BF −Ostrowski Type Inequalties via $\phi-\lambda$ −convex functions

Ali Hassan [1†], Asif Raza Khan [2††], Faraz Mehmood [3†††], Maria Khan [4††††]

Shah Abdul Latif University Khairpur, University of Karachi, Dawood University of Engineering and Technology

Summary

We are introducing very first time that the class of $\phi-\lambda-$ convex function, which is the generalization of class of ϕ −convex, h −convex, Godunova-Levin s −convex, s −convex in the 2^{nd} kind and convex, P −convex and GL-convex functions. Next, we would like to state well-known Ostrowski inequality via $\phi-\lambda$ −convex function by using the bifuzzy Reimann integrals. In addition, we establish some bifuzzy Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $\phi-\lambda-$convex functions by Hölder's and power mean inequalities. In this way we also capture the results with respect to convexity of functions.

Key words:

Ostrowski inequality, convex functions, Fuzzy set, Bifuzzy sets.

1. Introduction

In this section, from literature, we recall and introduce some definitions for various convex functions.

Definition 1.1 *[3] A function $\eta: I\subset\mathbb{R}\to\mathbb{R}$ is said to be convex, if*

$$\eta(tx+(1-t)y)\leq t\eta(x)+(1-t)\eta(y),$$

$\forall x,y\in I, t\in[0,1]$.

Definition 1.2 *[3] A function $\eta: I\subset\mathbb{R}\to\mathbb{R}$ is said to be MT −convex, if η is a non-negative and*

$$\eta(tx+(1-t)y)\leq\frac{\sqrt{t}}{2\sqrt{1-t}}\eta(x)+\frac{\sqrt{1-t}}{2\sqrt{t}}\eta(y),$$

$\forall x,y\in I, t\in[0,1]$.

Definition 1.3 *[17] We say that $\eta: I\subset\mathbb{R}\to\mathbb{R}$ is a P −convex function, if η is a non-negative and $\forall x,y\in I$ and $t\in[0,1]$ we have*

$$\eta(tx+(1-t)y)\leq\eta(x)+\eta(y).$$

Definition 1.4 *[20] We say that $\eta: I\subset\mathbb{R}\to\mathbb{R}$ is a Godunova-Levin convex function, if η is non-negative and $\forall x,y\in I$ and $t\in(0,1)$ we have*

$$\eta(tx+(1-t)y)\leq\frac{1}{t}\eta(x)+\frac{1}{1-t}\eta(y).$$

Definition 1.5 *[4] Let $s\in[0,1]$. A function $\eta: I\subset[0,\infty)\to\mathbb{R}$ is said to be s −convex in the 2^{nd} kind, if*

$$\eta(tx+(1-t)y)\leq t^s\eta(x)+(1-t)^s\eta(y),$$

$\forall x,y\in I, t\in[0,1]$.

Manuscript received October 5, 2021
Manuscript revised October 20, 2021
https://doi.org/10.22937/IJCSNS.2021.21.10.24

Definition 1.6 *[9] We say that the function $\eta: I\subset\mathbb{R}\to[0,\infty)$ is of Godunova-Levin s −convex function, with $s\in[0,1]$, if*

$$\eta(tx+(1-t)y)\leq\frac{1}{t^s}\eta(x)+\frac{1}{(1-t)^s}\eta(y),$$

$\forall t\in(0,1)$ and $x,y\in I$.

Definition 1.7 *[30] Let $h: J\subseteq\mathbb{R}\to[0,\infty)$ with h not identical to 0. We say that η is an h −convex function if $\forall x,y\in I$, we have*

$$\eta(tx+(1-t)y)\leq h(t)\eta(x)+h(1-t)\eta(y),$$

$\forall t\in[0,1]$.

Definition 1.8 *[10] Let $\phi:(0,1)\to(0,\infty)$ be a measurable function. We say that the $\eta: I\to[0,\infty)$ is a ϕ −convex(concave) function on the interval I if for all $x,y\in I$ we have*

$$\eta(tx+(1-t)y)\leq(\geq)t\phi(t)\eta(x)+(1-t)\phi(1-t)\eta(y), \quad (1.1)$$

$\forall t\in(0,1)$.

