Develop Integrability on Weighted Bergman Spaces

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Abstract

We investigate composition operators between weighted Bergman spaces with reduced exponents. It is demonstrated that both compactness and boundedness of such an operator can be described by the integrability condition on a generalized Nevanlinna counting function of the inducing map. By selecting specific weights, composition operators mapping into the Hardy spaces are provide.

Keywords:

(Bergman and Hardy spaces, composition operators, function of Nevanlinna counting)

1. Introduction

if : $\varphi: \mathbb{D} \to \mathbb{D}$ be an analytical over $\mathbb{D}$, we define $\mathbb{D}$ is an open unit disk on complex plane. A space of analytic functions over $\mathbb{D}$ then is provided of linear operator C_φ, defined by $C_\varphi(f) = f \circ \varphi$. Much focus has been placed on the limitations a composition operator of this kind to a variety of Banach spaces of analytical functions on $\mathbb{D}$. Every composition operator, is known to map each Hardy and Bergman's own domain in particular. Littlewood's Subordination Principle has an impact on this. (in [6] or [9]). Here, we carry on the investigation begun by Hunziker and Jarchow [3], Riedl [7], and the original author [6], and examine composition operators that arguably enhance integrability. The measure of Lebesgue over $\mathbb{D}$ is denoted by dA, normal to $A(\mathbb{D}) = 1$. $dA_{\mathcal{E}-1}$ is a measure over $\mathbb{D}$ for $\mathcal{E} > 0$ by :

$$dA_{\mathcal{E}-1}(w) = [\log(1/|w|)]^{\mathcal{E}-1} dA(w).$$

if f is analytic over $\mathbb{D}$ and take the condition $0 < p < \infty$ and $\mathcal{E} > 0$ are found in the weighted Bergman space $A^p_{\mathcal{E}-1}$.

$$\|f\|^p_{A^p_{\mathcal{E}-1}} = \int_{\mathbb{D}} |f(w)|^p dA_{\mathcal{E}-1}(w) < \infty.$$

Let $(1-|w|)^{\mathcal{E}-1}$, $[\log(1/|w|)]^{\mathcal{E}-1}$ is a similar of $1/2 \le |w| < 1$ and the universe of $dA_{\mathcal{E}-1}$ is integrable at the origin., replacing the measure $dA_{\mathcal{E}-1}$ with
$(1-|w|)^{\mathcal{E}-1} dA(w)$, yields the same functional space and an equivalent norm.
A^p_0, which stands for the unweighted Bergman space, will also be used to refer to it.
Let $p > 0$, The functions that are analytic on $\mathbb{D}$ and satisfying make up the Hardy space H^p.

$$\|f\|^p_{H^p} = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^p d\theta < \infty.$$

In [9] provided the formula for composition operator's fundamental rule from H^2 to H^2 caused by φ, [9] using a conventional Nevanlinna counting function $N_{\varphi,1}$. In the same article, Shapiro also discussed to itself from a weighted Bergman space composition

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operator. He also presented an extended Nevanlinna counting functions $N_{\varphi,\mathcal{E}-1}$, defined for $\mathcal{E} > 1$ by

$$N_{\varphi,\mathcal{E}-1}(w) = N_{\mathcal{E}-1}(w) = \sum_{z\in\varphi^{-1}\{w\}} [\log(1/|z|)]^{\mathcal{E}-1}, \quad w \in \mathbb{D}\backslash\{\phi(0)\},$$

The first author then used the expansion of these counting functions in [10] to describe when $C_\varphi: A^p_{k-2} \to A^q_\beta$ is compact or bound, that is, if $q \geq p$. By specifying A^p_{-1} to be H^p, the Hardy spaces were taken into account in these findings; for more information, see the notes under Lemma (2.1). In the remaining instance, when $q < q$, a growth condition was similarly demonstrated in [10] to be sufficient for boundedness and compactness, but a characterization was left unspecified. In this study, we demonstrate that this characterization is provided by an inerrability constraint about the counting operations.

Theorem (1.1)

if $0 < q < p$, $k > 1$, Assume that $\mathbb{D}$ is analytical self-map φ is. Consequently, these are equivalent.

