Subdivision of Certain Barbell Operation of Origami Graphs has Locating-Chromatic Number Five

Agus Irawan[1*], Asmiati[2], La Zakaria[2], Kurnia Muludi[3] and Bernadhita Herindri Samodra Utami[1],
[1]Information System, STMIK Pringsewu, Jl. Wismarini No. 09 Pringsewu, Lampung, Indonesia
[2]Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Lampung, Jl. Brodjonegoro No.1 Bandar Lampung, Indonesia.
[3]Computer Sciences, Faculty of Mathematics and Natural Sciences, University of Lampung, Jl. Brodjonegoro No.1 Bandar Lampung, Indonesia.
[*]Coreponding author

Abstract
The locating-chromatic number denote by $\chi_L(G)$, is the smallest t such that G has a locating t-coloring. In this research, we determined locating-chromatic number for subdivision of certain barbell operation of origami graphs.
Key words:
locating-chromatic number, sudivision, certain barbell operation, origami graphs.

1. Introduction

The concept of partition dimension was introduced by Chartrand et al. [1] as the development of the concept of metric dimension. The application of metric dimension can be found in robotic navigation [2], chemical data classification [3], and the optimization of threat detecting sensors [4]. The locating-chromatic number was first discovered by Chartrand et al. [5] in 2002, with obtained two graph concepts, coloring vertices and partition dimension of a graph. The locating-chromatic number denote by $\chi_L(G)$, is the smallest t such that G has a locating t-coloring. Next, investigated the locating-chromatic number for a path graph P_n, a cycle graph C_n, and double star graph $S_{a,b}$. Furthermore, Chartrand et al. [6] characterized all graphs of order n with locating-chromatic number $n-1$. Baskoro and Asmiati [7] characterized all trees with locating-chromatic number 3.

The locating-chromatic number of the join of graphs was introduced by Behtoei and Anbarloei [8]. Purwasih et al. [9], obtained locating-chromatic number for a subdivision of a graph on one edge. For graph with dominant vertices have been studied in [10]. In [11], Asmiati found the locating-chromatic number of non-homogeneous caterpillar and firecrackers graph, [12] certain barbell graphs $B_{m,n}$ and $B_{P(n,1)}$. In 2019, Irawan et al. [13] obtained the locating-chromatic number for certain operation of generalized Petersen graphs $sP(4,2)$. Furthermore, in [14] determined the locating-chromatic number for $sP(n,1)$, origami graphs [15] and certain barbell origami graphs [16]. The locating-chromatic number of a graph is a newly interesting topic to study because there is no general theorem for determining the locating-chromatic number of any graph. In this research, we specifying about locating-chromatic number for subdivision of certain barbell operation of origami graphs, called $B^s_{O_n}$. This study is a continuation of previous research.

The following definition of the locating-chromatic number of a graph, dominant vertices, origami graph, and certain barbell origami graphs is taken from [5, 17, 18, 16]. We use some theorems that is basics to work out a lower bound of the locating-chromatic number of a graph is taken from [5, 15]. The set of neighbours of a vertex l in G, denoted by $N(l)$.

Theorem 1.1. [5] *Let c be a locating coloring in a connected graph G. If k and l are distinct vertices of G such that $d(k,w) = d(l,w)$ for all $w \in V(G) - \{k,l\}$, then $c(k) \neq c(l)$. In particular, if k and l are non-adjacent vertices of Gsuch that $N(k) \neq N(l)$, then $c(k) \neq c(l)$.*

Theorem 1.2. [15] *Let O_n be an origami graph for $n \geq 3$. The locating chromatic number of an origami graphs O_n is 4 for n=3 and 5 otherwise.*

2. Results and Discussion

In this section, we will discuss the locating-chromatic number for subdivision of certain barbell operation of origami graphs, denoted by $B^s_{O_n}$.

Theorem 2.1. *Let $B^s_{O_n}$ be a subdivision of certain barbell operation of origami graphs for $n \geq 3, s \geq 1$. Then the locating-chromatic number of $B^s_{O_n}$ is five, $\chi_L(B^s_{O_n}) = 5$.*

Proof. Let $B^s_{O_n}$ be a subdivision of certain barbell operation of origami graphs for $n \geq 3$, $s \geq 1$, with $V(B^s_{O_n}) = \{u_i, u_{n+i}, v_i, v_{n+i}, w_i, w_{n+i}: i \in \{1, \dots, n\}\} \cup \{x_i:$

Manuscript received September 5, 2021
Manuscript revised September 20, 2021
https://doi.org/10.22937/IJCSNS.2021.21.9.9

$i \in \{1, \dots, s\}\}$ and $E(B_{O_n}^s) = \{u_i w_i, u_i v_i, v_i w_i, u_i u_{i+1}, w_i u_{i+1} \colon i \in \{1, \dots, n\}\} \cup \{u_{n+i} w_{n+i}, u_{n+i} v_{n+i}, v_{n+i}, w_{n+i}, u_{n+i} u_{n+i+1}, w_{n+i} u_{n+i+1} \colon i \in \{1, \dots, n-1\}\} \cup \{u_n x_1, x_s u_{n+1}\} \cup \{x_i x_{i+1} \colon i \in \{1, \dots, s-1\}\}$.

To prove the theorem, we will be divided into two cases :

Cases 1. *For* $n = 3$
First, we determine lower bound of $\chi_L(B_{O_3}^s)$. Since subdivision of certain barbell operation of origami graphs, containing origami graphs O_3, then by Theorem 1.2. $\chi_L(B_{O_3}^s) \geq 4$. Next, we will show that 4 colors are not enough. Origami graph $B_{O_3}^s$ there are six complete graph with four vertices, denote by K_4. Without loss of generality, we assign three colors for any K_4 in $B_{O_3}^s$, and then the six vertices are dominant vertices. As a result, if we use four colors it is not enough because there are more than one K_4 in $B_{O_3}^s$. So $\chi_L(B_{O_3}^s) \geq 5$.