Theorem 1.9 *[29] Let $\varphi:[\rho_a,\rho_b]\to\mathbb{R}$ be differentiable function on (ρ_a,ρ_b) with the property that $|\varphi'(t)|\leq M$ for all $t\in(\rho_a,\rho_b)$. Then*

$$\left|\varphi(x)-\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt\right|\leq M(\rho_b-\rho_a)\left[\frac{1}{4}+\left(\frac{x-\frac{\rho_a+\rho_b}{2}}{\rho_b-\rho_a}\right)^2\right], \quad (1.2)$$

for all $x\in(\rho_a,\rho_b)$. The constant $\frac{1}{4}$ is the best possible in the kind that it cannot be replaced by a smaller quantity.

Now we present the extension of definitions of fuzzy numbers and their results as from the [6], [7], [26] and [19].

Definition 1.10 *A BF-Number is $\phi:\mathbb{R}\to[0,1]$ can be defined as*

1. $[\phi]^0=\text{Closure}(\{r\in\mathbb{R}: T\phi(r)>0, F\phi(r)>0\})$ is compact.
2. ϕ is Normal.(i.e, $\exists\, r_0\in\mathbb{R}$ such that $T\phi(r_0)=1$ and $\varnothing F\phi(r_0)=0$).
3. ϕ is BF-convex, i.e, $\forall r_1,r_2\in\mathbb{R}, \eta\in[0,1]$
$$T\phi(\eta r_1+(1-\eta)r_2)\geq\min\{T\phi(r_1),T\phi(r_2)\},$$
$$F\phi(\eta r_1+(1-\eta)r_2)\leq\max\{F\phi(r_1),F\phi(r_2)\}.$$
4. $\forall r_0\in R$ and $\epsilon>0$, $\exists$ Neighborhood $V(r_0)$, such that $\forall r\in\mathbb{R}, T\phi(r)\leq T\phi(r_0)+\epsilon$ and $F\phi(r)\geq\phi(r_0)-\epsilon$,

Definition 1.11 *For any $(\zeta_1,\zeta_2)\in[0,1]^2$, and ϕ be any BF-number, then $\zeta-$ level set $[\phi]^{(\zeta_1,\zeta_2)}=\{r\in\mathbb{R}: T\phi(r)\geq\zeta_1, F\phi(r)\leq\zeta_2\}$. Moreover $[\phi]^\zeta=\left[\phi_-^{(\zeta_1,\zeta_2)},\phi_+^{(\zeta_1,\zeta_2)}\right],\forall(\zeta_1,\zeta_2)\in[0,1]^2$.*

Proposition 1.12 *Let $\phi,\varphi\in BF_{\mathbb{R}}$(Set of all BF-Numbers) and $\eta\in\mathbb{R}$, then the following properties holds:*

1. $[\phi + \varphi]^{(\zeta_1,\zeta_2)} = [\phi]^{(\zeta_1,\zeta_2)} + [\varphi]^{(\zeta_1,\zeta_2)}$.
2. $[\eta \odot \phi]^{(\zeta_1,\zeta_2)} = \eta[\phi]^{(\zeta_1,\zeta_2)}$.
3. $\phi \oplus \varphi = \varphi \oplus \phi$.
4. $\eta \odot \phi = \phi \odot \eta$.
5. $\tilde{1} \odot \phi = \phi$.

$\forall \zeta \in [0,1]$, where $\tilde{1} \in BF_{\mathbb{R}}$, defined by $\forall r \in \mathbb{R}, \tilde{1}(r) = (1,0)$.

Definition 1.13 *Let $D: BF_{\mathbb{R}} \times BF_{\mathbb{R}} \to \mathbb{R}_+ \cup \{0\}$, defined as*

$$D(\phi,\varphi) = \sup_{\zeta\in[0,1]} \max\left\{\left|T\phi_-^{(\zeta)}, T\phi_+^{(\zeta)}\right|, \left|T\varphi_-^{(\zeta)}, T\varphi_+^{(\zeta)}\right|\right\} + \inf_{\zeta\in[0,1]} \min\left\{\left|F\phi_-^{(\zeta)}, F\phi_+^{(\zeta)}\right|, \left|F\varphi_-^{(\zeta)}, F\varphi_+^{(\zeta)}\right|\right\}.$$

$\forall \phi, \varphi \in BF_{\mathbb{R}}$. Then D is metric on $BF_{\mathbb{R}}$.