(i) $C_\varphi: A^p_{k-2} \to A^q_\beta$ is bounded;

(ii) $C_\varphi: A^p_{k-2} \to A^q_\beta$ is compact;

(iii) $\frac{N_{\varphi,\beta+2}(z)}{(1-|z|^2)^k} \in L^{\frac{p}{p-q}}(dA_{k-2})$.

Proof

We start by noting that under the supposition that $\varphi(0) = 0$, it is sufficient to prove the theorem. Replacing with $\sigma_{\varphi(0)} \circ \varphi$ will demonstrate this. It should be noted that C_φ is closed and bounded if and only if $C_\varphi C_{\sigma_{\varphi(0)}})$ is closed and bounded, as $C_{\sigma_{\varphi(0)}}$ has invertible or bounded on each Bergman space of weighted.

It is similarly simple to confirm that this operates meets case (iii) of theorem (1.1) only applies if and when $N_{\varphi,\beta+2}$ does.

$$N_{\sigma_{\varphi(0)}\circ\varphi,\beta+2} = N_{\varphi,\beta+2},\circ\, \sigma_{\varphi(0)}.$$

First, we make the assumption that $\frac{N_{\varphi,\beta+2}}{(1-|z|^2)^2} + k - 2 \in L^{\frac{p}{p-q}}(dA_{k-2})$ and show $C_\varphi: A^p_{k-2} \to A^q_\beta$ is constrained. It is sufficient to demonstrate how C_φ transfers A^p_{k-2} into A^q_β. using the Closed Graph Theorem. Lemma (2.1) states that we must demonstrate that if $f \in A^p_{k-2}$, then

$$\int_{\mathbb{D}} |f|^{q-2}|f'|^2 N_{\varphi,\beta+2}\, dA < \infty. \quad (2)$$

We use the above $\frac{1}{4}\mathbb{D}$ and $\mathbb{D}\backslash\frac{1}{4}\mathbb{D}$ integral independently to demonstrate this.

In terms of the integral over $\frac{1}{4}\mathbb{D}$, Littlewood's Inequality states

$$N_{\varphi,\beta+2}(z) \leq (\log(1/|z|))^{\beta+2}.$$

We define$\psi(z) = z/2$, we have

$$N_{\psi,\beta+2}(z) = (\log(1/|z|))^{\beta+2}, \qquad |z| \leq 1/2,$$

but so

$$N_{\varphi,\beta+2}(z) \leq 2^{\beta+2} N_{\psi,\beta+2}(z), \qquad |z| \leq 1/4.$$

As a result, this estimation and Lemma (2.1) provide

$$\int_{\frac{1}{4}\mathbb{D}} |f|^{q-2}|f'|^2 N_{\varphi,\beta+2}\, dA \leq 2^{\beta+2}\int_{\frac{1}{4}\mathbb{D}} |f|^{q-2}|f'|^2 N_{\psi,\beta+2}\, dA \leq C\int_{\mathbb{D}} |f\circ\psi|^q\, dA_\beta. \quad (3)$$

The f is constrained on $\psi(\mathbb{D})$, hence it follows that

$$\int_{\frac{1}{4}\mathbb{D}} |f|^{q-2}|f'|^2 N_{\varphi,\beta+2}\, dA < \infty. \quad (4)$$

After that, using Lemma (2.3) to $w \in\backslash\frac{1}{4}\mathbb{D}$,

$$\int_{\mathbb{D}\backslash\frac{1}{4}\mathbb{D}} |f|^{q-2}|f'|^2 N_{\varphi,\beta+2}\, dA \leq C\int_{\mathbb{D}} |f(w)|^{q-2}|f'(w)|^2 \frac{1}{(1-|w|^2)^2}\int_{D(w,\frac{1}{4})} N_{\varphi,\beta+2}(z)\, dA(z)\, dA$$

Applying Fubini's Theorem to the fact that $\chi_{D(w,1/4)}(z) = \chi_{D(z,1/4)}(w)$, $1-|w|^2 \approx 1-|z|^2$ for $w \in D(z,1/4)$, and Lemma (2.3) to determine that the aforementioned is less t

$$C\int_{\mathbb{D}} \int_{D(z,\frac{1}{4})} |f(w)|^{q-2}|f'(w)|^2 dA(w) \frac{N_{\varphi,\beta+2}(z)}{(1-|z|^2)^2}\, dA(z) \leq C\int_{\mathbb{D}} \int_{D(z,\frac{1}{2})} |f(w)|^q dA(w)\, N_{\varphi,\beta+2}(z)(1-|z|^2)^4\, dA(z).$$

