The Place and Role of Problems Combinations In The Methodical System of The Pupils' Logical Thinking

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Summary
We explain that it is crucial for mathematics teachers to take care of the conditions of formation and development of pupils' logical thinking. We indicate the method developed and tested by us for solving special problems and their combinations to be one of the means of forming such thinking. The article offers a specific example of such a technique implementation. Our experimental research with mathematics teachers reveals the benefits of using geometric problems for study and highlight several guidelines for teachers on the pupils' logical thinking formation.
Keywords:
logical thinking, methods of solving problems, teacher's methodical activity, educational goals.

1. Introduction

Schools around the world are now emphasizing the need to develop a wide range of pupils’ competencies. It is considered to be a necessary factor for success in the 21st century. For example, Singaporean schools have higher demands for pupils - they are required to reflect and justify their actions while learning mathematics [3]. It is essential for the effective teaching of mathematics to design such a learning environment so that pupils can improve and develop mathematical competence [11]. Educational programs currently try to include problem-solving skills in the teaching mathematics goals, as training the ability to solve problems develops pupils' logical thinking, reasoning skills and mathematical competence [12].

In this article, we try to explain why it is necessary and possible to teach mathematics at school to create unique conditions for forming the pupils' logical thinking. As you know, logical thinking includes mastery of methods of analysis and synthesis, abstraction and generalization, the ability to prove and disprove, draw the correct conclusions, make informed and rational decisions.

2. The role of problems in teaching mathematics to pupils

We fully agree with Miroslava Sovičová [12] that knowledge of mathematics should be formed through an active approach to learning. It is essential for mathematics teachers to understand that pupils' acquisition of mathematical competencies is effective if education is organized by studying problems, analysing situations, and setting topical issues. It is also vital to involve pupils in experiments, organize hypotheses analysis, and actively discuss them with others. Teaching mathematics by solving appropriate problems is increasingly common in primary and secondary schools [12].

Every problem involves a certain contradiction, for the solution of which thinking processes are activated until a way to solve it is found. Therefore, it is vital to use different content in various forms in teaching mathematics problems of varying complexity. We agree with Malcolm Swan [14], in learning mathematics, pupils often focus more on getting the correct answer than learning how to solve a problem. Therefore, we support the idea of using “rich tasks” that engage in reflection, promote discussion and debate, allow pupils to make their own decisions, encourage originality and ingenuity, enable the use of questions “What if?”, “What if otherwise?” [4]. Simple

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tasks do not cause the need to learn, and “rich tasks” can arouse interest in learning [14].

We understand that when using such “rich tasks”, a teacher along with pupils can solve much fewer different tasks in class than in traditional learning. However, when solving such problems, much better conditions are created for mastering the methods of solving problems, developing mental techniques, and forming a pupils' logical thinking. Jonas Jäder states that problems are a central aspect of teaching mathematics, as solving them enables pupils to understand, comprehend, and learn [5]. We also believe that the effectiveness of teaching mathematics depends significantly on the design of each problem, the method of solving it, and the combinations of problems selected by the teacher to achieve a specific learning goal. It is important to consider the particular task types cumulative effect [13].

Selection, evaluation, and possibly processing of a learning problem are essential for achieving specific learning objectives [5]. In particular, it is crucial to assess the difficulty of the problem correctly. Problems can be complex, even from a cognitive perspective. Along with their cognitive significance, other challenges for pupils' apprehension can be designed. A problem should reveal the thinking in its entirety and ensure the relevance of the solution process itself. It should also be interesting for a pupil, capture all the pupil' attention, and stimulate his activity. Depending on the choice and the specific change in the task, different learning goals can be achieved [13].
Jonas Jäder focuses on problem design with an emphasis on conceptual and creative tasks. He offers a specific structure for the selection and (re)design of problems. For example, if a problem at a particular stage of learning is too complex, it is worth making changes to the problem, which will create good conditions for achieving learning goals [5].