Next, we determined the upper bound of $\chi_L(B_{O_3}^s) \leq 5$. To show that $\chi_L(B_{O_3}^s) \leq 5$, consider the 5-coloring c on $B_{O_3}^s$ as follow,
$C_1 = \{u_1, w_2, u_6, v_5\}$;
$C_2 = \{u_4, w_1, w_5\}$;
$C_3 = \{u_2, v_1, w_3, u_5, v_4, v_6\} \cup \{x_i | \text{for odd } i, i \geq 1\}$;
$C_4 = \{u_3, v_2, w_4, w_6\} \cup \{x_i | \text{for even } i, i \geq 2\}$;
$C_5 = \{v_3\}$;

The coloring c will create partition Π on $V(B_{O_3}^s)$. We shall show that the color codes of all vertices in $B_{O_3}^s$ are diferent. We have $c_\Pi(u_1) = (0, 2, 1, 1, 1)$; $c_\Pi(u_2) = (1, 1, 0, 1, 2)$; $c_\Pi(u_3) = (1, 2, 1, 0, 1)$; $c_\Pi(u_4) = (1, 0, 1, 1, s + 3)$; $c_\Pi(u_5) = (1, 1, 1, 0, s + 4)$; $c_\Pi(u_6) = ((0, 1, 1, 1, s + 4)$; $c_\Pi(v_1) = (1, 3, 2, 0, 1)$; $c_\Pi(v_2) = (1, 3, 0, 1, 2)$; $c_\Pi(v_3) = (2, 0, 1, 1, 3)$; $c_\Pi(v_4) = (2, 1, 1, 0, s + 4)$; $c_\Pi(v_5) = (0, 1, 2, 1, s + 5)$; $c_\Pi(v_6) = (1, 2, 1, 0, s + 5)$; $c_\Pi(w_1) = (1, 3, 2, 1, 0)$; $c_\Pi(w_2) = (0, 2, 1, 1, 2)$; $c_\Pi(w_3) = (1, 1, 1, 0, 2)$; $c_\Pi(w_4) = (2, 1, 0, 1, s + 4)$; $c_\Pi(w_5) = (1, 0, 2, 1, s + 5)$; $c_\Pi(w_6) = (1, 1, 0, 1s + 4)$. For $s = 1$, we have $c_\Pi(x_i) = (i + 1, 1, 1, 0, i + 2)$ .For i odd, $i \leq \lfloor \frac{s}{2} \rfloor$, $s \geq 2$, we have $c_\Pi(x_i) = (i + 1, i + 1, 1, 0, i + 2)$. For i even, $i \leq \lfloor \frac{s}{2} \rfloor$, $s \geq 2$, we have $c_\Pi(x_i) = (i + 1, i + 1, 0, 1, i + 2)$. For i odd, $i > \lfloor \frac{s}{2} \rfloor$, $s \geq 2$, we have $c_\Pi(x_i) = (s - i + 2, s - i + 1, 1, 0, i + 2)$. For i even, $i > \lfloor \frac{s}{2} \rfloor$, $s \geq 2$, we have $c_\Pi(x_i) = (s - i + 2, s - i + 1, 0, 1, i + 2)$.

Since the color codes of all vertices $B_{O_3}^s$ are diferent, thus c is a locating coloring. So $\chi_L(B_{O_3}^s) \leq 5$.

Case 2. *For* $n \geq 4$
First, we determine lower bound of $\chi_L(B_{O_n}^s)$ for $n \geq 4$. Since subdivision of certain barbell operation of origami graphs, containing origami graphs O_n, then by Theorem 1.2 it is clear that $\chi_L(B_{O_n}^s) \geq 5$.

To show the upper bound for the locating-chromatic number for subdivison of certain barbell operation of origami graphs $\chi_L(B_{O_n}^s) \geq 5$ for $n \geq 4$. Let us diferent some subcases.

Subcase 2.1. *(odd n), for* $\lceil \frac{n}{2} \rceil$ *odd,* $n \geq 5$
Let c be a coloring for subdivison of certain barbell operation of origami graph $B_{O_n}^s$, for $\lceil \frac{n}{2} \rceil$ odd, $n \geq 5$ we make the partition Π of $V(B_{O_n}^s)$:
$C_1 = \{w_1 | 1 \leq i \leq n\} \cup \{u_{n+1}\}$;
$C_2 = \{u_i | \text{for odd } i, 3 \leq i \leq n\} \cup \{v_i | \text{for even } i, 2 \leq i \leq n-1\} \cup \{u_{n+i} \; | \text{for odd } i, 3 \leq i \leq \lceil \frac{n}{2} \rceil - 2\} \cup \{u_{n+i} | \text{for odd } i, \lceil \frac{n}{2} \rceil + 2 \leq i \leq n\} \cup \{v_{n+i} | \text{ for even } i, 2 \leq i \leq n-1\} \cup \{x_i | \text{for even } i, i \geq 2\}$;
$C_3 = \{u_i | \text{for even } i, 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1\} \cup \{u_i | \text{for even } i, \lceil \frac{n}{2} \rceil + 3 \leq i \leq n-1\} \cup \{v_i | \text{for odd } i, 1 \leq i \leq n\} \cup \{u_{n+i} | \text{for even } i, 2 \leq i \leq n-1\} \cup \{v_{n+i} | \text{for odd } i, 1 \leq i \leq n\}$;
$C_4 = \{u_1\} \cup \{w_{n+i} | 1 \leq i \leq n\} \cup \{x_i | \text{for odd } i, i \geq 1\}$;
$C_5 = \{u_{\lceil \frac{n}{2} \rceil + 1}\} \cup \{u_{n + \lceil \frac{n}{2} \rceil}\}$.

For $\lceil \frac{n}{2} \rceil$ odd $n \geq 5$, the color codes of all the vertices of $V(B_{O_n}^s)$ are :

$$c_\Pi(u_i) = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 3 \leq i \leq n, n \geq 5 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1, n \geq 5 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, \lceil \frac{n}{2} \rceil + 3 \leq i \leq n-1, n \geq 9 \\ & \text{for } 4^{th} \text{ ordinate, } i = 1 \\ & \text{for } 5^{th} \text{ ordinate, } i = \lceil \frac{n}{2} \rceil + 1 \\ 2, & \text{for } 3^{rd} \text{ ordinate, } i = \lceil \frac{n}{2} \rceil + 1 \\ i - 1, & \text{for } 4^{th} \text{ ordinate, } 2 \leq i \leq \lceil \frac{n}{2} \rceil, n \geq 5 \\ n - i + 1, & \text{for } 4^{th} \text{ ordinate, } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n, n \geq 5 \\ i - \lceil \frac{n}{2} \rceil - 1, & \text{for } 5^{th} \text{ ordinate, } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n, n \geq 5 \\ \lceil \frac{n}{2} \rceil - i + 1, & \text{for } 5^{th} \text{ ordinate, } 2 \leq i \leq \lceil \frac{n}{2} \rceil, n \geq 5 \\ \lceil \frac{n}{2} \rceil - i, & \text{for } 5^{th} \text{ ordinate, } i = 1 \\ 1, & \text{otherwise.} \end{cases}$$