Thus, another use of Fubini's Theorem results in

$$\int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} |f|^{q-2}|f'|^2 N_{\varphi,\beta+2}dA$$

$$\leq C\int_{\mathbb{D}} |f|^q H dA_\alpha, \qquad (5)$$

where

$$H(w)=\int_{(w,\frac{1}{2})}\frac{N_{\varphi,\beta+2}(z)}{(1-|z|^2)^4}dA(z)\cdot\frac{1}{(1-|w|^2)^{k-2}}.$$

As shown in (1), the lower bound of $M[f]$results in that

$$H(w)\leq C\cdot M\left[\frac{N_{\varphi,\beta+2}}{(1-|z|^2)^{\frac{q}{p}k}}\right](w)\cdot\frac{1}{(1-|w|^2)^{\frac{p-q}{p}(k-2)}}.$$

Assuming that case (iii) applies and the widely accepted estimate that, for $s>1, \|M[f]\|_s\leq C\|f\|_s$, now prove:

$$\int H(w)^{\frac{p}{p-q}}dA_{k-2}(w) \leq C\left\|M\left[(1-|z|^2)^{-\frac{q}{p}k-4}N_{\varphi,\beta+2}(z)\right]\right\|_{\frac{p}{p-q}}^{\frac{p}{p-q}} \leq C\left\|\frac{N_{\varphi,\beta+2}}{(1-|z|^2)^k}\right\|_{L^{\frac{p}{p-q}}(dA_{k-2})}^{\frac{p}{p-q}} <\infty,$$

Consequently, $H\in L^{\frac{p}{p-q}}(dA_{k-2})$. Since we're assuming that $k>1$ and $A^p_{k-2}, \int|f|^p dA_{k-2}<\infty$. In light of Hölder's inequality and (5), it follows that.

$$\int_{\mathbb{D}\setminus\frac{1}{4}\mathbb{D}} |f|^{q-2}|f|2N_{\varphi,\beta+2}dA<\infty.$$

When combined with (4), this demonstrates that (2) is true and that $C_\varphi: A^p_{k-2}\to A^q_\beta$ is bounded. The demonstration that case (iii) entails theorem (1.1) is now complete case (i). Now, we demonstrate that $\frac{N_{\varphi,\beta+2}}{(1-|z|^2)^k}\in L^{\frac{p}{p-q}}(dA_{k-2})$Actually, (dA (k-2)) suggests that $C_\varphi: A^p_{k-2}\to A^q_\beta$is compact. We must demonstrate that there is a subsequence of $\{C_\varphi f_n\}$ that converges in A^q_β *and* $\{f_n\}\subset A^p_{k-2}$ and $\|f_n\|_{A^p_{k-2}}\leq 1$. Functions on the unit ball of A^p_{k-2} are members of a normal family because according to Lemma 2.5 in [10], they are bounded uniformly on closed and bounded subsets over $\mathbb{D}$. We may then suppose that $\{f_n\}$convergent to f uniformly over compact subsets over $\mathbb{D}$ by moving to a subsequence, and by looking at integrals over $r\ \mathbb{D}$ and letting r increasing to 1, if $f\in A^p_{k-2}$. Given that we have established that $C_\varphi: A^p_{k-2}\to A^q_\beta$is bounded, $C_\varphi f\in A^q_\beta$ We can therefore assume that the sequence $\{f_n\}$, uniformly convergent to 0 on closed and bounded to subsets f of $\mathbb{D}$ by deducting f from each member of the sequence, and it is sufficient to demonstrate that $\lim_{n\to\infty}\|C_\varphi f_n\|_{A^q_\beta}=0$. It follows from Lemma (2.1) that we must demonstrate

$$\lim_{n\to\infty}\int_{\mathbb{D}} |f_n|^{q-2}|f_n'|^2 N_{\varphi,\beta+2}dA=0. \qquad (6)$$