3. Methodology

This article originates from a broader scientific study [8; 10], which aims to develop methods for forming and developing pupils' logical thinking. To understand the conditions created for the formation of pupils' logical thinking in the process of solving specially selected problems, we chose the following task [1]:

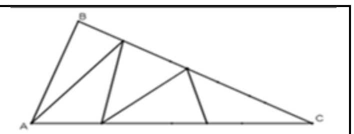
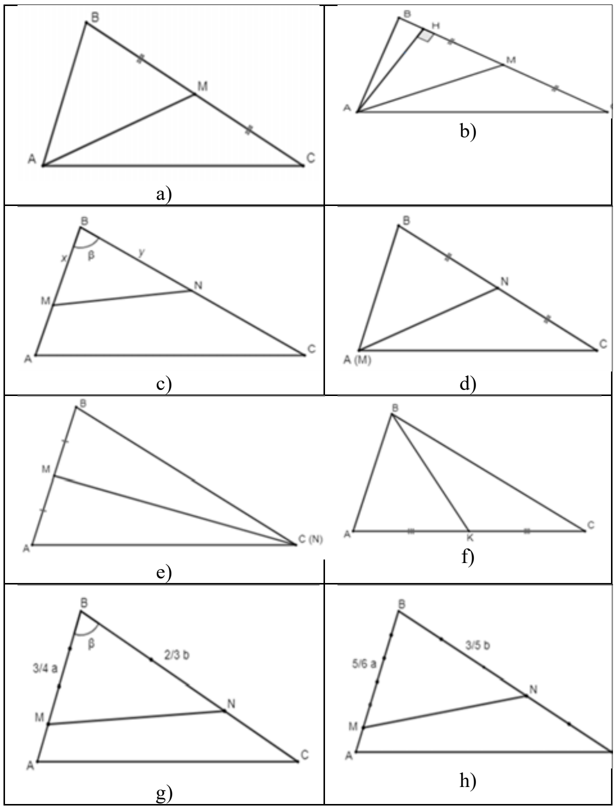
Divide the given triangle by means of a zigzag into 5 equal parts.	

Fig. 1. The problem with geometric content.

On the one hand, we understood the usefulness of this task to develop pupils' thinking. On the other hand, our initial experiment showed that the problem was too difficult for pupils. The pupils didn't even know where to start looking for a solution. The problem was proposed as an experiment to seven different mathematics teachers who teach geometry to ninth graders. All participants of the experiment claimed that without the special methodological activities of a mathematics teacher, pupils could not even begin to put forward ideas actively. To create conditions for the pupils' logical thinking formation, we offered teachers key guidelines for a deep understanding of all aspects of the methodology of solving the chosen problem. First, we should consider the problem 1: *Using one segment, divide the triangle into two equal halves.*

We should note that according to Ukrainian mathematics curricula, 9th-grade pupils should know: the property of the median of a triangle to divide it into two equal ones; formulas for calculating the area of a triangle: $S_{\Delta} = \frac{1}{2} \cdot a \cdot h_a$, $S_{\Delta} = \frac{1}{2} \cdot a \cdot b \cdot sin\, sin\, \gamma$; definitions and signs of similar triangles; main problems of construction; definition and properties of inequalities.
Our observations in one of the experimental classes recorded the following dialogue between the mathematics teacher and his pupils. It was built based on our guidelines.



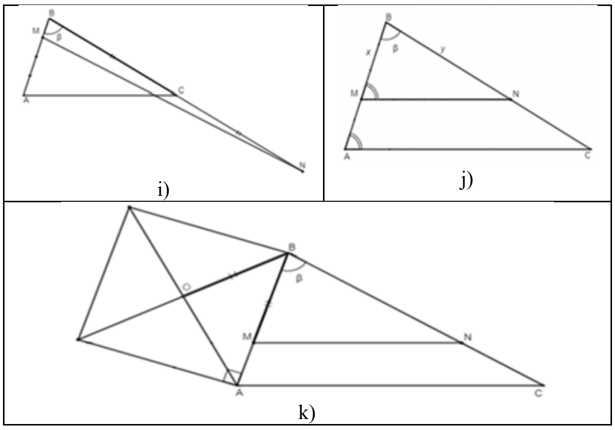


Fig. 2. Figures for solving problem 1.

Teacher: What are your thoughts on solving this problem?

Pupil A: We know the property of the median of a triangle to divide the triangle into two levels by area, so the solution of the problem looks like in Figure 2 a).

Teacher: Can anyone explain why the median *AM* of the triangle has the given property?

Pupil B: If you lower the height from the vertex A to the opposite side *BC*, the *AN* is the height for both the triangle AVM and the triangle AFM. Then we have: $S_{\Delta ABM} = \frac{1}{2} \cdot BM \cdot AH$, $S_{\Delta ACM} = \frac{1}{2} \cdot CM \cdot AH$.

AS *BM*=*CM*, then $S_{\Delta ABM} = S_{\Delta ACM}$, it can be seen in Figure 2 b).