$c_\Pi(v_i) =$

$$\begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \le i \le n-1, n \ge 5 \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, 1 \le i \le n, n \ge 5 \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = 1 \\ 3, & \text{for } 3^{rd} \text{ ordinate, } i = \left\lceil \frac{n}{2} \right\rceil + 1 \\ i, & \text{for } 4^{th} \text{ ordinate, } 2 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 5 \\ n-i+2, & \text{for } 4^{th} \text{ ordinate, } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 5 \\ \left\lceil \frac{n}{2} \right\rceil, & \text{for } 5^{th} \text{ ordinate, } i = 1 \\ \left\lceil \frac{n}{2} \right\rceil - i + 2, & \text{for } 5^{th} \text{ ordinate, } 2 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 5 \\ i - \left\lceil \frac{n}{2} \right\rceil, & \text{for } 5^{th} \text{ ordinate, } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 5 \\ 1, & \text{otherwise.} \end{cases}$$

$c_\Pi(w_i) =$

$$\begin{cases} 0, & \text{for } 1^{st} \text{ ordinate, } 1 \le i \le n, n \ge 5 \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = 1 \\ & \text{for } 3^{rd} \text{ ordinate, } i = \left\lceil \frac{n}{2} \right\rceil \\ i, & \text{for } 4^{th} \text{ ordinate, } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 5 \\ n-i+1, & \text{for } 4^{th} \text{ ordinate, } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 5 \\ \left\lceil \frac{n}{2} \right\rceil - i + 1, & \text{for } 5^{th} \text{ ordinate, } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 5 \\ i - \left\lceil \frac{n}{2} \right\rceil, & \text{for } 5^{th} \text{ ordinate, } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 5 \\ 1, & \text{otherwise.} \end{cases}$$

$c_\Pi(u_{n+i}) =$

$$\begin{cases} i-1, & \text{for } 1^{st} \text{ ordinate, } 2 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 5 \\ n-i+1, & \text{for } 1^{st} \text{ ordinate, } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 5 \\ 0, & \text{for } 1^{st} \text{ ordinate, } i = 1 \\ & \text{for } 2^{nd} \text{ ordinate, odd } i, 3 \le i \le \left\lceil \frac{n}{2} \right\rceil - 2, n \ge 9 \\ & \text{for } 2^{nd} \text{ ordinate, odd } i, \left\lceil \frac{n}{2} \right\rceil + 2 \le i \le n, n \ge 5 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 3 \le i \le n-1, n \ge 5 \\ & \text{for } 5^{th} \text{ ordinate, } i = \left\lceil \frac{n}{2} \right\rceil + 1 \\ \left\lceil \frac{n}{2} \right\rceil - 1, & \text{for } 1^{st} \text{ ordinate, } i = \left\lceil \frac{n}{2} \right\rceil \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = \left\lceil \frac{n}{2} \right\rceil \\ \left\lceil \frac{n}{2} \right\rceil - i, & \text{for } 5^{th} \text{ ordinate, } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1, n \ge 5 \\ i - \left\lceil \frac{n}{2} \right\rceil, & \text{for } 5^{th} \text{ ordinate, } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 5 \\ 1, & \text{otherwise.} \end{cases}$$

$c_\Pi(v_{n+i}) =$

$$\begin{cases} i, & \text{for } 1^{st} \text{ ordinate, } 2 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 5 \\ n-i+2, & \text{for } 1^{st} \text{ ordinate, } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 5 \\ 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \le i \le n-1, n \ge 5 \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, 1 \le i \le n, n \ge 5 \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = 1 \\ 3, & \text{for } 2^{nd} \text{ ordinate, } i = \left\lceil \frac{n}{2} \right\rceil \\ \left\lceil \frac{n}{2} \right\rceil - i + 1, & \text{for } 5^{th} \text{ ordinate, } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 5 \\ i - \left\lceil \frac{n}{2} \right\rceil + 1, & \text{for } 5^{th} \text{ ordinate, } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 5 \\ 1, & \text{otherwise.} \end{cases}$$

$c_\Pi(w_{n+i}) =$

$$\begin{cases} i, & \text{for } 1^{st} \text{ ordinate, } 2 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 5 \\ n-i+1, & \text{for } 1^{st} \text{ ordinate, } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 5 \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = 1 \text{ and } i = \left\lceil \frac{n}{2} \right\rceil \\ 0, & \text{for } 4^{th} \text{ ordinate, } 1 \le i \le n, n \ge 5 \\ \left\lceil \frac{n}{2} \right\rceil - i, & \text{for } 5^{th} \text{ ordinate, } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1, n \ge 5 \\ i - \left\lceil \frac{n}{2} \right\rceil + 1, & \text{for } 5^{th} \text{ ordinate, } \left\lceil \frac{n}{2} \right\rceil \le i \le n, n \ge 5 \\ 1, & \text{otherwise.} \end{cases}$$

$c_\Pi(x_i) =$

$$\begin{cases} s-i+1, & \text{for } 1^{st} \text{ ordinate, } i > \left\lfloor \frac{s}{2} \right\rfloor, s \ge 2 \\ i+1, & \text{for } 1^{st} \text{ ordinate, } i \le \left\lfloor \frac{s}{2} \right\rfloor, s \ge 2 \\ & \text{for } 3^{rd} \text{ ordinate, } i \le \left\lfloor \frac{s}{2} \right\rfloor \\ 1, & \text{for } 1^{st} \text{ ordinate, } i = s \\ & \text{for } 2^{nd} \text{ ordinate, odd } i, i \ge 1 \\ & \text{for } 4^{th} \text{ ordinate, even } i, i \ge 2 \\ s-i+2, & \text{for } 3^{rd} \text{ ordinate, } i > \left\lfloor \frac{s}{2} \right\rfloor \\ i + \left\lceil \frac{n}{2} \right\rceil - 2, & \text{for } 5^{th} \text{ ordinate, } i < \left\lfloor \frac{s}{2} \right\rfloor \\ s - i + \left\lceil \frac{n}{2} \right\rceil + 1, & \text{for } 5^{th} \text{ ordinate, } i \ge \left\lfloor \frac{s}{2} \right\rfloor \\ 0, & \text{otherwise.} \end{cases}$$

Since for odd n all vertices have different color codes, c is a locating coloring for subdivison of certain barbell operation of origami graphs $B_{O_n}^s$, so that $\chi_L(B_{O_n}^s) \le 5$, for $\left\lceil \frac{n}{2} \right\rceil$ odd, $n \ge 5$.