Proposition 1.14 *Let $\phi_1, \phi_2, \phi_3, \phi_4 \in BF_{\mathbb{R}}$ and $\eta \in BF_{\mathbb{R}}$, we have*
1. $(BF_{\mathbb{R}}, D)$ is complete.
2. $D(\phi_1 \oplus \phi_3, \phi_2 \oplus \phi_3) = D(\phi_1, \phi_2)$.
3. $D(\eta \odot \phi_1, \eta \odot \phi_2) = |\eta| D(\phi_1, \phi_2)$.
4. $D(\phi_1 \oplus \phi_2, \phi_3 \oplus \phi_4) = D(\phi_1, \phi_3) + D(\phi_2, \phi_4)$.
5. $D(\phi_1 \oplus \phi_2, \tilde{0}) = D(\phi_1, \tilde{0}) + D(\phi_2, \tilde{0})$.
6. $D(\phi_1 \oplus \phi_2, \phi_3) = D(\phi_1, \phi_3) + D(\phi_2, \tilde{0})$,

where $\tilde{0} \in BF_{\mathbb{R}}$, defined by $\forall r \in \mathbb{R}, \tilde{0}(r) = (0,1)$.

Definition 1.15 *Let $\phi, \varphi \in BF_{\mathbb{R}}$, if $\exists\, \theta \in BF_{\mathbb{R}}$, such that $\phi = \varphi \oplus \theta$, then θ is $H-$difference of ϕ and φ, denoted by $\theta = \phi \ominus \varphi$.*

Definition 1.16 *A function $\phi: [r_0, r_0 + \epsilon] \to BF_{\mathbb{R}}$ is $H-$differentiable at r, if $\exists\, \phi'(r) \in BF_{\mathbb{R}}$, i.e both limits*

$$\lim_{h\to 0^+} \frac{\phi(r+h)\ominus\phi(r)}{h}, \lim_{h\to 0^+} \frac{\phi(r)\ominus\phi(r-h)}{h}$$

exists and are equal to $\phi'(r)$.

Definition 1.17 *Let $\phi: [\rho_a, \rho_b] \to BF_{\mathbb{R}}$, if $\forall \zeta > 0, \exists \eta > 0$, for any partition $P = \{[u,v]: \delta\}$ of $[\rho_a, \rho_b]$ with norm $\Delta(P) < \eta$, we have*

$$D\left(\sum_P^* (v-u)\phi(\delta), \varphi\right) < \zeta,$$

then we say that ϕ is BF-Riemann integrable to $\varphi \in BF_{\mathbb{R}}$, we write it as

$$\varphi = (BFR)\int_{\rho_a}^{\rho_b} \phi(x)dx.$$

2 $BF-$ Ostrowski type inequalities via $\phi - \lambda-$convex functions

In this section, we are introducing very first time the concept of $\phi - \lambda-$convex function, which contain many classes of convex functions in litrature.

Definition 2.1 *Let $\lambda \in (0,1], \phi: (0,1) \to (0,\infty)$ be a measurable function. We say that the $\eta: I \to [0,\infty)$ is a $\phi - \lambda-$convex(concave) function on the interval I if for all $x, y \in I$ we have*

$$\eta(tx + (1-t)y) \le (\ge) t^\lambda \phi(t)\eta(x) + (1-t)^\lambda \phi(1-t)\eta(y), \quad (2.1)$$

$\forall t \in (0,1)$.

Remark 2.2 *In Definition* 2.1, *one can see the following.*
1. If we put $\lambda = 1$, in (2.1), then we get the concept of $\phi-$convex (concave) function.
2. If we denote $l(t) = t$, and by taking $\lambda = 1$, $h = l\phi$ in (2.1), we get $h-$convex (concave) function.
3. If we take $\lambda = 1$, $\phi(t) = \frac{1}{t^{s+1}}$ with $s \in [0,1]$ in (2.1), then we get the class of Godunova-Levin $s-$convex (concave) functions.
4. If we put $\lambda = 1$, $\phi(t) = \frac{1}{t^2}$ in (2.1), then we get the concept of Godunova-Levin convex (concave) function.
5. if we put $\lambda = 1$, $\phi(t) = t^{s-1}$ with $s \in [0,1]$ in (2.1), then we get the concept of $s-$convex (concave) in 2^{nd} kind.
6. If we put $\lambda = 1$, $\phi(t) = \frac{1}{t}$ in (2.1), then we get the concept of $P-$convex (concave) function.
7. If we put $\lambda = 1$, $\phi(t) = 1$ in (2.1), then we get the concept of ordinary convex (concave) function.
8. If we put $\lambda = 1$, $\phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (2.1), then we get the concept of $MT-$convex (concave) function.

In order to prove our main results in next section, we need the following lemma.