Therefore, to divide the triangle by one segment into two equal ones, you need to draw its median.

Teacher: Indeed, we have divided this triangle into two equal parts. And is it possible to divide the triangle into two equal parts by a segment that is not the median of the triangle?

Pupil C: I assumed it was possible. So I drew Figure 2 c).

If the segment MN divides the triangle ABC into two equal parts, then $S_{\Delta MBN} = \frac{1}{2} S_{\Delta ABC}$..Since these triangles have a common angle β, consider the area by the formula $S_{\Delta} = \frac{1}{2} \cdot a \cdot b \cdot \sin\gamma$.

Let AB=*a*, BC=*b*, BM=*x*, BN=*y*, then

$S_{\Delta ABC} = \frac{1}{2} \cdot a \cdot b \cdot \sin\beta$, $S_{\Delta MBN} = \frac{1}{2} \cdot x \cdot y \cdot \sin\beta$.

As, under the condition of the problem must be $S_{\Delta MBN} = \frac{1}{2} S_{\Delta ABC}$, then

$\frac{1}{2} \cdot x \cdot y \cdot \sin\beta = \frac{1}{2} \cdot (\frac{1}{2} \cdot a \cdot b \cdot \sin\beta)$, so $x \cdot y = \frac{1}{2} \cdot a \cdot b$.

It is easy to discern that if $x = \frac{1}{2} \cdot a$, and $y = b$, then the equality holds. Or vice versa, $x = a$, i $y = \frac{1}{2} \cdot b$. This will be as shown in the Figure 2 d) and Figure 2 e), i.e. *MN* is necessarily the median of the triangle.

Teacher: Good. How many medians can be drawn in a triangle?

Pupil D: Three. That is, you can also specify the Figure 2 f).

Teacher: Let us review the equality $x \cdot y = \frac{1}{2} \cdot a \cdot b$. Why have you decided that this equality is possible only in cases $x = \frac{1}{2} \cdot a$, $y = b$ (or $x = a$, $y = \frac{1}{2} \cdot b$)? And if, for example, I take that $x = \frac{3}{4} a$, then $y = \frac{1}{2} \cdot a \cdot b \div \frac{3}{4} \cdot a = \frac{2}{3} \cdot b$. That is, I divide the segment AB into four equal parts, I choose three of them and specify a point M, and I divide the segment BC into three equal parts and choose two of them, I get the point N (Figure 2 g). I find the area of triangle MBN:

$$S_{\Delta MBN} = \frac{1}{2} \cdot \left(\frac{3}{4} \cdot a\right) \cdot \left(\frac{2}{3} \cdot b\right) \cdot \sin\beta = \frac{1}{4} \cdot a \cdot b \cdot \sin\beta =$$
$$= \frac{1}{2} \cdot \left(\frac{1}{2} \cdot a \cdot b \cdot \sin\beta\right) = \frac{1}{2} \cdot S_{\Delta ABC}.$$

$$S_{AMNC} = S_{\Delta ABC} - S_{\Delta MBN} = S_{\Delta ABC} - \frac{1}{2} S_{\Delta ABC} = \frac{1}{2} S_{\Delta ABC}$$

So, $S_{\Delta MBN} = S_{AMNC}$. This confirms that not only the median divides the triangle into two equal parts. I chose $x = \frac{3}{4} a$ as a specific example.

Pupil E: I get it! That is, if $x = \frac{5}{6} a$, then $y = \frac{1}{2} ab \div \frac{5}{6} a = \frac{3}{5} b$. The segment MN also divides this triangle into two equal parts, it can be seen in Figure 2 h).

Pupil F: But I can't, because I chose $x = \frac{1}{4} a$, then $y = \frac{1}{2} ab \div \frac{1}{4} a = 2b$, as shown in Figure 2 i). Although $S_{\Delta MBN} = \frac{1}{2} \cdot S_{\Delta ABC}$, we cannot say that the segment MN divides the triangle ABC into two equal parts, can we?

Teacher: Yes!!! This means that we should consider the condition in which the interval should be x, so that the point N belongs to the segment BC. What do you think about this?

Pupil N: I noticed when $BM = \frac{1}{2} a$ or $BM = \frac{3}{4} a$, or $BM = \frac{5}{6} a$, then everything was good; and if $BM = \frac{1}{4} a$, then the pont N went beyond the segment BC. Maybe $BM \geq \frac{1}{2} a$?