Subcase 2.2. *(odd n), for* $\left\lceil \frac{n}{2} \right\rceil$ *even,* $n \ge 7$

Let c be a coloring for subdivison of certain barbell operation of origami graph $B_{O_n}^s$, for $\left\lceil \frac{n}{2} \right\rceil$ even, $n \ge 7$ we make the partition Π of $V(B_{O_n}^s)$:

$C_1 = \{w_1 | 1 \le i \le n\} \cup \{u_{n+1}\}$;

$C_2 = \{u_i | \text{for odd } i, 3 \le i \le n\} \cup \{v_i | \text{for even } i, 2 \le i \le n-1\} \cup \{u_{n+i} | \text{for even } i, 2 \le i \le \left\lceil \frac{n}{2} \right\rceil - 2\} \cup \{u_{n+i} |$

for even $i, \left\lceil \frac{n}{2} \right\rceil + 2 \le \; i \le n-1\}$ $\cup$ $\{v_{n+i} |$for odd $i, 1 \le i \le n\}$;
$C_3 = \{u_i |$for even $i, 2 \le i \le \left\lceil \frac{n}{2} \right\rceil - 2\} \cup \{u_i |$for even $i, \left\lceil \frac{n}{2} \right\rceil + 2 \le i \le n-1\} \cup \{v_i |$for odd $i, 1 \le i \le n\}$ $\cup$ $\{u_{n+i} |$ for odd $i,\; 3 \le i \le n\}$ $\cup$ $\{v_{n+i} |$for even $i, 2 \le i \le n-1\}$ $\cup \{x_i |$for even $i, i \ge 2\}$;
$C_4 = \{u_1\} \cup \{w_{n+i} | 1 \le i \le n\} \cup \{x_i |$for odd $i, i \ge 1\}$;
$C_5 = \{u_{\lceil \frac{n}{2} \rceil + 1}\} \cup \{u_{n+\lceil \frac{n}{2} \rceil}\}$.

For $\left\lceil \frac{n}{2} \right\rceil$ even $n \ge 7$, the color codes of all the vertices of $V(B^s_{O_n})$ are :

$$c_\Pi(u_i) = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate, odd } i, 3 \le i \le n, n \ge 7 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 2 \le i \le \left\lceil \frac{n}{2} \right\rceil - 2, n \ge 7 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, \left\lceil \frac{n}{2} \right\rceil + 2 \le i \le n-1, n \ge 7 \\ & \text{for } 4^{th} \text{ ordinate}, i = 1 \\ & \text{for } 5^{th} \text{ ordinate}, i = \left\lceil \frac{n}{2} \right\rceil \\ 2, & \text{for } 3^{rd} \text{ ordinate}, i = \left\lceil \frac{n}{2} \right\rceil \\ i-1, & \text{for } 4^{th} \text{ ordinate}, 2 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 7 \\ n-i+1, & \text{for } 4^{th} \text{ ordinate}, \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 7 \\ i - \left\lceil \frac{n}{2} \right\rceil, & \text{for } 5^{th} \text{ ordinate}, \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 7 \\ \left\lceil \frac{n}{2} \right\rceil - i, & \text{for } 5^{th} \text{ ordinate}, 1 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1, n \ge 7 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\Pi(v_i) = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \le i \le n-1, n \ge 7 \\ & \text{for } 3^{rd} \text{ ordinate odd } i, 1 \le i \le n, n \ge 7 \\ 2, & \text{for } 2^{nd} \text{ ordinate}, i = 1 \\ 3, & \text{for } 3^{rd} \text{ ordinate}, i = \left\lceil \frac{n}{2} \right\rceil \\ i, & \text{for } 4^{th} \text{ ordinate}, 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 7 \\ n-i+2, & \text{for } 4^{th} \text{ ordinate}, \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 7 \\ \left\lceil \frac{n}{2} \right\rceil - i + 1, & \text{for } 5^{th} \text{ ordinate}, 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 7 \\ i - \left\lceil \frac{n}{2} \right\rceil + 1, & \text{for } 5^{th} \text{ ordinate}, \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 7 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\Pi(w_i) = \begin{cases} 0, & \text{for } 1^{st} \text{ ordinate}, 1 \le i \le n, n \ge 7 \\ 2, & \text{for } 2^{nd} \text{ ordinate}, i = 1 \\ & \text{for } 3^{rd} \text{ ordinate}, i = \left\lceil \frac{n}{2} \right\rceil \\ i, & \text{for } 4^{th} \text{ ordinate}, 2 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 7 \\ n-i+1, & \text{for } 4^{th} \text{ ordinate}, \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 7 \\ \left\lceil \frac{n}{2} \right\rceil - i, & \text{for } 5^{th} \text{ ordinate}, 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 7 \\ i - \left\lceil \frac{n}{2} \right\rceil + 1, & \text{for } 5^{th} \text{ ordinate}, \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 7 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\Pi(u_{n+i}) = \begin{cases} i-1, & \text{for } 1^{st} \text{ ordinate}, 2 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 7 \\ n-i+1, & \text{for } 1^{st} \text{ ordinate}, \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 7 \\ 0, & \text{for } 1^{st} \text{ ordinate}, i = 1 \\ & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \le i \le \left\lceil \frac{n}{2} \right\rceil - 2, n \ge 7 \\ & \text{for } 2^{nd} \text{ ordinate, even } i, \left\lceil \frac{n}{2} \right\rceil + 2 \le i \le n-1, n \ge 7 \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, 3 \le i \le n, n \ge 7 \\ & \text{for } 5^{th} \text{ ordinate}, i = \left\lceil \frac{n}{2} \right\rceil \\ 2, & \text{for } 2^{nd} \text{ ordinate}, i = \left\lceil \frac{n}{2} \right\rceil \\ \left\lceil \frac{n}{2} \right\rceil - i, & \text{for } 5^{th} \text{ ordinate}, 1 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1, n \ge 7 \\ i - \left\lceil \frac{n}{2} \right\rceil, & \text{for } 5^{th} \text{ ordinate}, \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 7 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\Pi(v_{n+i}) = \begin{cases} i, & \text{for } 1^{st} \text{ ordinate}, 2 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1, n \ge 7 \\ n-i+2, & \text{for } 1^{st} \text{ ordinate}, \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 7 \\ 0, & \text{for } 2^{nd} \text{ordinate, odd } i, 1 \le i \le n, n \ge 7 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 2 \le i \le n-1, n \ge 7 \\ 2, & \text{for } 3^{rd} \text{ ordinate}, i = 1 \\ 3, & \text{for } 2^{nd} \text{ ordinate}, i = \left\lceil \frac{n}{2} \right\rceil \\ \left\lceil \frac{n}{2} \right\rceil - i + 1, & \text{for } 5^{th} \text{ ordinate}, 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 7 \\ i - \left\lceil \frac{n}{2} \right\rceil + 1, & \text{for } 5^{th} \text{ ordinate}, \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 7 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\Pi(w_{n+i}) = \begin{cases} i, & \text{for } 1^{st} \text{ ordinate}, 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, n \ge 7 \\ n-i+1, & \text{for } 1^{st} \text{ ordinate}, \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, n \ge 7 \\ 2, & \text{for } 2^{nd} \text{ ordinate}, i = \left\lceil \frac{n}{2} \right\rceil \\ & \text{for } 3^{rd} \text{ ordinate}, i = 1 \\ 0, & \text{for } 4^{th} \text{ ordinate}, 1 \le i \le n, n \ge 7 \\ \left\lceil \frac{n}{2} \right\rceil - i, & \text{for } 5^{th} \text{ ordinate}, 1 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1, n \ge 7 \\ i - \left\lceil \frac{n}{2} \right\rceil + 1, & \text{for } 5^{th} \text{ ordinate}, \left\lceil \frac{n}{2} \right\rceil \le i \le n, n \ge 7 \\ 1, & \text{otherwise.} \end{cases}$$

