A Computer Oriented Solution for the Fractional Boundary Value Problem with Fuzzy Parameters with Application to Singular Perturbed Problems

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Summary

A treatment based on the algebraic operations on fuzzy numbers is used to replace the fuzzy problem into an equivalent crisp one. The finite difference technique is used to replace the continuous boundary value problem (BVP) of arbitrary order $1<\alpha\leq 2$, with fuzzy boundary parameters into an equivalent crisp (algebraic or differential) system. Three numerical examples with different behaviors are considered to illustrate the treatment of the singular perturbed case with different fractional orders of the BVP (α=1.8, α=1.9) as well as the classical second order (α=2). The calculated fuzzy solutions are compared with the crisp solutions of the singular perturbed BVP using triangular membership function (r-cut representation in parametric form) for different values of the singular perturbed parameter (ε=0.8, ε=0.9, ε=1.0). Results are illustrated graphically for the different values of the included parameters.

Key words: *fractional order linear BVP, finite difference method, Fuzzy, Fuzzy linear systems.*

1. Introduction

Solving systems of linear n × n equations with some parameters assigned as fuzzy numbers has several applications in the areas of applied computing such as mathematics, physics, statistics, bioinformatics, information, and finance. A systematic technique for solving fuzzy linear systems FLS numerically was first proposed by Friedman and his colleagues in 1998 [1]. The considered algebraic systems have crisp values for its coefficient matrix; right -hand side vector column with arbitrary fuzzy number. Allahviranloo and Ghanbari introduced approach for solving FLS based on interval theory and interval inclusion theory, to obtain the solution linear algebraic system,[2]. The finite difference method is a technique for solving differential equations by replacing the derivatives included in the equation of any order (integer or fractional) by a corresponding finite difference representation with an acceptable accuracy, [3]. To obtain a corresponding finite difference approximation at each point of a well-defined grid imposed over the given continuous domain of the differential equation. Accordingly, an algebraic system is obtained when the problem under consideration is a linear boundary value problem with fuzzy boundary conditions the resultant algebraic system is a fuzzy linear system [4].

Although, fractional order derivatives considered an old concept since introduced by LoHoptal and leibentz (1695) the modern applications in engineering, physical, biological as well as economical areas have illustrated the effectiveness of the mathematical models with fractional order derivatives, [5,6,7,8]. Fractional Calculus is a powerful tool which has been recently employed to model complex natural systems even those which have non-linear behavior and require long-term memory, Yousef Jamali Mohammad and Amirian Matlob [9] presented clear descriptions of the fractional calculus, its techniques and its implementation. They also apply the fractional tool to investigate and to model the mechanical response of phenomena pertain a dynamic system that investigate efficiently the mechanical behavior of a cell.

The modern applications in which fuzzy concepts appear specially in cases of computing under uncertain definition for a given variables [10,11,12,13], illustrated the need to use fuzzy concepts in the mathematical formulations of many practical problems [14,15,16,17].

In this work a fractional order singular perturbed boundary value problem with fuzzy boundary conditions is considered for different values of the order of the fractional derivative $(1<\alpha\leq 2)$, also different values singular perturbation parameter $0.8\leq \varepsilon \leq 1.0$ are considered [18,19,20,21]. Triangular shape of membership function in parametric form is used for the representation of fuzzy numbers. It is well known that implementation of the extension principle is equivalent to the solution of a nonlinear programming problem.

1.2. Theoretical Consideration

We consider the theoretical background needed in this work. Background from fuzzy mathematics, from fractional calculus, from singular perturbed problems and the finite difference method.

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2.1 Fuzzy Numbers and their arithmetic

The concept of fuzzy numbers is used efficiently in many applications to describe the vagueness and lack of precision of data [1, 2, 4]. Fuzzy numbers reflect the human perception of uncertain numerical quantification. There are different equivalent representations of fuzzy numbers.

Definition

A fuzzy number, u is a real valued function with the interval [0, 1] as its range $u: R \rightarrow [0,1]$

A fuzzy number u in a parametric form is represented as an ordered pair ($\underline{u}$, $\overline{u}$), of functions $\underline{u}(r)$ *and* $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfies the following requirements:

1. $\underline{u}(r)$ is a bounded monotonic non-de-creasing left-continuous function over $[0,1]$,
2. $\overline{u}(r)$ is a bounded monotonic non-increasing (1) left-continuous function over $[0,1]$.
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

Two forms of fuzzy numbers are commonly used in many applications the trapezoidal and the triangular fuzzy numbers. In both forms a linear function approximation is used for both $\underline{u}(r)$ and $\overline{u}(r)$. A parametric representation of a trapezoidal fuzzy number $u = (a, b, \sigma, \beta)$ with two de-fuzzifiers a, b and left (width) fuzziness $\sigma > 0$ and right (width) fuzziness $\beta > 0$. Hence, the membership function in the parametric form is:

where the parametric form of a trapezoidal fuzzy number is:

$$\underline{u}(r) = a - \sigma + \sigma r, \ \overline{u}(r) = b + \beta - \beta r \qquad (2)$$

For example, the fuzzy number $(1 + r, 6 - 2r)$ has $a = 2, b = 4, \sigma = 1$ *and* $\beta = 2$.