Lemma 2.3 *Let $\varphi: [\rho_a, \rho_b] \to BF_{\mathbb{R}}$ be an absolutely continuous mapping on (ρ_a, ρ_b) with $\rho_a < \rho_b$. If $\varphi' \in C_F[\rho_a, \rho_b] \cap L_F[\rho_a, \rho_b]$, then for $x \in (\rho_a, \rho_b)$ the following identity holds:*

$$\frac{1}{\rho_b - \rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt \oplus \frac{(x-\rho_a)^2}{\rho_b-\rho_a} \odot (BFR)\int_0^1 t \odot \varphi'(tx + (1-t)\rho_a)dt$$
$$= \varphi(x) \oplus \frac{(\rho_b - x)^2}{\rho_b - \rho_a} \odot (BFR)\int_0^1 t \odot \varphi'(tx + (1-t)\rho_b)dt. \quad (2.2)$$

We make use of the beta function of Euler type, which is for $x, y > 0$ defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(x) = \int_0^\infty e^{-u}u^{x-1}du$.

Theorem 2.4 *Suppose all the assumptions of Lemma* 2.3 *hold. Additionally, $\lambda \in (0,1], \phi: (0,1) \to (0,\infty)$ be a measurable function with $\phi(t) \neq \frac{1}{t^2}$, $D(\varphi', \tilde{0})$ be a $\phi - \lambda-$convex function on $[\rho_a, \rho_b]$ and $D(\varphi'(x), \tilde{0}) \le M$. Then for each $x \in (\rho_a, \rho_b)$ the following inequality holds:*

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \le M\left(\int_0^1 \left(t^{\lambda+1}\phi(t) + t(1-t)^\lambda \phi(1-t)\right)dt\right) I(x), \quad (2.3)$$

where $I(x) = \frac{(x-\rho_a)^2 + (\rho_b - x)^2}{\rho_b - \rho_a}$.

Proof. From the Lemma 2.3,

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right)$$
$$\le D\left(\frac{(x-\rho_a)^2}{\rho_b - \rho_a} \odot (BFR)\int_0^1 t \odot \varphi'(tx + (1-t)\rho_a)dt, \frac{(\rho_b - x)^2}{\rho_b - \rho_a} \odot (BFR)\int_0^1 t \odot \varphi'(tx + (1-t)\rho_b)dt\right),$$

$$\le D\left(\frac{(x-\rho_a)^2}{\rho_b - \rho_a} \odot (BFR)\int_0^1 t \odot \varphi'(tx + (1-t)\rho_a)dt, \tilde{0}\right)$$
$$+ D\left(\frac{(\rho_b - x)^2}{\rho_b - \rho_a} \odot (BFR)\int_0^1 t \odot \varphi'(tx + (1-t)\rho_b)dt, \tilde{0}\right),$$

$$= \frac{(x-\rho_a)^2}{\rho_b-\rho_a} D\left((BFR)\int_0^1 t \odot \varphi'(tx+(1-t)\rho_a)dt, \tilde{0}\right)$$
$$+ \frac{(\rho_b-x)^2}{\rho_b-\rho_a} D\left((BFR)\int_0^1 t \odot \varphi'(tx+(1-t)\rho_b)dt, \tilde{0}\right),$$
$$\leq \frac{(x-\rho_a)^2}{\rho_b-\rho_a}\int_0^1 tD(\varphi'(tx+(1-t)\rho_a), \tilde{0})dt$$
$$+ \frac{(\rho_b-x)^2}{\rho_b-\rho_a}\int_0^1 tD(\varphi'(tx+(1-t)\rho_b), \tilde{0})dt, \quad (2.4)$$

Since $D(\varphi', \tilde{0})$ be ϕ −convex function and $D(\varphi'(x), \tilde{0}) \leq M$, we have

$$D \leq t^\lambda \phi(t) D(\varphi'(x), \tilde{0}) + (1-t)^\lambda \phi(1-t) D(\varphi'(\rho_a), \tilde{0})$$
$$\leq M[t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t)] \quad (2.5)$$

$$D \leq t^\lambda \phi(t) D(\varphi'(x), \tilde{0}) + (1-t)^\lambda \phi(1-t) D(\varphi'(\rho_b), \tilde{0})$$
$$\leq M[t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t)]. \quad (2.6)$$

Now using (2.5) and (2.6) in (2.4) we get (2.3).