The proof now continues the line of reasoning used to demonstrate that C_φ was constrained by taking the integrals over $\frac{1}{4}\mathbb{D}$ and $\mathbb{D}\setminus\frac{1}{4}\mathbb{D}$ into separate consideration. By using (3) with f_n in place of f, we obtain that

$$\lim_{n\to\infty}\int_{\frac{1}{4}\mathbb{D}} |f_n|^{q-2}|f_n'|^2 N_{\varphi,\beta+2}dA=0, \qquad (7)$$

since the sequence $\{f_n\}$ uniformly converges to $0, \psi(\mathbb{D})$. It is a straightforward task to demonstrate that, for the integral over $\mathbb{D}\setminus\frac{1}{4}\mathbb{D}$, Hölder's inequality applied to (5) implies that since $H\in L^{\frac{p}{p-q}}(dA_{k-2})$, the assumptions regarding the sequence $\{f_n\}$

$$\lim_{n\to\infty}\int_{\mathbb{D}\setminus\frac{1}{4}\mathbb{D}} |f_n|^{q-2}|f_n'|^2 N_{\varphi,\beta+2}dA=0.$$

Combined with (7) this shows that (6) holds, and therefore that $C_\varphi: A^p_{k-2}\to A^q_\beta$ is closed and bounded. Theorem (1.1) case (iii) is now fully demonstrated to imply Theorem (1.1) case (ii). In order to demonstrate that theorem (1.1) case I implies theorem (1.1) case, it must be shown that any compact operator is bounded (iii). The notion of atomic decomposition of the A^p_{k-2} spaces is used in this proof. If the pseudo hyperbolic distance between points in the series, let $\{z_n\}\subset\mathbb{D}$, is constrained below, then the separation constant for the sequence is said to be non-negative.

$$\inf_{n\neq m}|\sigma_{z_n}(z_m)|>0.$$

Let $f(z)=\sum_n a_n k_n(z)$, where

$$k_n(z)=\bar{z}_n^{-1}\frac{(1-|z_n|^2)^{M-2/p-k+2/p}}{(1-\bar{z}_n z)^M}$$

and $M > (k)\max(1, 1/p)$. The fact that if $\{z_n\}$ is a non-negative separation constant, then a constant C exists such that

$$\|f\|_{A^p_{k-2}} \le C\left(\sum_n |a_n|^p\right)^{1/p}.$$

In Theorem (2.2) in [8]. (let $k > 1$) We use this disparity is being replaced by $a_n r_n(t)$, if r_n is an introducing the Rademacher functions in. let $C_\varphi: A^p_{k-2} \to A^q_\beta$ is bounded, and suppose $q \ge 2$. Then, recalling that $|r_n(t)| = 1$,

$$\int_{\mathbb{D}} \left|\sum_n a_n r_n(t) k_n \circ \varphi\right|^q dA_\beta \le \|C_\varphi\|^q \|f\|^q_{A^p_{k-2}}$$

$$\le C\left(\sum_n |a_n|^p\right)^{q/p}.$$

This disparity from 0 to 1 is integrated with regard to t to give us

$$\int_{\mathbb{D}} \sum_n |a_n|^q |k_n \circ \varphi|^q \, dA_\beta$$

$$\le \int_{\mathbb{D}} \left(\sum_n |a_n|^2 |k_n \circ \varphi|^2\right)^{q/2} dA_\beta$$

$$\le C\left(\sum_n |a_n|^p\right)^{q/p}.$$

The second inequality resulted from Khinchine's Inequality, while the first inequality was based solely on the assumption that $q \ge 2$.

Lemma (2.4) leads to the conclusion that

$$\sum_n |a_n|^q \int_{\mathbb{D}} |k_n|^{q-2} |k_n'|^2 N_{\varphi,\beta+2} dA$$

$$\le C\left(\sum_n |a_n|^p\right)^{q/p}.$$

It is easily verified that

$$\frac{1}{(1-|z_n|^2)^{(k)\frac{q}{p}+2}} \le C|k_n|^{q-2}|k_n'|^2$$

on $D(zn, 1/2)$, and thus

$$\sum_n |a_n|^q \frac{\int_{D(z_n,\frac{1}{2})} N_{\varphi,\beta+2} dA}{(1-|z_n|^2)^{(k)\frac{q}{p}+2}} \le C\left(\sum_n |a_n|^p\right)^{q/p}.$$