$c_\Pi(x_i) =$

$$\begin{cases} s-i+1, & \text{for } 1^{st} \text{ ordinate}, i > \left\lfloor \frac{s}{2} \right\rfloor, s \geq 2 \\ i+1, & \text{for } 1^{st} \text{ ordinate}, i \leq \left\lfloor \frac{s}{2} \right\rfloor, s \geq 2 \\ 1, & \text{for } 1^{st} \text{ ordinate}, i = s \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, i \geq 1 \\ & \text{for } 4^{th} \text{ ordinate, even } i, i \geq 2 \\ i, & \text{for } 2^{nd} \text{ ordinate}, i \leq \left\lceil \frac{s}{2} \right\rceil \\ s-i+2, & \text{for } 2^{nd} \text{ ordinate}, i > \left\lceil \frac{s}{2} \right\rceil \\ 2, & \text{for } 3^{rd} \text{ ordinate}, s = 1 \\ i + \left\lceil \frac{n}{2} \right\rceil - 2, & \text{for } 5^{th} \text{ ordinate}, i \leq \left\lceil \frac{s}{2} \right\rceil \\ s - i + \left\lceil \frac{n}{2} \right\rceil, & \text{for } 5^{th} \text{ ordinate}, i > \left\lceil \frac{s}{2} \right\rceil \\ 0, & \text{otherwise.} \end{cases}$$

Since for odd n all vertices have different color codes, c is a locating coloring for subdivison of certain barbell operation of origami graphs $B^s_{O_n}$, so that $\chi_L(B^s_{O_n}) \leq 5$, for $\left\lceil \frac{n}{2} \right\rceil$ even, $n \geq 7$.

Subcase 2.3. *(even n), for $\frac{n}{2}$ odd, $n \geq 6$*

Let c be a coloring for subdivison of certain barbell operation of origami graph $B^s_{O_n}$, for $\frac{n}{2}$ odd, $n \geq 6$ we make the partition Π of $V(B^s_{O_n})$:

$C_1 = \{w_i | 1 \leq i \leq \frac{n}{2} - 1\} \cup \{w_i | \frac{n}{2} + 1 \leq i \leq n\} \cup \{u_{n+1}\}$;

$C_2 = \{u_i | \text{for odd } i, 3 \leq i \leq n-1\} \cup \{v_i | \text{for even } i, 2 \leq i \leq n\} \cup \{u_{n+i} | \text{for even } i, 2 \leq i \leq n\} \cup \{v_{n+i} | \text{for odd } i, 1 \leq i \leq n-1\}$;

$C_3 = \{u_i | \text{for even } i, 2 \leq i \leq n\} \cup \{v_i | \text{for odd } i, 3 \leq i \leq n-1\} \cup \{u_{n+i} | \text{for odd } i, 3 \leq i \leq n-1\} \cup \{v_{n+i} | \text{ for even } i, 2 \leq i \leq n\} \cup \{x_i | \text{for odd } i, i \geq 1\}$;

$C_4 = \{u_1\} \cup \{w_{n+i} | 1 \leq i \leq \frac{n}{2} - 1\} \cup \{w_{n+i} | \frac{n}{2} + 1 \leq i \leq n\} \cup \{x_i | \text{for odd } i, i \geq 2\}$;

$C_5 = \{w_{\frac{n}{2}}\} \cup \{w_{n+\frac{n}{2}}\}$.