Corollary 2.5 *In Theorem* 2.4, *one can see the following.*

1. If one takes $\lambda = 1$, in (2.3), one has the $BF-$ Ostrowski inequality for ϕ −convex function:

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq M\left(\int_0^1 (t^2\phi(t) + t(1-t)\phi(1-t))dt\right) I(x).$$

2. If one takes $\lambda = 1, l(t) = t$, the identity function, then by taking $h = l\phi$, in (2.3), one has the $BF-$ Ostrowski inequality for h −convex function:

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq M\left(\int_0^1 (th(t) + th(1-t))dt\right) I(x).$$

3. If one takes $\lambda = 1, \phi(t) = t^{-(s+1)}$ in (2.3), then one has the $BF-$ Ostrowski inequality for Godunova-Levin $s-$ convex functions:

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq M\left(\frac{1}{1-s}\right) I(x).$$

4. If one takes $\lambda = 1, \phi(t) = t^{s-1}$ where $s \in (0,1]$ in (2.3), then one has the $BF-$ Ostrowski inequality for s −convex functions in 2^{nd} kind:

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq M\left(\frac{1}{1+s}\right) I(x).$$

5. If one takes $\lambda = 1, \phi(t) = t^{-1}$ in (2.3), then one has the $BF-$ Ostrowski inequality for P −convex function:

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq MI(x).$$

6. If one takes $\lambda = \phi(t) = 1$ in (2.3), then one has the $BF-$ Ostrowski inequality for convex function:

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2} I(x).$$

7. If one takes $\lambda = 1, \phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (2.3), then one has the $BF-$ Ostrowski inequality for MT −convex function:

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M\pi}{4} I(x).$$

Theorem 2.6 *Suppose all the assumptions of Lemma* 2.3 *hold. Additionally,* $\lambda \in (0,1], \phi: (0,1) \to (0,\infty)$ *be a measurable function with* $\phi(t) \neq \frac{1}{t^2}$, $[D(\varphi', \tilde{0})]^q$ *for* $q \geq 1$ *be* $\phi - \lambda$ *−convex function on* $[\rho_a, \rho_b]$ *and* $D(\varphi'(x), \tilde{0}) \leq M$. *Then* $\forall x \in (\rho_a, \rho_b)$ *the following inequality holds:*

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right)$$
$$\leq \frac{M}{2^{1-\frac{1}{q}}}\left(\int_0^1 (t^{\lambda+1}\phi(t) + t(1-t)^\lambda \phi(1-t))dt\right)^{\frac{1}{q}} I(x). \quad (2.7)$$

Proof. From the inequality (??) and power mean inequality [31]

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq$$
$$\frac{(x-\rho_a)^2}{\rho_b-\rho_a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t[D(\varphi'(tx+(1-t)\rho_a), \tilde{0})]^q dt\right)^{\frac{1}{q}} +$$
$$\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t[D(\varphi'(tx+(1-t)\rho_b), \tilde{0})]^q dt\right)^{\frac{1}{q}}. \quad (2.8)$$

Since $[D(\varphi', \tilde{0})]^q$ be ϕ −convex function and $D(\varphi'(x), \tilde{0}) \leq M$, we have

$$[D(\varphi'(tx+(1-t)\rho_a), \tilde{0})]^q \leq t^\lambda \phi(t)[D(\varphi'(x), \tilde{0})]^q + (1-t)^\lambda \phi(1-t)[D(\varphi'(\rho_a), \tilde{0})]^q$$
$$\leq M^q[t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t)], \quad (2.9)$$

$$[D(\varphi'(tx+(1-t)\rho_b), \tilde{0})]^q \leq$$
$$t^\lambda \phi(t)[D(\varphi'(x), \tilde{0})]^q + (1-t)^\lambda \phi(1-t)[D(\varphi'(\rho_b), \tilde{0})]^q$$
$$\leq M^q[t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t)], \quad (2.10)$$

Now using (2.9) and (2.10) in (2.8) we get (2.7).