Now, the discussion goes just as it did in [5]. We provide a summary of the defense in order to be thorough; See in [5] for more information. The inequality above leads us to the conclusion that the sequence

$$\frac{\int_{D(z_n,\frac{1}{2})} N_{\varphi,\beta+2} dA}{(1-|z_n|^2)^{(k)\frac{q}{p}+2}}$$

belongs to $\ell^{p/q}$ dual or equivalently

$$\sum_n \left(\frac{\int_{D(z_n,\frac{1}{2})} N_{\varphi,\beta+2} dA}{(1-|z_n|^2)^{(k)\frac{q}{p}}}\right)^{\frac{p}{p-q}} A\left(D\left(z_n, \frac{1}{2}\right)\right) < \infty.$$

We select the sequence $\{z_n\}$ has von-negative a constant length and such that the disks $D(z_n, 1/4$ over $\mathbb{D}$ in order to obtain the integrability condition we need from this. Then:

$$\int_{\mathbb{D}} \left(\frac{\int_{D(z_n,\frac{1}{4})} N_{\varphi,\beta+2} dA}{(1-|z|^2)^{4+\frac{q}{p}(k-2)}}\right)^{\frac{p}{p-q}} dA$$

$$\le \sum \int_{D(z_n,1/4)} \left(\frac{\int_{D(z,\frac{1}{24})} N_{\varphi,\beta+2} dA}{(1-|z|^2)^{4+\frac{q}{p}(k-2)}}\right)^{\frac{p}{p-q}} dA$$

$$\le C\sum_n \left(\frac{\int_{D(z_n,\frac{1}{2})} N_{\varphi,\beta+2} dA}{(1-|z_n|^2)^{4+(k-2)\frac{q}{p}}}\right)^{\frac{p}{p-q}} A\left(D\left(z_n, \frac{1}{4}\right)\right),$$

This is limited by the previously shown inequality, and as a result

$$\frac{\int_{D(z,\frac{1}{4})} N_{\varphi,\beta+2} dA}{(1-|z|^2)^{4+\frac{q}{p}(k-2)}} \in L^{\frac{p}{p-q}}(dA).$$

Given that $\frac{1}{4}\mathbb{D}$ is integrability is obvious, Lemma (2.3) now demonstrates that

$$\frac{N_{\varphi,\beta+2}}{(1-|z|^2)^{4+\frac{q}{p}(k-2)}} \in L^{\frac{p}{p-q}}(dA),$$

Additionally, it is simple to verify that this is identical to

$$\frac{N_{\varphi,\beta+2}}{(1-|z|^2)^k} \in L^{\frac{p}{p-q}}(dA_{k-2}).$$

With this, the theorem's proof is complete. Theorem (1.1) case (i) implies theorem (1.1) case (iii) in the case that $q \ge 2$. Let $q < 2$, let m be is an integer such that $mq \ge 2$. Assume that theorem (1.1) case (i) holds, i.e. $C_\varphi: A^p_{k-2} \to A^q_\beta$ is bounded, and let $f \in A^{mp}_{k-2}$. Then $f^m \in A^p_{k-2}$, and so $C_\varphi f^m = (f \circ \varphi)^m \in A^q_\beta$. This is equivalent to $f \circ \varphi \in A^{mq}_\beta$, and so $C_\varphi: A^{mp}_{k-2} \to A^{mq}_\beta$ is

Theorem of Closed Graphs limits. if $mq \geq 2$, we comprehend that

$$\frac{N_{\varphi,\beta+2}}{(1-|z|^2)^k} \in L^{\frac{mp}{mp-mq}}(dA_{k-2}) = L^{\frac{p}{p-q}}(dA_{k-2}),$$

We remark that the theorem f continues to fail if $k = 1$. For particular, let $\varphi(z) = z$, then $C_\varphi: H^2 \to H^1$ is not closed but bounded, and theorem (1.1) case (iii) not true. We also see that Littlewood's Inequality's easy side effect is that, stated above, that theorem (1.1) case (iii) complete for each analytical self-map φ of $\mathbb{D}$ whenever $q/p < (1+\beta)/(k-1)$ .In special case , this satisfy $\beta \geq k-2$, $q/p < 1$. The case of this theorem where $k = 2$ and $\beta = -1$ is the following corollary.