The r-level of u, $[u]_r = \left(\underline{u}(r), \overline{u}(r)\right) \ \forall \, r \in [0,1]$. For any $0 \leq r \leq 1, [u]_r$ is a bounded closed interval $[a_r, b_r]$. If $a = b$ then it is a triangular fuzzy number $u = (a, \sigma, \beta)$ where there exist exactly one $a \in \Re^1$ with $u(x) = 1$ The core of u is $core\,(u) = \{x \in \Re^1 : u(x) = 1\}$, it contains the elements with memberships u(x) =1. A crisp number k is simply represented by $\overline{u}(r) = \underline{u}(r)) = k$, $0 \leq r \leq 1$. Each crisp number is a single point while a fuzzy number is a set with degree of membership.

By appropriate definitions, the fuzzy number space $\{\overline{u}(r), \underline{u}(r)\}$ becomes a convex cone space E^1. The set E^1 is all (real) fuzzy numbers on $\Re^1$ which are normal, upper semi-continuous, convex, and compactly supported fuzzy sets.

Definition

The algebraic operations such as addition, subtraction, and multiplication by a real number k between fuzzy numbers are defined as follows: Let $u = (\underline{u}(r), \overline{u}(r))$ and $v = (\underline{v}(r), \overline{v}(r))$ be two arbitrary fuzzy numbers, hence:

1. $u = v$ if and only if $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$.
2. $u + v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$.
3. $ku = \begin{cases} (k\underline{u}(r), \ k\overline{u}(r)), & k \geq 0, \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0, \end{cases}$

2.2 Fractional calculus

Nowadays the fractional integrals FI and fractional order derivatives FD become essential tools in formulating mathematical models in many fields of sciences physical, engineering and economics [5-7,22,23]. In some sense the fractional order models generalize the classical standard integer order models. We briefly summaries the essential tools required in this work.

Definition

The Riemann-Liouville integral of fractional order α, of a function $f(t)$ denoted by $I^\alpha f(t)$is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau \qquad (3)$$

$\alpha > 0, t > 0$.

Definition

The Riemann-Liouville fractional Derivative, of a function $f(t)$ of order α denoted by $D_R^\alpha f(t)$ is defined as

$$D_R^\alpha f(t) = \frac{d^n}{dt^n} I^{n-\alpha} f(t) \qquad (4)$$

$(n-1) \leq \alpha < n, \ \ t > 0, \ \ \ n \in N$

Definition

The Caputo fractional Derivative, of a function $f(t)$ of order α denoted by $D_R^\alpha f(t)$ is defined as

$$D_C^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)^{(n)}\, d\tau \qquad (5)$$

Where:

$(n-1) \leq \alpha < n, \ t > 0, \ n \in N$.

It is well known that the Riemann-Liouville Derivative of the power function is the as the Caputo Derivative and is given by

$$D_R^\alpha t^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} t^{p-\alpha} = D_C^\alpha t^p, p \in N \qquad (6)$$

2.3 Singular perturbed Boundary Value Problem

We consider singular perturbed boundary value problem of the form, [14,18]:

$$\varepsilon y''(x) - p\, y'(x) \ - q\, y(x) = f(x), \qquad (7)$$

$(a \leq x \leq b)$,

$y(a) = A, y(b) = B,$

where, ε is small positive parameter, and $q, |p| > 0$ are constants, this Equation gives rise to a singular perturbation problem, when ε is very small. Moreover, its solutions contain boundary-layers of width $O(\varepsilon)$ at one or both boundaries, $x = a,\ x = b.$

Generalization, for such problems can be extended in many forms

The first form considers the fractional derivative, the second considers the fuzzy boundary conditions, and the third considers both the fractional derivative and the fuzzy boundary conditions

We study the effect of the parameter on the behavior of solution of this problem.