For $\frac{n}{2}$ odd $n \geq 6$, the color codes of all the vertices of $V(B^s_{O_n})$ are :

$c_\Pi(u_i) =$

$$\begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate, odd } i, 3 \leq i \leq n-1, n \geq 6 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 2 \leq i \leq n, n \geq 6 \\ & \text{for } 4^{th} \text{ ordinate}, i = 1 \\ 2, & \text{for } 3^{rd} \text{ ordinate}, i = 1 \\ i-1, & \text{for } 4^{th} \text{ ordinate}, 2 \leq i \leq \frac{n}{2}, n \geq 6 \\ n-i+1, & \text{for } 4^{th} \text{ ordinate}, \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ \frac{n}{2} - i + 1, & \text{for } 5^{th} \text{ ordinate}, 1 \leq i \leq \frac{n}{2}, n \geq 6 \\ i - \frac{n}{2}, & \text{for } 5^{th} \text{ ordinate}, \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ 1, & \text{otherwise.} \end{cases}$$

$c_\Pi(v_i) =$

$$\begin{cases} 3, & \text{for } 2^{nd} \text{ ordinate}, i = 1 \\ 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \leq i \leq n, n \geq 6 \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, 1 \leq i \leq n-1, n \geq 6 \\ i, & \text{for } 4^{th} \text{ ordinate}, 1 \leq i \leq \frac{n}{2}, n \geq 6 \\ n-i+2, & \text{for } 4^{th} \text{ ordinate}, \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ \frac{n}{2} - i + 2, & \text{for } 5^{th} \text{ ordinate}, 1 \leq i \leq \frac{n}{2} - 1, n \geq 6 \\ i - \frac{n}{2} + 1, & \text{for } 5^{th} \text{ ordinate}, \frac{n}{2} \leq i \leq n, n \geq 6 \\ 1, & \text{otherwise.} \end{cases}$$

$c_\Pi(w_i) =$

$$\begin{cases} 0, & \text{for } 1^{st} \text{ ordinate}, 1 \leq i \leq \frac{n}{2} - 1, n \geq 6 \\ & \text{for } 1^{st} \text{ ordinate}, \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ & \text{for } 5^{th} \text{ ordinate}, i = \frac{n}{2} \\ 2, & \text{for } 2^{nd} \text{ ordinate}, i = 1 \\ i, & \text{for } 4^{th} \text{ ordinate}, 1 \leq i \leq \frac{n}{2}, n \geq 6 \\ n-i+1, & \text{for } 4^{th} \text{ ordinate}, \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ \frac{n}{2} - i + 1, & \text{for } 5^{th} \text{ ordinate}, 1 \leq i \leq \frac{n}{2} - 1, n \geq 6 \\ i - \frac{n}{2} + 1, & \text{for } 5^{th} \text{ ordinate}, \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ 1, & \text{otherwise.} \end{cases}$$

$c_\Pi(u_{n+i}) =$

$$\begin{cases} i-1, & \text{for } 1^{st} \text{ ordinate}, 2 \leq i \leq \frac{n}{2}, n \geq 6 \\ n-i+1, & \text{for } 1^{st} \text{ ordinate}, \frac{n}{2} \leq i \leq n, n \geq 6 \\ 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \leq i \leq n, n \geq 6 \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, 3 \leq i \leq n-1, n \geq 6 \\ & \text{for } 1^{st} \text{ ordinate}, i = 1 \\ 2, & \text{for } 3^{rd} \text{ ordinate}, i = 1 \\ \frac{n}{2} - i + 1, & \text{for } 5^{th} \text{ ordinate}, 1 \leq i \leq \frac{n}{2}, n \geq 6 \\ i - \frac{n}{2}, & \text{for } 5^{th} \text{ ordinate}, \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ 1, & \text{otherwise.} \end{cases}$$

$c_\Pi(v_{n+i}) =$

$$\begin{cases} i, & \text{for } 1^{st} \text{ ordinate}, 2 \leq i \leq \frac{n}{2}, n \geq 6 \\ n-i+2, & \text{for } 1^{st} \text{ ordinate}, \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd } i, 1 \leq i \leq n-1, n \geq 6 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 2 \leq i \leq n, n \geq 6 \\ 3, & \text{for } 2^{nd} \text{ ordinate}, i = 1 \\ \frac{n}{2} - i + 1, & \text{for } 5^{th} \text{ ordinate}, 1 \leq i \leq \frac{n}{2} - 1, n \geq 6 \\ i - \frac{n}{2} + 1, & \text{for } 5^{th} \text{ ordinate}, \frac{n}{2} \leq i \leq n, n \geq 6 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\Pi(w_{n+i}) = \begin{cases} i, & \text{for } 1^{st} \text{ ordinate}, 1 \le i \le \frac{n}{2}, n \ge 6 \\ n-i+1, & \text{for } 1^{st} \text{ ordinate}, \frac{n}{2}+1 \le i \le n, n \ge 6 \\ 2, & \text{for } 2^{nd} \text{ ordinate}, i = 1 \\ 0, & \text{for } 4^{th} \text{ ordinate}, 1 \le i \le \frac{n}{2}-1, n \ge 6 \\ & \text{for } 4^{th} \text{ ordinate}, \frac{n}{2}+1 \le i \le n, n \ge 6 \\ & \text{for } 5^{th} \text{ ordinate}, \text{i} = \frac{n}{2} \\ \frac{n}{2}-i+1, & \text{for } 5^{th} \text{ ordinate}, 1 \le i \le \frac{n}{2}-1, n \ge 6 \\ i-\frac{n}{2}+1, & \text{for } 5^{th} \text{ ordinate}, \frac{n}{2}+1 \le i \le n, n \ge 6 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\Pi(x_i) = \begin{cases} s-i+1, & \text{for } 1^{st} \text{ ordinate}, i > \left\lfloor \frac{s}{2} \right\rfloor, s \ge 2 \\ i+1, & \text{for } 1^{st} \text{ ordinate}, i \le \left\lfloor \frac{s}{2} \right\rfloor, s \ge 2 \\ 0, & \text{for } 3^{rd} \text{ ordinate, even } i, i \ge 2 \\ & \text{for } 4^{th} \text{ ordinate, odd } i, i \ge 1 \\ i, & \text{for } 2^{nd} \text{ ordinate}, i \le \left\lceil \frac{s}{2} \right\rceil, s \ge 2 \\ s-i+2 & \text{for } 2^{nd} \text{ ordinate}, i > \left\lceil \frac{s}{2} \right\rceil, s \ge 2 \\ i+\frac{n}{2}-1 & \text{for } 5^{th} \text{ ordinate}, i \le \left\lceil \frac{s}{2} \right\rceil, s \ge 2 \\ s-i+\frac{n}{2} & \text{for } 5^{th} \text{ ordinate}, i > \left\lceil \frac{s}{2} \right\rceil, s \ge 2 \\ 1, & \text{otherwise.} \end{cases}$$

Since for odd n all vertices have different color codes, c is a locating coloring for subdivison of certain barbell operation of origami graphs $B^s_{O_n}$, so that $\chi_L(B^s_{O_n}) \le 5$, for $\frac{n}{2}$ odd , $n \ge 6$.