Corollary (1.2)

if $q > 0, p > q$, Assume that φ $\mathbb{D}$ is analytical self-map on $\mathbb{D}$. In this situation, the following are comparable.

(i) $C_\varphi: A^p \to H^q$ is bounded;

(ii) $C_\varphi: A^p \to H^q$ is bounded and closed;

(iii) $\frac{N_{\varphi,1}(z)}{(1-|z|^2)^2} \in L^{\frac{p}{p-q}}(dA)$.

This corollary raises a natural query: The circumstance $N_{\varphi,1}(z)/(1-|z|^2)^2 \in L^1(dA)$?

denotes what operator theoretic property of C_φ? The first author and R. Zhao recently replied to this in Theorem (1.3) of [11], where it is proved that this case is equal to $C_\varphi: \mathcal{B} \to H^2$ is and closed bounded. Bloch space is denoted here by $\mathcal{B}$. We give an example of how Corollary (1.2) is applied a maps of polygonal. Assume that $P \subset \overline{\mathbb{D}}$ is a polygon, and that π/η is the biggest angle at a P number of vertices of is located on $\partial\mathbb{D}$. It was established in Theorem (6.7) in [10] is $C_\varphi: A^p \to H^q$ is bounded if is an analytical self-map over $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset P$, $\eta \geq 2q/p \geq 2$. The sharpness of this conclusion is that C_φ need not be closed , bounded and the lower bound for η decreasing. We are now able to obtain a precise solution for the example $p > q$ using Corollary (1.2).

Theorem (1.3)

if $q > 0, p > q$, let P and η to the above. If φ is a bounded or equivalently compact $C_\varphi: A^p \to H^q$ analytic self-map of $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset p$, $\eta > 2q/p$, then is an analytical self-map of D. Additionally, this not true $2q/p$.

We simply provide a brief outline of the proof; for more information, read the proof of Theorem (6.7) in [10]. It is sufficient to investigate the situation where φ is a Riemann map from $\mathbb{D}$ onto P in accordance with Littlewood's Subordination Principle. Since $N_{\varphi,1}$ is supported on P, all that is left to do is verify that each of P's infinitely numerous vertices on $\partial\mathbb{D}$ satisfies the integrability requirement (1.1)(iii). The Schwarz Reflection Principle's simple implication that $N_{\varphi,1}(w) = O((1-|w|^2)^\eta)$, as $|w| \to 1$, putting these numbers in a sector that is acceptable and has a vertex at the maximum angle that P can make with $\partial\mathbb{D}$. Thus, using polar coordinates at each vertex in turn, we see theorem (1.1) case (iii) holds if and only if $(2-\eta)p/(p-q) < 2$, or equivalently $\eta > 2q/p$.

Therefore, we can observe that the case (iii) of theorem (1.1) applies if and only if $(2-\eta)p/(p-q) < 2$, or alternatively if $\eta > 2q/p$. We demonstration on using the strategy D. H. Luecking developed in [5].

$$\left(\int_{\mathbb{D}} \left|f^{(n)}\right|^q d\mu\right)^{1/q} \leq C\|f\|_{A_p},$$

if $p > q$. The measures that satisfy this inequality are described by Luecking, but it is the necessity portion of the argument that interests us the most and is the one that we use in this case. When working with composition operators that map to A^q_β, We calculate form integrals.

$$\int_{\mathbb{D}} |f|^{q-2}|f'|^2 d\mu;$$

see below, Lemma (2.1). The proof of Theorem (1.1) would be easier if we could make the assumption that $q = 2$. Luecking's result would then be applicable. Although it doesn't seem possible, Luecking's method still holds true. In this context, we point out that, for an integer $n \geq 1$, $k \geq 2$, $C_\varphi: A^p_{k-2} \to A^q_\beta$is bounded if and only if $C_\varphi: A^{np}_{k-2} \to A^{nq}_\beta$ is bounded. This derives from the work of C. Horowitz. For functions in $A^p_{k-2}, k \geq 2$, one path is straightforward, and the other relies on Horowitz's factorization theorem in Theorem 1 in [2]. Even with this outcome, it does not appear to be simple to reduce to assuming that $q = 2$ if $k \geq 2$. We Describe the generalized counting functions and weighted Bergman spaces in general. Review is also given to Khinchine's Inequality, which was crucial in [5] and is employed in our prved of Theorem (1.1). The prove of Theorem (1.1) will follows. The following lemma explains how the generalized counting functions can be used to study composition operators between weighted Bergman spaces.