2.4 Finite Difference Methods

As in the integer case the derivatives are approximated by a combination of function vales at certain points within the interval of definition so, the interval [a, b] is partitioned into N subintervals by the $N+1$points

$$x_i = a + ih\ ,\quad i = 0,1,\dots,N.$$
$$h = \frac{b-a}{N}, \tag{8}$$

To approximate the fractional order derivatives, using the shifted Grünwald-Letnikov formula, [5, 6, 16].

$$D_{SG}^{\alpha}u(x) \cong \frac{1}{h^{\alpha}}\sum_{k=0}^{\left[\frac{x-\alpha}{\Delta x}\right]} g_i^{\alpha}\, u(x-(i-1)h) + O(h^{\alpha}), \tag{9}$$

were, $g_i^{\alpha} = (-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha-i+1)}.$

If $\alpha = 2$ (integer number) then,

$g_0 = 1, g_1 = -2, g_2 = 1, g_3 = 0$

$$u''(x_i) \cong \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} + O(h^2), \tag{10}$$

The classical central difference approximation

2.5 Fuzzy Linear Systems

We consider $n \times n$, fuzzy linear systems of equations of the form, [1, 2,4].

$$\sum_{j=1}^{n} a_{ij}\, x_j = y_i,\ i = 0,1,\cdots,n \tag{11}$$

Where, a_{ij} are real numbers and y_i are fuzzy numbers written in the parametric form, $y_i = \left(\underline{y_i}(r),\ \overline{y_i}(r)\right)$.

Which can be written in matrix form

$$A\,X = B \tag{12}$$

Friedman, used the embedding approach to replace $n \times n$, fuzzy linear systems by another $2n \times 2n$ crisp system

$$S\,X^c = B^c \tag{13}$$

S is a matrix of nonnegative real numbers defined from the elements a_{ij} of the original matrix A as follows,

If $a_{ij} \geq 0$ then $s_{i,j} = a_{ij}$ and $s_{i+n,j+n} = a_{ij}$

If $a_{ij} < 0$ then $s_{i,j+n} = -a_{ij}$ and $s_{i+n,j} = -a_{ij}$

Otherwise, $s_{i,j} = 0$

$(B^c)_i = \underline{y_i}(r)\,,\quad i \leq n,$

$(B^c)_i = -\overline{y_i}(r)\,,\quad i > n,$

In the same way the unknown vector X^c

$(X^c)_i = \underline{x_i}(r)\,,\quad i \leq n,$

$(X^c)_i = -\overline{x_i}(r)\,,\quad i > n,$

3. Experimental Consideration

The study of fractional order boundary value problems with fuzzy parameters is a recently growing area of research.

Mathematical modelling of life applications will remain a core part in many modern academic studies. The complications in realistic behaviors make the theoretical models far from reality.

The subject of boundary value problems and their numerical solutions has a long history [24,25] and there is a vast literature on it, investigating this problem in various functional settings

As a preliminary attempt, in this work we study the efficient solution of large linear algebraic systems by iterative techniques and use the imbedding approach to handle fuzzy algebraic systems.

Example 1

This is a standard example,[1] mentioned to illustrate the performance of the technique:

$$x_1 - x_2 = (r, 2-r),$$
$$x_1 + 3x_2 = (4+r, 7-2r) \tag{14}$$

Solution

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix},\ B = \begin{bmatrix} (r, 2-r) \\ (4+r, 7-2r) \end{bmatrix} \tag{15}$$

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix},\ B^c = \begin{bmatrix} r \\ 4+r \\ r-2 \\ 2r-7 \end{bmatrix} \tag{16}$$

It an easy task to solve the system

$S\,X^c = B^c$

To obtain the fuzzy number solutions

$$x_1 = (1.375 + 0.625r, 2.875 - 0.875r),$$
$$x_2 = (0.875 + 0.125\,r, 1.375 - 0.375r) \tag{17}$$

Example 2

We consider a second order singular perturbed problem with fuzzy initial conditions of the form

$$-\varepsilon\, y'' + y = x,$$
$$y(0) = (0.8 + 0.2r, 1.2 - 0.2r),$$
$$y'(0) = (1 - \tfrac{1}{\sqrt{\varepsilon}} + 0.2r, 1.2 - \tfrac{1}{\sqrt{\varepsilon}} - 0.2r)\,, \tag{18}$$

Based on the mathematical operations on fuzzy numbers one can obtain the fuzzy solution shown in figures 3-6 shown below

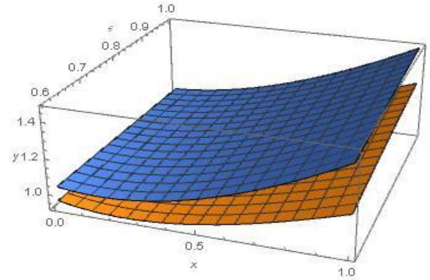

Figure 1 Fuzzy solution at $\in = 0.8$

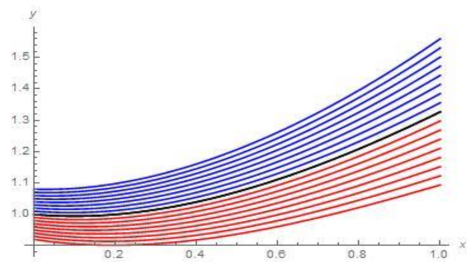

Figure 2 fuzzy solution at $\varepsilon = 0.8$ *for different values of r =0.6(0.05)1 the centered black curve occurs at r =1, represents the crisp solution as in figure 2*