Corollary 2.7 *In Theorem* 2.6, *one can see the following.*

1. If one takes $q = 1$, one has the Theorem 2.4.
2. If one takes $\lambda = 1$ in (??), one has the $BF-$ Ostrowski inequality for ϕ −convex function:

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}}\left(\int_0^1 (t^2\phi(t) + t(1-t)\phi(1-t))dt\right)^{\frac{1}{q}} I(x).$$

3. If one takes $\lambda = 1, l(t) = t$, the identity function, then by taking $h = l\phi$, in (??), one has the $BF-$ Ostrowski inequality for h −convex function:

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}}\left(\int_0^1 (t\, h(t) + t\, h(1-t))dt\right)^{\frac{1}{q}} I(x).$$

4. If one takes $\lambda = 1, \phi(t) = t^{-(s+1)}$ in (??), then one has $BF-$ Ostrowski inequality for Godunova-Levin $s-$convex functions:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (BFR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}}\left(\frac{1}{1-s}\right)^{\frac{1}{q}} I(x).$$

5. If one takes $\lambda = 1, \phi(t) = t^{s-1}$ where $s \in [0,1]$ in (??), then one has $BF-$ Ostrowski inequality for $s-$convex functions in 2^{nd} kind:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (BFR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}}\left(\frac{1}{1+s}\right)^{\frac{1}{q}} I(x).$$

6. If one takes $\lambda = 1, \phi(t) = t^{-1}$, in (??), then one has the $BF-$ Ostrowski inequality for $P-$convex function:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (BFR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}} I(x).$$

7. If one takes $\lambda = \phi(t) = 1$, in (??), then one has the $BF-$ Ostrowski inequality for convex function:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (BFR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2} I(x).$$

8. If one takes $\lambda = 1, \phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (??), then one has the $BF-$ Ostrowski inequality for $MT-$convex function:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (BFR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M\pi^{\frac{1}{q}}}{2^{1+\frac{1}{q}}} I(x).$$

Theorem 2.8 *Suppose all the assumptions of Lemma* 2.3 *hold. Additionally, assume that* $\lambda \in (0,1], \phi: (0,1) \to (0,\infty)$ *be a measurable function with* $\phi(t) \neq \frac{1}{t^2}$, $[D(\varphi', \tilde{0})]^q$ *be a* $\phi-\lambda-$ *convex function on* $[\rho_a, \rho_b], q > 1$ *and* $D(\varphi'(x), \tilde{0}) \leq M$. *Then for each* $x \in (\rho_a, \rho_b)$, *the following inequality holds:*

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (BFR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right)$$
$$\leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\int_0^1 \left(t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t)\right)dt\right)^{\frac{1}{q}} I(x), \qquad (2.11)$$

where $p^{-1} + q^{-1} = 1$.

Proof. From the inequality (??) and Hölder's inequality [32]

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (BFR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right)$$
$$\leq \frac{(x-\rho_a)^2}{\rho_b - \rho_a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[D\left(\varphi'(tx + (1-t)\rho_a), \tilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{(\rho_b - x)^2}{\rho_b-\rho_a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[D\left(\varphi'(tx + (1-t)\rho_b), \tilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}. \qquad (2.12)$$

Since $[D(\varphi', \tilde{0})]^q$ be $\phi-$convex function and $D(\varphi'(x), \tilde{0}) \leq M$, we have

$$\left[D\left(\varphi'(tx + (1-t)\rho_a), \tilde{0}\right)\right]^q \leq t^\lambda \phi(t)\left[D\left(\varphi'(x), \tilde{0}\right)\right]^q + (1-t)^\lambda \phi(1-t)\left[D\left(\varphi'(\rho_a), \tilde{0}\right)\right]^q$$
$$\leq M^q\left[t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t)\right], \qquad (2.13)$$

$$\left[D\left(\varphi'(tx + (1-t)\rho_b), \tilde{0}\right)\right]^q \leq t^\lambda \phi(t)\left[D\left(\varphi'(x), \tilde{0}\right)\right]^q + (1-t)^\lambda \phi(1-t)\left[D\left(\varphi'(\rho_b), \tilde{0}\right)\right]^q$$
$$\leq M^q\left[t^\lambda \phi(t) + (1-t)^\lambda \phi(1-t)\right], \qquad (2.14)$$

Now using (2.13) and (2.14) in (2.12) we get (2.11).

Corollary 2.9 *In Theorem* 2.8, *one can see the following.*

1. If one takes $\lambda = 1$ in (2.11), one has the $BF-$ Ostrowski inequality for $\phi-$convex function:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (BFR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\int_0^1 (t\phi(t) + (-t)\phi(-t))dt\right)^{\frac{1}{q}} I(x).$$

2. If one takes $\lambda = 1, l(t) = t$, then by taking $h = l\phi$, in (2.11), one has the $BF-$ Ostrowski inequality for $h-$convex function:

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (BFR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\int_0^1 (h(t) + h(1-t))dt\right)^{\frac{1}{q}} I(x).$$