Teacher: Let us test your guess: $x \geq \frac{1}{2} a$, then $\frac{1}{2} a \leq x$. We know that if $x \cdot y = \frac{1}{2} \cdot a \cdot b$, then $\frac{1}{2} \cdot a \cdot y \leq \frac{1}{2} \cdot a \cdot b$, so, $y \leq b$. If $x \geq \frac{1}{2} a$, then $y \leq b$, and this means that N∈BC.

Pupil F: Thus, I have that MN∥AC, as shown in Figure 2 j), because MN∥AC, so ∠BMA=∠BAC, that is why ΔABC~ΔMBN, and this means that $\frac{a}{x} = \frac{b}{y}$. If $S_{\Delta MBN} = \frac{1}{2} \cdot S_{\Delta ABC}$, so $x \cdot y = \frac{1}{2} \cdot a \cdot b$. That is why it results in

$$\begin{cases} x \cdot y = \frac{1}{2} \cdot a \cdot b \\ \frac{a}{x} = \frac{b}{y} \end{cases} \longrightarrow \begin{cases} x \cdot y = \frac{1}{2} \cdot a \cdot b \\ y = \frac{x \cdot b}{a} \end{cases}$$

So, $x \cdot \frac{x \cdot b}{a} = \frac{1}{2} \cdot a \cdot b$,

$\frac{x^2}{a} = \frac{a}{2}$, $2 \cdot x^2 = a^2$, $x^2 = \frac{a^2}{2}$, $x = \frac{a}{\sqrt{2}}$. How to choose the point M now?

Teacher: $x = \frac{a}{\sqrt{2}} = \frac{a \cdot \sqrt{2}}{2}$. You need to construct the segment $a \cdot \sqrt{2}$, if the segment a is known.

Pupil Z: I know that if the side of the square is a, then its diagonal is $a \cdot \sqrt{2}$,. Therefore, we construct a square on the side a of the triangle ABC; let us draw its diagonals, which are bisected by the point of intersection, and obtain the segment $x = \frac{a \cdot \sqrt{2}}{2}$. As depicted in Figure 2 k), you need:

1) $BM = BO = \frac{a \cdot \sqrt{2}}{2}$.
2) $MN \parallel AC$

As $\Delta ABC \sim \Delta MBN$, then $\frac{S_{\Delta MBN}}{S_{\Delta ABC}} = k^2 = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} = \frac{1}{2}$.

So, $S_{\Delta MBN} = \frac{1}{2} \cdot S_{\Delta ABC}$.

Teacher: Let us summarize the results of our study. This triangle can be divided into two equal parts by one segment. We describe the segment position: on one of the sides of this triangle, let us choose a point *M*, so the distance from the vertex of the triangle is *x*:

- if $x = \frac{1}{2}a$, then the segment is the median of this triangle;
- if $x = \frac{\sqrt{2}}{2} \cdot a$, then the segment is parallel to the side of the triangle;
- if $x = a$, then the segment is the median of this triangle;
- if $x \in \left(\frac{1}{2}a \,;\, \frac{\sqrt{2}}{2}a\right) \cup \left(\frac{\sqrt{2}}{2}a \,;\, a\right)$, then on one of the other two sides of the triangle we choose a point *N*, so the distance from the same vertex of the triangle (see point *M*) is equal to y, and $y = \frac{1}{2} \cdot a \cdot b \div x$.

4. Discussion

The primary purpose of solving problem one method was to provide conditions for the pupils logical thinking formation. Pupils were placed in the conditions of a certain study under the guidance of a mathematics teacher. The question is: can a math teacher afford to spend so much time solving one problem? The answer to this question requires consideration of many different factors. When discussing with the experimenter-teachers the method of solving problem one, we found that the following factors should be taken into account: the level of academic achievement of pupils in a particular geometry class; the level of cognitive activity of pupils in the class; the teacher's priorities in determining the goals of a specific lesson; teacher's awareness of the need for pupils' logical thinking formation.

The results of our survey of 35 mathematics teachers suggest that the vast majority of teachers (78%) believe that the proposed combinations of tasks and relevant guidelines are very useful for organizing project activities of pupils in teaching mathematics with emphasis on the formation and development of their logical thinking. Accordingly, the project we recommend can be called "Divide the triangle into equal parts" and can be performed by a group of pupils capable of learning mathematics. In this case, their research activity is in solving several problems. In particular:

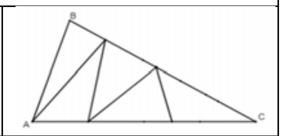
Problem 1. Divide the triangle into two equal parts using one segment.	
Problem 2. Divide the triangle into three equal parts using two segments.	
Problem 3. Divide a given triangle with a zigzag into five equal parts.	