Subcase 2.4. *(even n), for $\frac{n}{2}$ even, $n \ge 4$*

Let c be a coloring for subdivison of certain barbell operation of origami graph $B^s_{O_n}$, $\frac{n}{2}$ even, $n \ge 4$ we make the partition Π of $V(B^s_{O_n})$:

$C_1 = \{w_i | 1 \le i \le \frac{n}{2}-1\} \cup \{w_i | \frac{n}{2}+1 \le i \le n\} \cup \{u_{n+1}\}$;

$C_2 = \{u_i | \text{for odd } i, 1 \le i \le n-1\} \cup \{v_i | \text{for even } i, 2 \le i \le n\} \cup \{u_{n+i} | \text{for odd } i, 3 \le i \le n\} \cup \{v_{n+i} | \text{for even } i, 1 \le i \le n-1\}$;

$C_3 = \{u_i | \text{for even } i, 2 \le i \le n-2\} \cup \{v_i | \text{for odd } i, 1 \le i \le n-1\} \cup \{u_{n+i} | \text{for even } i, 3 \le i \le n-1\} \cup \{v_{n+i} | \text{for odd } i, 2 \le i \le n\} \cup \{x_i | \text{for odd } i, i \ge 1\}$;

$C_4 = \{u_n\} \cup \{w_{n+i} | 1 \le i \le \frac{n}{2}\} \cup \{w_{n+i} | \frac{n}{2}+2 \le i \le n\} \cup \{x_i | \text{for even } i, i \ge 2\}$;

$C_5 = \{w_{\frac{n}{2}}\} \cup \{w_{n+\frac{n}{2}+1}\}$.

For $\frac{n}{2}$ even $n \ge 4$, the color codes of all the vertices of $V(B^s_{O_n})$ are :

$$c_\Pi(u_i) = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate, odd } i, 1 \le i \le n-1, n \ge 4 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 2 \le i \le n, n \ge 4 \\ & \text{for } 4^{th} \text{ ordinate}, i = n \\ i, & \text{for } 4^{th} \text{ ordinate}, 1 \le i \le \frac{n}{2}, n \ge 4 \\ n-i, & \text{for } 4^{th} \text{ ordinate}, \frac{n}{2}+1 \le i \le n-1, n \ge 4 \\ \frac{n}{2}-i+1, & \text{for } 5^{th} \text{ ordinate}, 1 \le i \le \frac{n}{2}, n \ge 4 \\ i-\frac{n}{2}, & \text{for } 5^{th} \text{ ordinate}, \frac{n}{2}+1 \le i \le n, n \ge 4 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\Pi(v_i) = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \le i \le n, n \ge 4 \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, 1 \le i \le n-1, n \ge 4 \\ i+1, & \text{for } 4^{th} \text{ ordinate}, 1 \le i \le \frac{n}{2}, n \ge 4 \\ n-i+1, & \text{for } 4^{th} \text{ ordinate}, \frac{n}{2}+1 \le i \le n, n \ge 4 \\ \frac{n}{2}-i+2, & \text{for } 5^{th} \text{ ordinate}, 1 \le i \le \frac{n}{2}-1, n \ge 4 \\ i-\frac{n}{2}+1, & \text{for } 5^{th} \text{ ordinate}, \frac{n}{2} \le i \le n, n \ge 4 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\Pi(w_i) = \begin{cases} 0, & \text{for } 1^{st} \text{ ordinate}, 1 \le i \le \frac{n}{2}-1, n \ge 4 \\ & \text{for } 1^{st} \text{ ordinate}, \frac{n}{2}+1 \le i \le n, n \ge 4 \\ & \text{for } 5^{th} \text{ ordinate}, i = \frac{n}{2} \\ 2, & \text{for } 1^{st} \text{ ordinate}, i = \frac{n}{2} \\ & \text{for } 3^{rd} \text{ ordinate}, i = n \\ i+1, & \text{for } 4^{th} \text{ ordinate}, 1 \le i \le \frac{n}{2}, n \ge 4 \\ n-i, & \text{for } 4^{th} \text{ ordinate}, \frac{n}{2} \le i \le n-1, n \ge 4 \\ \frac{n}{2}-i+1, & \text{for } 5^{th} \text{ ordinate}, 1 \le i \le \frac{n}{2}, n \ge 4 \\ i-\frac{n}{2}+1, & \text{for } 5^{th} \text{ ordinate}, \frac{n}{2}+1 \le i \le n, n \ge 4 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\Pi(u_{n+i}) = \begin{cases} i-1, & \text{for } 1^{st} \text{ ordinate}, 2 \le i \le \frac{n}{2}, n \ge 4 \\ n-i+1, & \text{for } 1^{st} \text{ ordinate}, \frac{n}{2}+1 \le i \le n, n \ge 4 \\ 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \le i \le n, n \ge 4 \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, 3 \le i \le n-1, n \ge 4 \\ & \text{for } 1^{st} \text{ ordinate}, i = 1 \\ 2, & \text{for } 3^{rd} \text{ ordinate}, i = 1 \\ \frac{n}{2}-i+1, & \text{for } 5^{th} \text{ ordinate}, 1 \le i \le \frac{n}{2}, n \ge 4 \\ i-\frac{n}{2}, & \text{for } 5^{th} \text{ ordinate}, \frac{n}{2}+1 \le i \le n, n \ge 4 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\Pi(v_{n+i}) = \begin{cases} i, & \text{for } 1^{st} \text{ ordinate}, 2 \le i \le \frac{n}{2}+1, n \ge 4 \\ n-i+2, & \text{for } 1^{st} \text{ ordinate}, \frac{n}{2}+2 \le i \le n, n \ge 4 \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd } i, 1 \le i \le n-1, n \ge 4 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 2 \le i \le n, n \ge 4 \\ 3, & \text{for } 2^{nd} \text{ ordinate}, i = 1 \\ \frac{n}{2}-i+3, & \text{for } 5^{th} \text{ ordinate}, 1 \le i \le \frac{n}{2}, n \ge 4 \\ i-\frac{n}{2}, & \text{for } 5^{th} \text{ ordinate}, \frac{n}{2}+1 \le i \le n, n \ge 4 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\Pi(w_{n+i}) = \begin{cases} i, & \text{for } 1^{st} \text{ ordinate}, 1 \le i \le \frac{n}{2}, n \ge 4 \\ n-i+1, & \text{for } 1^{st} \text{ ordinate}, \frac{n}{2}+1 \le i \le n, n \ge 4 \\ 0, & \text{for } 4^{th} \text{ ordinate}, 1 \le i \le \frac{n}{2}, n \ge 4 \\ & \text{for } 4^{th} \text{ ordinate}, \frac{n}{2}+2 \le i \le n, n \ge 4 \\ & \text{for } 5^{th} \text{ ordinate}, i = \frac{n}{2}+1 \\ 2, & \text{for } 4^{th} \text{ ordinate}, i = \frac{n}{2}+1 \\ \frac{n}{2}-i+2, & \text{for } 5^{th} \text{ ordinate}, 1 \le i \le \frac{n}{2}, n \ge 4 \\ i-\frac{n}{2}, & \text{for } 5^{th} \text{ ordinate}, \frac{n}{2}+2 \le i \le n, n \ge 4 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\Pi(x_i) = \begin{cases} s-i+1, & \text{for } 1^{st} \text{ ordinate}, i > \left\lfloor \frac{s}{2} \right\rfloor, s \ge 2 \\ i+1, & \text{for } 1^{st} \text{ ordinate}, i \le \left\lfloor \frac{s}{2} \right\rfloor, s \ge 2 \\ & \text{for } 2^{nd} \text{ ordinate}, i < \left\lceil \frac{s}{2} \right\rceil, s \ge 2 \\ s-i+2, & \text{for } 2^{nd} \text{ ordinate}, i \ge \left\lceil \frac{s}{2} \right\rceil, s \ge 2 \\ 0, & \text{for } 3^{rd} \text{ ordinate, odd } i, i \ge 1 \\ & \text{for } 4^{th} \text{ ordinate, even } i, i \ge 2 \\ i+\frac{n}{2}, & \text{for } 5^{th} \text{ ordinate}, i < \left\lceil \frac{s}{2} \right\rceil, s \ge 2 \\ s-i+\frac{n}{2}+1, & \text{for } 5^{th} \text{ ordinate}, i \ge \left\lceil \frac{s}{2} \right\rceil, s \ge 2 \\ 1, & \text{otherwise.} \end{cases}$$