3. If one takes $\lambda = 1, \phi(t) = t^{-(s+1)}$ where $s \in [0,1)$ in (2.11), then one has the $BF-$ Ostrowski inequality for Godunova-Levin $s-$ convex functions:

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (BFR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{2}{1-s}\right)^{\frac{1}{q}} I(x).$$

4. If one takes $\lambda = 1, \phi(t) = t^{s-1}$, where $s \in (0,1]$ in (2.11), then one has the $BF-$ Ostrowski inequality for $s-$ convex functions in 2^{nd} kind:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (BFR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{2}{1+s}\right)^{\frac{1}{q}} I(x).$$

5. If one takes $\lambda = 1, \phi(t) = t^{-1}$, in (2.11), then one has the $BF-$ Ostrowski inequality for $P-$convex function:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (BFR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{2^{\frac{1}{q}}M}{(p+1)^{\frac{1}{p}}} I(x).$$

6. If one takes $\lambda = \phi(t) = 1$, in (2.11), then one has the $BF-$ Ostrowski inequality for convex function:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (BFR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{(p+1)^{\frac{1}{p}}} I(x).$$

7. If one takes $\lambda = 1, \phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (2.11), then one has the $BF-$ Ostrowski inequality for $MT-$convex function:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (BFR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M\left(\frac{\pi}{2}\right)^{\frac{1}{q}}}{(1+p)^{\frac{1}{p}}} I(x).$$

3. $BF-$ Ostrowski type midpoint inequalties via $\phi-\lambda-$convex functions

Remark 2.10 *In Theorem* 2.6, *one can see the following.*

1. If one takes $x=\frac{\rho_a+\rho_b}{2}$ in (2.7), one has the $BF-$ Ostrowski Midpoint inequality for $\phi-\lambda-$ convex function: $D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(BFR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$
$$\leq\frac{M}{2^{2-\frac{1}{q}}}\left(\int_0^1\left(t^{\lambda+1}\phi(t)+t(1-t)^{\lambda}\phi(1-t)\right)dt\right)^{\frac{1}{q}}(\rho_b-\rho_a).$$

2. If one takes $\lambda=1,x=\frac{\rho_a+\rho_b}{2}$ in (2.7), one has the $BF-$ Ostrowski Midpoint inequality for $\phi-$convex function:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(BFR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq\frac{M}{2^{2-\frac{1}{q}}}\left(\int_0^1\left(t^2\phi(t)+t(1-t)\phi(1-t)\right)dt\right)^{\frac{1}{q}}(\rho_b-\rho_a).$$

3. If one takes $\lambda=1,x=\frac{\rho_a+\rho_b}{2}$, $l(t)=t$ and $h=l\phi$ in (2.7), then one has the $BF-$ Ostrowski Midpoint inequality for $h-$convex function: $D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(BFR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$
$$\leq\frac{M}{2^{2-\frac{1}{q}}}\left(\int_0^1\left(th(t)+th(1-t)\right)dt\right)^{\frac{1}{q}}(\rho_b-\rho_a).$$

4. If one takes $\lambda=1,x=\frac{\rho_a+\rho_b}{2}$ and $\phi(t)=t^{-(s+1)}$ in (2.7), then one has $BF-$ Ostrowski Midpoint inequality for Godunova-Levin $s-$ convex functions: $D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(BFR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\leq\frac{M}{2^{2-\frac{1}{q}}}\left(\frac{1}{1-s}\right)^{\frac{1}{q}}(\rho_b-\rho_a).$

5. If one takes $\lambda=1,x=\frac{\rho_a+\rho_b}{2}$ and $\phi(t)=t^{s-1}$ where $s\in[0,1]$ in (2.7), then one has $BF-$ Ostrowski Midpoint inequality for $s-$convex functions in 2^{nd} kind:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(BFR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\leq\frac{M}{2^{2-\frac{1}{q}}}\left(\frac{1}{1+s}\right)^{\frac{1}{q}}(\rho_b-\rho_a).$$

6. If one takes $\lambda=1,x=\frac{\rho_a+\rho_b}{2}$ and $\phi(t)=t^{-1}$ in (2.7), then one has the $BF-$ Ostrowski Midpoint inequality for $P-$convex function:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(BFR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\leq\frac{M}{2^{2-\frac{1}{q}}}(\rho_b-\rho_a).$$

7. If one takes $\lambda=1,x=\frac{\rho_a+\rho_b}{2}$ and $\phi(t)=1$ in (2.7), then one has the $BF-$ Ostrowski Midpoint inequality for convex function:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(BFR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\leq\frac{M}{4}(\rho_b-\rho_a).$$