Fig. 3. Tasks of the research project.

Problem 3 in this combination of problems falls into the best conditions to achieve the goal - forming a pupils' logical thinking. The experience gained by pupils in solving problems 1 and 2 allows us to understand the main idea: if we first consider the triangle *ACD* (I), the area of which is 1/5 of the triangle *ABC* area: it is enough to choose point D so that $CD = \frac{1}{5}CB$. The area of a triangle is equal to the half-product of the side and the height drawn to it! If we take the fifth part of the side, we will have the fifth part of the area! The height stays the same!

Continuing to act in the same way, let us review the triangle ADE (II), the area of which is $\frac{1}{4}$of the remaining part of the original triangle - the triangle ABD, it is enough to choose a point E, so that $AE = \frac{1}{4}AB$. Then it is enough to select a point F so that $DF = \frac{1}{3}DB$ and, finally, a point G so that $EG = \frac{1}{2}EB$, as shown in Figure 4.

This task is especially relevant at the stage of systematization and generalization of pupils' knowledge and skills on the topic "Areas of figures".

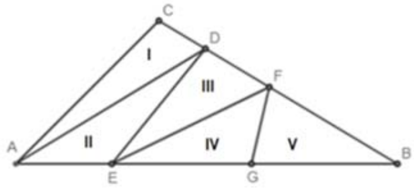


Fig. 4. To solve problem 3.

Based on our observations of the teachers' methodological activities and pupils' educational and cognitive activities, we concluded that the research tasks alone are insufficient to develop pupils' logical thinking. The methodical activity of the teacher is crucial for the organization of the active mental activity of pupils [7]. The teacher should help pupils understand someone's reasoning, ask for alternative justifications, and focus on explaining “why?”. In addition, the teacher must encourage pupils to share their ideas and different versions of their reasoning, trying to take into account the incorrect or partial contribution of pupils and expand their real contribution [2].

Our recommendations for math teachers emphasize that it is important to keep a clear eye on what pupils are saying while solving problems, be aware of the differences between different reasoning strategies, and use this reflection to regulate their understanding of problem-solving [11]. Learning to solve certain problems makes it possible to stimulate pupils' logical thinking and increase the need for argumentation and discussion as the most important part of teaching mathematics [12]. We agree with Boon Liang Chua that it is not necessary to reduce all the justifications to strict theoretical arguments in the process of solving each problem. Some explanations lend themselves well to experimental justification, which is mainly confirmed by specific examples and illustrations [3]. The tasks proposed in this article allow us to shift the emphasis from the usual situation of “getting an answer” to a position where a mathematics teacher can encourage pupils to compare different ideas in the process of solving the problem.

The problems characteristic is fundamental to provide a learning environment where problem-solving becomes accessible to pupils [6]. To solve the problem of forming the pupils' logical thinking, we pay special attention to geometric problems. In particular, we focus on the problems of proof and the problems of research.

5. Conclusion

In our recommendations for mathematics teachers, tested in the conditions of the pedagogical experiment, we, in particular, point out:

- It is worth striving to turn the process of learning mathematics into an attractive study, to provide the opportunity to profoundly understand the unknown and the chance to think flexibly and comprehensively...

- It is important that the techniques, methods and means of forming pupils' logical thinking in learning mathematics are diversified and gradually become more complex so that the share of active mental actions increases.

- Solving any problem is already a condition for the formation of pupils' logical thinking, which the teacher can improve through methodological skills.

- The successfully selected combination of tasks creates good conditions for the formation of pupils' mathematical skills. Secondly, it allows you to use and thus activate, consolidate, systematize, and develop mathematical knowledge. Third, solving a well-chosen problem, or combinations of them, will enable you to create better conditions for pupils' development of logical thinking.

We consider it essential to form and develop future mathematics teachers with a specific taste for solving the problem, building methodically sound combinations of problems, constructing problems and their systems, and collecting a particular treasure trove of original problems on various grounds. We consider it necessary to form in future teachers of mathematics deep methodical beliefs about the place and role of each task. Therefore, we have prepared manuals for future and current mathematics teachers [9; 10]. By providing guidance to teachers, we do not seek to impose our vision on them. We strive to show effective and enjoyable ways of teaching mathematics, methods of developing pupils' thinking in the process of teaching mathematics, which are tested in our experimental research.

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