Since for odd n all vertices have different color codes, c is a locating coloring for subdivison of certain barbell operation of origami graphs $B^s_{O_n}$, so that $\chi_L(B^s_{O_n}) \le 5$, for $\frac{n}{2}$ even , $n \ge 4$. This completes the proof of the theorem. □

Acknowledgments

This work was partially supported by a research grant from DRPM Dikti 2021.

References

[1] G. Chartrand, P. Zhang, E. Salehi, On the partition dimension of a graph, Congressus Numerantium, vol.130, pp.157-168, 1998.

[2] V. Saenpholphat and P. Zhang, Conditional resolvability: a survey, International Journal of Mathematics and Mathematical Sciences, vol.38, pp.1997-2017, 2004.

[3] G. Chartrand and P. Zhang, The theory and applications of resolvability in graphs. A survey,Congr.Numer.160, pp.47–68, 2003.

[4] M. Johnson, Conditional resolvability: a survey, Structure-activity maps for visualizing the graph variables arising in drug design, Journal of Biopharmaceutical Statistics, vol.3, no.2, pp.203-236, 1993.

[5] G. Chartrand, D. Erwin, M. A, Henning, P. J. Slater, P. Zhang, The locating-chromatic number of a graphs, Bulletin of the Institute of Combinatorics and its Applications, vol. 36, pp.89-101, 2002.

[6] G. Chartrand, D. Erwin, M. A, Henning,P. J. Slater, P. Zhang, Graf of order n with locating-chromatic number n − 1, Discrate Mathematics, vol.269, no.1-3, pp.65 – 79, 2003

[7] E. T. Baskoro and Asmiati, Characterizing all trees with locating-chromatic number 3, Electronic Journal of Graph Theory and Applications *(*EJGTA*)*, vol.1, no.2, pp.109-117, 2013.

[8] A. Behtoei, M. Anbarloei, The locating chromatic number of the join of graphs, Bulletin of the Iranian Mathematical Society, vol.40, no.6, pp.1491-1504, 2014.

[9] I. A. Purwasih, E. T. Baskoro, H. Assiyatun, D. Suprijanto, The bounds on the locating-chromatic number for a subdivision of a graph on one edge, Procedia Computer Science, vol.74, pp.84 – 88, 2015.

[10] D. Welyyanti , E. T. Baskoro, R. Simanjuntak, S. Uttunggadewa, On locating-chromatic number for graphs with dominant vertices, Procedia Computer Science, vol.74, pp.89-92, 2015.

[11] Asmiati, On the locating-chromatic numbers of non-homogeneous caterpillars and firecrackers graphs, Far East Journal of Mathematical Sciences (FJMS), vol.100, no.8, pp.1305-1316, 2016.

[12] Asmiati, I. K. G. Yana, and L. Yulianti, On The Locating Chromatic Number of Certain Barbell Graphs, International Journal of Mathematics and Mathematical Sciences, pp.1-5, 2018.

[13] A. Irawan, Asmiati, S. Suharsono, K. Muludi, The locating-chromatic number for certain operation of generalized petersen graphs sP(4,2), Journal of Physics: Conference Series, 1338, pp.1-6, 2019.

[14] A. Irawan, Asmiati, S. Suharsono, K. Muludi, L. Zakaria, Certain operation of generalized petersen graphs having locating-chromatic number five, Advances and Applications in Discrete Mathematics, vol.24, no.2, pp.83-97, 2020.

[15] A. Irawan, Asmiati, L. Zakaria, K. Muludi, The locating-chromatic number of origami graphs, Algorithms, vol.14, no.167, pp.1-15, 2021.

[16] A. Irawan, Asmiati, S. Suharsono, K. Muludi, The Locating-Chromatic Number of Certain Barbell Origami Graphs, Journal of Physics: Conference Series, 1750, pp.1-13, 2021.

[17] Asmiati, E. T. Baskoro, Characterizing all graphs containing cycles with locating-chromatic number 3, AIP Conference Proceedings, 1450, pp.351-357, 2012.

[18] S. Nabila, A. N. M.Salman, The Rainbow Conection Number of Origami Graphs and Pizza Graphs, Procedia Computer Science, 74, pp.162-167, 2015.