2. Methodology

Lemma (2.1)

If $\mathcal{E} \geq 0$, φ is analytical self-map over $\mathbb{D}$ and let f be analytic on D.,

$$\|f \circ \varphi\|^p_{A^p_{\mathcal{E}-1}} \approx |f(\phi(0))|^p + \int_{\mathbb{D}} |f|^{p-2}|f'|^2 N_{\varphi,\mathcal{E}+1} dA.$$

In this case, the sign "≈" denotes that the right-hand side's continuous multiples, which are non-negative and related to$f$, delimited the left-hand side above and below. A comment is necessary for the case where $\mathcal{E} = 0$. Remember that the initial definition of the spaces $A^p_{\mathcal{E}-1}$ only included $\mathcal{E} > 0$?

This is because it is clear that $A^p_{\mathcal{E}-1}$ only contains the zero function if and only if $\mathcal{E} \leq 0$, then $dA_{\mathcal{E}-1}(D) = \infty$. However, the right side of the graphic above behaves nicely with $\mathcal{E} = 0$. In fact, it is an exception to C. S. Stanton's Analytical functions with integral methods formula [12] (also in [1] and [10]) that

$$\|f \circ \varphi\|^p_{H^p} = |f(\phi(0))|^p + \frac{p^2}{2}\int_{\mathbb{D}} |f|^{p-2}|f'|^2 N_{\varphi,1} dA.$$

This formula and the assertion in Lemma (2.1) can be compared to demonstrate how the definition $A^p_{-1} = H^p$ allows us to take the Hardy spaces, into account when presenting our findings. We are now compiling a list of generalized counting function characteristics that we will need. First is Littlewood's Inequality, a constraint for $N_{\psi,\mathcal{E}-1}$. Littlewood set this up for $\mathcal{E} = 2$ in [4], and Shapiro expanded it to $\mathcal{E} > 2$ in Proposition (6.3) in [9]. If ψ is an analytic self-map of $\mathbb{D}$ and let $\mathcal{E} \geq 2$. If $\psi(0) = 0$, then

$$N_{\psi,\mathcal{E}-1}(z) \leq (\log(1/|z|))^{\mathcal{E}-1}, \qquad z \in D\backslash\{0\}.$$

If $\lambda \in \mathbb{D}$ let $\sigma_{\mathcal{E}-1}(w) = (\mathcal{E} - 1 - w)/(1 - \bar{\lambda}w)$ be $\mathbb{D}$ is automorphism, which switches0 and λ 0 and. The following lemma demonstrates how these automorphisms modify the counting functions under composition. It is a straightforward result of how counting functions are defined; for more information, see Lemma (4.2) in [10].

Lemma (2.2)

Let ψ be an analytic self-map of $\mathbb{D}$ and let $\mathcal{E} - 1 \in D$. Then

$$(N_{\psi,\mathcal{E}-1}) \circ \sigma_{\mathcal{E}-1} = N_{\sigma(\mathcal{E}-1)\circ\psi,\mathcal{E}-1}.$$

The fact that these counting functions meet a subharmonic mean value feature is a crucial characteristic of them. This was demonstrated in [1] for the classical case =2 and by Shapiro as Corollary 6.7 of [9] for the generalized counting functions.