8. If one takes $\lambda=1,\phi(t)=\frac{1}{2\sqrt{t(1-t)}}$ in (2.7), then one has the $BF-$ Ostrowski inequality for $MT-$convex function:
$$D\left(\varphi(x),\frac{1}{\rho_b-\rho_a}\odot(BFR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\leq\frac{M\pi^{\frac{1}{q}}}{2^{1+\frac{1}{q}}}I(x).$$

Remark 2.11 *In Theorem* 2.8, *one can see the following.*

1. If one takes $x=\frac{\rho_a+\rho_b}{2}$ in (2.11), one has the $BF-$ Ostrowski Midpoint inequality for $\phi-\lambda-$convex function:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(BFR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq\frac{M}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1\left(t^{\lambda}\phi(t)+(1-t)^{\lambda}\phi(1-t)\right)dt\right)^{\frac{1}{q}}(\rho_b-\rho_a),$$

2. If one takes $\lambda=1,x=\frac{\rho_a+\rho_b}{2}$ in (2.11), one has the $BF-$ Ostrowski Midpoint inequality for $\phi-$convex function:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(BFR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq\frac{M}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1\left(t\phi(t)+(1-t)\phi(1-t)\right)dt\right)^{\frac{1}{q}}(\rho_b-\rho_a),$$

3. If one takes $\lambda=1,x=\frac{\rho_a+\rho_b}{2}$, $l(t)=t$ and $h=l\phi$ in (2.11), one has the $BF-$ Ostrowski Midpoint inequality for $h-$convex function:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(BFR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)$$
$$\leq\frac{M}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1\left(h(t)+h(1-t)\right)dt\right)^{\frac{1}{q}}(\rho_b-\rho_a).$$

4. If one takes $\lambda=1,x=\frac{\rho_a+\rho_b}{2}$ and $\phi(t)=t^{-(s+1)}$ where $s\in[0,1)$ in (2.11), then one has the $BF-$ Ostrowski Midpoint inequality for Godunova-Levin $s-$convex functions:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(BFR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\leq\frac{2^{\frac{1}{q}-1}M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{1-s}\right)^{\frac{1}{q}}(\rho_b-\rho_a).$$

5. If one takes $\lambda=1,x=\frac{\rho_a+\rho_b}{2}$ and $\phi(t)=t^{s-1}$, where $s\in(0,1]$ in (2.11), then one has the $BF-$ Ostrowski Midpoint inequality for $s-$convex functions in 2^{nd} kind:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right),\frac{1}{\rho_b-\rho_a}\odot(BFR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\leq\frac{2^{\frac{1}{q}-1}M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{1+s}\right)^{\frac{1}{q}}(\rho_b-\rho_a).$$

6. If one takes $\lambda = 1, x = \frac{\rho_a+\rho_b}{2}$ and $\phi(t) = t^{-1}$ in (2.11), then one has the $BF-$ Ostrowski Midpoint inequality for $P-$convex function:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{2^{\frac{1}{q}-1}M}{(p+1)^{\frac{1}{p}}}(\rho_b-\rho_a).$$

7. If one takes $\lambda = 1, x = \frac{\rho_a+\rho_b}{2}$ and $\phi(t) = 1$ in (??), then one has the $BF-$ Ostrowski Midpoint inequality for convex function:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2(p+1)^{\frac{1}{p}}}(\rho_b-\rho_a).$$

8. If one takes $\lambda = 1, \phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (??), then one has the $BF-$ Ostrowski inequality for $MT-$convex function:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (BFR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M\pi^{\frac{1}{q}}}{2^{\frac{1}{q}+1}(1+p)^{\frac{1}{p}}}(\rho_b-\rho_a).$$

4. Conclusion

Ostrowski inequality is one of the most celebrated inequalities, we can find its various generalizations and variants in literature. In this paper, we presented the generalized notion of $\phi-$ convex function which is the generalization of many important classes including class of $h-$convex [30], Godunova-Levin $s-$convex [9], $s-$convex in the 2^{nd} kind [4] (and hence contains class of convex functions [3]). It also contains class of $P-$convex functions [17] and class of Godunova-Levin functions [20]. We would like to state the BF-Ostrowski inequality via $\phi-$ convex function. In addition, we establish some BF-Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $\phi-$convex functions by using different techniques including Hölder's inequality[32] and power mean inequality[31].

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