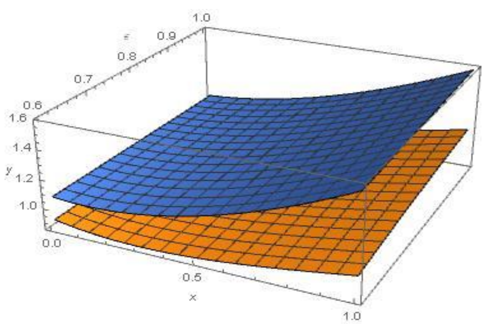

Figure 3fuzzy solution at $\in = 0.6$

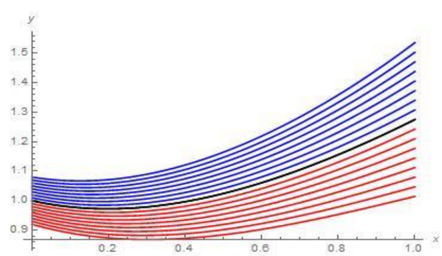

Figure 4Fuzzy solution at $\varepsilon = 0.6$ *for different values of r =0.6(0.05)1 the centered black curve occurs at r =1, represents the crisp solution.*

Example 3

We consider a fractional order singular perturbed problem with fuzzy conditions of the form,

$$\begin{aligned} &-\varepsilon\, y^{(\alpha)} + 16y = 47 - 8\,x^2, \\ &1 < \alpha \le 2;\ 0 \le x \le 2 \\ &y(0) = (2 + r, 3.5 - 0.5r), \\ &y(2) = (0.5 + 0.5r, 1.5 - 0.5r) \end{aligned} \tag{19}$$

This example is a generalization to the example considered in [4, 14, 21] ($\alpha = 2,\ \varepsilon = 1$).

Based on the mathematical operations on fuzzy numbers and the finite difference technique introduced above on the theoretical part one can obtain a corresponding fuzzy system of algebraic equations which can be solved for different values of α and ε the fuzzy solution shown in figures 7-15 shown below

Case 1: $\alpha = 2$ and $\varepsilon = 1.0$

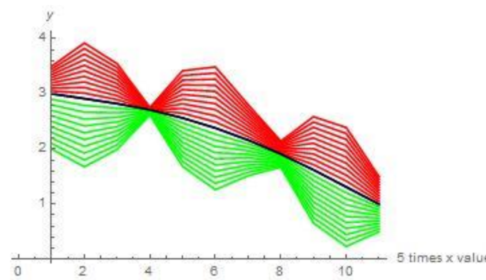

Figure 51 fuzzy solution at $\alpha = 2$ *and* $\varepsilon = 1$ *for different values of the fuzzy parameter r =0(0.1)1*

Case 2: $\alpha = 1.9$ and $\varepsilon = 0.8$

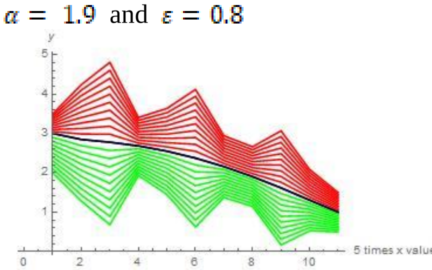

Figure 6 fuzzy solution at $\alpha = 1.9$ *and* $\varepsilon = 0.8$ *for different values of the fuzzy parameter r =0(0.1)1*

Case 3 $\alpha = 1.8$ and $\varepsilon = 0.8$

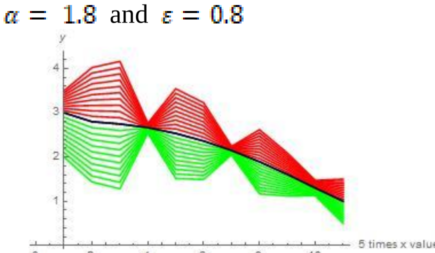

Figure 7 fuzzy solution at $\alpha = 1.8$ *and* $\varepsilon = 0.8$ *for different values of the fuzzy parameter r =0(0.1)1*

4. Conclusion

The solution of fuzzy algebraic system by the embedding approach was reviewed. The finite difference technique is used to replace a general linear boundary value problem of arbitrary fractional order with fuzzy boundary condition by a corresponding fuzzy linear system. The embedding approach is used to replace the fuzzy linear system by a corresponding crisp one. Graphs of the solution illustrates that the fuzzy solutions are generalizations of the classical ones ($r = 1.0$) as illustrated in the figures 1-7.

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