dealing with time series and non-stationary data, it is observed that mean and variance of data is variable in nature. The trend in the data is shown in the figure 8. ARIMA model is projected only after converting the series of observations (sample data) considered for forecasting into a stationary series. The stationary series imply the observations differ through time just about a constant variance and a constant mean. First it is ascertained whether the series of observations of the sample data are stationary or not, observing from figure 7, it is obvious that the series is not stationary. Non-stationary in mean is changed by proper differencing order of the data series under consideration.

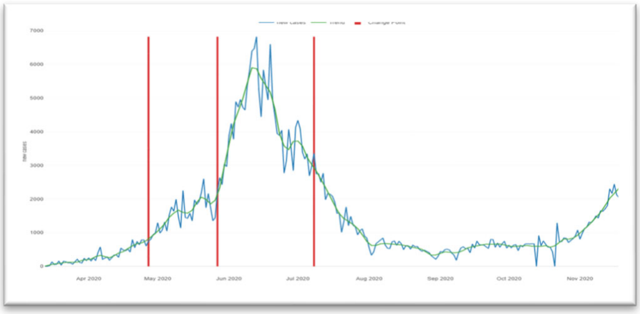

Figure 8 Trend of the Covid-19 confirmed cases in Pakistan.

From figure 9, it is obvious that the series at first difference is stationary. The stationarity of the series is examined through unit root test , the ADF (Augemented Dickey Fuller Unit Root Test) is obtained here. The estimated ADF test results are statistically significant having p=0.01 value. From the value it is inferred that the series of observations when converted into first difference is stationary. So d=1 for our ARIMA model (p,d,q) is adopted.

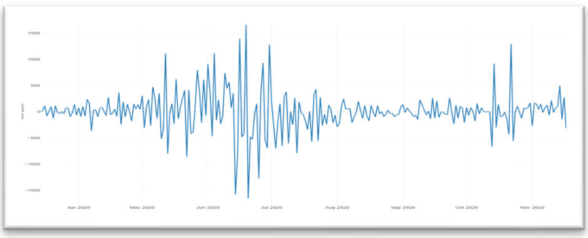

Figure 9 Trend of the Covid-19 confirmed cases in Pakistan.

Using Unit root test: Augmented Dickey-Fuller Test for stationarity of the data: p=0.01 significant for the data. ARIMA (2,1,2) has been applied on the number of confirmed cases on the basis of AIC with the number of days as shown in figure 8. Results from the model revealed that the number of confirmed cases show a rapid exponential growth which may increase by 6.5 times compared to current cases until end of November 2020. The 95% prediction interval for confirmed cases is from 5681 to 33079 which are at much higher growth as shown in figure 8.

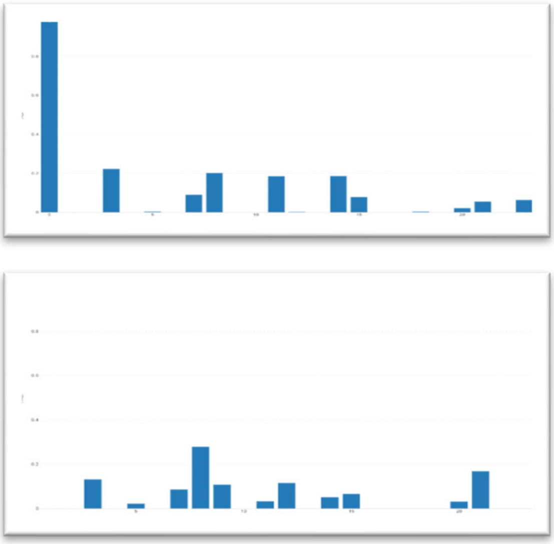

Figure 10 ACF and PACF plot of confirmed cases in Pakistan

From figure 8, it is found that the Auto Correlation Function is tailing off and Partial Auto Correlation Function cutoff with the order 1. By iterative process, various ARIMA models were fitted, the model with minimum normalized BIC values and AIC values are chosen. The model ARIMA (2,1,2) has the lowest AIC and BIC criteria, since it has two parameters following the principle of parsimony, this model is chosen, table 7 reports all the performance parameters of the ARIMA model.

RMSE	MAE	AIC	BIC	AICc
3.41615	217.09	3,642.13	3,666.78	3,642.60

Table 5. ARIMA model performance parameters

The models has been used for the prediction of the values x which is the number of days since first case appeared and also test the model and parameters significance using RMSE, MAE, AIC , BIC and AICc. . The goodness fit is observed through the small values of the RMSE. As evident from the results the ARIMA stands as the utmost best fit model developed. The fitting result shows that the ARIMA is the best forecasting model, table 5. The graphical representation of forecasting based on the model is found in Figure 11.

After performing the diagnostic checking, the task is to forecast ARIMA (2,1,2) model. Forecasting can imply making predictions for the observations of defined period. The predicted values are utilized to test the robustness of the model built and the forecasts more values. The specified period forecasting indicate that the number of cases will be rising as evident from daily observation.

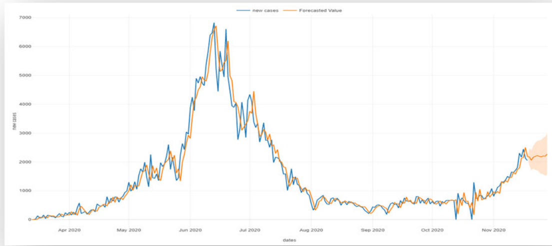

Figure 11 ARIMA model fitting on the confirmed cases in Pakistan.

The orange line with the shaded area show the 95% confidence interval of the predicted value at a given time (t). The results of ARIMA (2,1,2) model forecasting is compared with RMSE of the models. The goodness fit is observed through the small values of the RMSE. As evident from the results the ARIMA stands as the utmost best fit model developed. The fitting result shows that the ARIMA is the best forecasting model, Table 5.

5. Conclusion:

In this paper, we have compared three different growth curve models and time series model for the new cases reported due to COVID-19 in Pakistan. The growth models are good even in the early stages of the epidemic, which is generally said to increase exponentially and monotonically. This method of modelling has also been suggested in the situations where an estimation of the possible date of flattening the curve of the cases of infected individuals [32]. The time based model was also applied to the log of new cases that accurately predict the trajectory of the epidemic in China [31]. The results of this study revealed that the most fitted model for the data is ARIMA model and it shows that the three-fold increase in the number of confirmed cases at the end of December if the current trend continues. However, the number of confirmed cases is increasing in Pakistan at higher rate as compared to the number of recoveries as the disease is spreading to a wider region of the country. The modeling strategies is based on current trends and non-seasonal time series variations, assuming the data is accurate, and the trends will continue in the upcoming month of December. We used statistical approaches AIC [27] for model assessment and selection. Similar kind of forecasts has been conducted by other researchers such as [10] but with slightly different method of the fitting models and presentation of the results is not delivered in a simple way to be understood to a layman, while others are now planning to conduct or continue to conduct such a kind of analysis [19] . It is important to note that majority of the research studies are modeling the preparedness scenarios to inform planning rather predictions [20] that is they informed the actions to be taken to slow the spread and prepare the health system to respond to the pandemic. As Pakistan is a developing country, having a lack of medical facilities which resultantly is affecting the situation further no vaccine or medicine is developed yet to prevent or cure the COVID- 19 pandemic permanently. The public health officials and government should take hard decisions to control the rapid increase of the COVID-19. Besides officials, the public should keep social distancing and use precautions to ensure their safety and control the disease from further spreading.

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