If ψ is an analytic self-map of $\mathbb{D}$ and let $\mathcal{E} > 0$. let $\psi(0) \neq 0$, $0 < r < |\psi(0)|$, then

$$N_{\psi,\mathcal{E}-1}(0) \leq \frac{1}{r^2}\int_{rD} N_{\psi,\mathcal{E}-1} dA.$$

A conformally invariant definition of this characteristic is given in the lemma that follows, and it will be used to support the theorem (1.1). We use $D(\lambda, \delta)$ is a pseudo hyperbolic disk with center $\mathcal{E} - 1$ and radius δ:

$$D(\lambda, \delta) = \{w: |\sigma\lambda(w)| < \delta\}.$$

Lemma (2.3)

if ψ is an analytic self-map of $\mathbb{D}$ satisfying $\psi(0) = 0$ and let $\mathcal{E} \geq 2$. There is a constant C such that if $1 > |w| > 1/4$, then

$$N_{\psi,\mathcal{E}-1}(w) \leq \frac{C}{(1 - |w|^2)^2}\int_{D(w,\frac{1}{4})} N_{\psi,\mathcal{E}-1} dA.$$

Proof

A property known as subharmonic mean value, Lemma (2.2), and then Lemma (2.2) once more, we have

$$N_{\psi,\mathcal{E}-1}(w) = N_{\sigma_w\circ\psi,\mathcal{E}-1}(0) \leq 16\int_{\frac{1}{4}\mathbb{D}} N_{\sigma_w\circ\psi,\mathcal{E}-1} dA = 16\int_{\frac{1}{4}\mathbb{D}} N_{\psi,\mathcal{E}-1} \circ \sigma_w dA.$$

Note that a property known as subharmonic mean value was applied on a premise that $|\sigma_w \circ \psi(0)| = |w| > 1/4$.

a change in the variable $\zeta = \sigma w(z)$, along with the projection $|\sigma'_w(\zeta)| \approx (1 - |w|^2)^{-1}$ for $\zeta \in D(w, 1/4)$, now includes

$$N_{\psi,\mathcal{E}-1}(w) \leq \frac{C}{(1 - |w|^2)^2}\int_{D(w,\frac{1}{4})} N_{\psi,\gamma\mathcal{E}-1} dA.$$

Lemma (2.4)

let $\lambda \in \mathbb{D}$,, for every function f analytic on $\mathbb{D}$,

$$\int_{D(\lambda,1/4)} |f|^{q-2}|f'|^2 dA \leq \frac{C}{(1 - |\lambda|^2)^2}\int_{D(\lambda,1/2)} |f|^q dA.$$

Proof

Making the variable change $w = \sigma_\lambda(z)$ and let $|\sigma'_\lambda(z)| \approx (1-|\lambda|^2)^{-1}$, $z \in D(\lambda, 1/2)$, let $\lambda = 0$. Following this reduction, the resulting estimate follows naturally from Lemma (2.1), we use $\mathcal{E} - 1$, $\varphi(z) = z/2$.

Khinchine's Inequality is used in our proof of Theorem (1.1). The {rn (t)}, Rademacher functions, defined as follows, are involved in this inequality:

$$r_0(t) = \begin{cases} 1, & 0 \le t - [t] < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t - [t] < 1; \end{cases}$$
$$r_n(t) = r_0(2^n t), \qquad n > 0.$$

let $0 < p < \infty$ and let $\{c_j\}_{j=1}^{m}$ involves complex numbers. Constants exist. $0 < b_p \le B_p < \infty$, just depends on p, so that

$$b_p \left(\sum_{j=1}^{m} |c_j|^2 \right)^{p/2} \le \int_0^1 \left| \sum_{j=1}^{m} c_j r_j(t) \right|^p dt \le B_p \left(\sum_{j=1}^{m} |c_j|^2 \right)^{p/2}.$$

The Hardy-Littlewood maximum function can be used to provide a bound for the average of a function over a pseudohyperbolic disk, which is the last step in our proof of Theorem (1.1). Let $M[f]$ stand for the maximal Hardy-Littlewood function for f, that is

$$M[f](z) = \sup_{\delta > 0} \frac{1}{A(B(z,\delta))} \int_{B(z,\delta)} |f| dA.$$

Here $B(z,\delta) = \{w : |z - w| < \delta\}$ and f is extended to be 0 outside its original domain of definition. Since there is a constant C such that $D(\lambda, 1/2) \subset B(\lambda, C(1-|\lambda|^2))$ for all $\lambda \in \mathbb{D}$, it follows that

$$\frac{1}{(1-|\lambda|^2)^2} \int_{D(\lambda,1/2)} |f| dA \le C \cdot M[f](\lambda). \qquad (8)$$

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